A CASCADE OF DETERMINANTAL CALABI–YAU THREEFOLDS

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with Appendix by Piotr Pragacz

Abstract. We investigate geometric properties of Calabi–Yau threefolds defined as linear sections of determinantal varieties. By joining them via a natural sequence of conifold transitions, we compute their Hodge numbers and describe the morphisms corresponding to the faces of their Kähler–Mori cone.

1. Introduction

By a Calabi–Yau threefold we mean a smooth complex projective threefold with trivial canonical divisor, such that $H^1(\mathcal{O}_X) = 0$. The basic examples of such threefolds are complete intersections in appropriate projective spaces or in homogenous varieties. The aim of this paper is to enlarge the class of easy to handle Calabi–Yau threefolds by working out the geometric properties of Calabi-Yau threefolds whose ideals in projective spaces are defined by the minors of an appropriate matrix of linear forms.

In particular, using results of Namikawa [Nm], we obtain that the Calabi–Yau threefold defined in $\mathbb{P}^7$ by the 3×3 minors of a generic 4×4 matrix of linear forms has Hodge numbers $h^{1,1} = 2$ and $h^{1,2} = 34$. The Calabi–Yau threefolds defined in $\mathbb{P}^8$ by the 3×3 minors of a generic 4×5 partially symmetric matrix of linear forms in $\mathbb{P}^8$ has Hodge numbers $h^{1,1} = 2$ and $h^{1,2} = 26$, and the Calabi–Yau threefold defined in $\mathbb{P}^9$ by the 3×3 minors of a generic 5×5 symmetric matrix of linear forms in $\mathbb{P}^9$ has Hodge numbers $h^{1,1} = 1$ and $h^{1,2} = 27$. Moreover, we give the descriptions of primitive contractions corresponding to the faces of the Kähler-Mori cones of these threefolds, and joint them via a sequence of conifold transitions. We shall study the above varieties using the Grassmann blow up (see [CM]) whose general properties are discussed in Lemma 3.9.

The considered determinantal Calabi–Yau threefolds were already studied from a different point of view in [GP] (see also [Be]). In particular, M. Gross and S. Popescu observed an analogy between the descriptions of smooth del Pezzo surfaces D' embedded by their anti-canonical divisors and descriptions of some families of Calabi–Yau threefolds (we present it in Table 1). By the symbols $X_{d_1,d_2,\ldots}$, we denote a generic complete intersections of indicated degrees in the indicated manifold.

Table 1. The analogy

i	del Pezzo surfaces D'	Calabi–Yau threefolds X'
1	$D_6 \subset \mathbb{P}(1,1,2,3)$	
2	$D_4 \subset \mathbb{P}(1,1,1,2)$	$X_8 \subset \mathbb{P}(1,1,1,1,4)$
3	$D_3 \subset \mathbb{P}^3$	$X_5 \subset \mathbb{P}^4$
4	$D_{2,2} \subset \mathbb{P}^4$	$X_{3,3} \subset \mathbb{P}^5$
5	4×4 Pfaffians of a 5×5 skew-symmetric matrix	6×6 Pfaffians of a 7×7 skew-symmetric matrix
6	2×2 minors of a 3×3 matrix	3×3 minors of a 4×4 matrix
7	2×2 minors of a 3×4 matrix obtained by deleting one row from a symmetric matrix	3×3 minors of a 4×5 matrix obtained by deleting one row from a symmetric matrix
8	2×2 minors of a 4×4 symmetric matrix	3×3 minors of a 5×5 symmetric matrix
8'	2×2 minors of a 3×5 double-symmetric matrix (see 5)	3×3 minors of a 4×6 double-symmetric matrix

We give a kind of explanation of this analogy by joining the threefolds in Table 1 by a natural sequence (a "cascade") of conifold transitions (recall that Reid and Suzuki studied in [RS] "cascades" of del Pezzo surfaces). More precisely, we choose a del Pezzo surface of degree i. We embed it into a nodal Calabi–Yau threefold X' corresponding (in the table) to a del Pezzo surface of degree $i-1$. Then, by resolving the singularities of X' in such a way that the strict transform D of D' is isomorphic to D, we shall obtain a Calabi–Yau threefold X containing a del Pezzo surface of degree i. We next show that the surface D can be contracted by an appropriate linear system to a point that lies on a singular threefold belonging to the family of Calabi–Yau threefolds corresponding to del Pezzo surfaces of degree i. As a result, we obtain a conifold transition between the families of Calabi–Yau threefolds corresponding to the del Pezzo surfaces of degree i and $i-1$. This proves in particular that the above threefolds are Calabi–Yau threefolds. The above cascade can also be used to find mirrors to the considered determinantal Calabi–Yau threefolds (see [BCKS]).

Acknowledgements

We would like to express our gratitude to P. Pragacz for his encouragment and help. We thank J. Buczynski, S. Cynk, A. Langer for discussions and advice.

2. Del Pezzo of degree ≤ 5

As a starting point let us consider the case of a del Pezzo surface $D' \subset \mathbb{P}^3$ of degree 3 defined by the cubic $c(x_1, x_2, x_3, x_4) = 0$. We want to embed it into a singular Calabi–Yau threefold being a hypersurface $X' = X'_8 \subset \mathbb{P}(1,1,1,1,4)$ (with coordinates $x_1, \dots, x_4, u$). Let hence X' be defined by the equation $c(x_1, \dots, x_4)f(.) + ug(.)$, where f an g are a generic quintic and a generic quartic in $\mathbb{P}(1,1,1,1,4)$. Clearly X' contains the surface D' in its natural embeding.

Proposition 2.1. *The threefold X' is a nodal Calabi–Yau threefold with 60 nodes. Moreover, the small resolutions of X' has Picard group of rank 2.*

Proof. This theorem can be proved by arguing as in Section 2.4 from [K]. Instead, let us show how work the methods of this paper in this simplest example. First, the locus of singularities of X' is defined by the equations $u = f(x_1, \dots, x_4, u) = g(x_1, \dots, x_4, u) = c(x_1, \dots, x_4) = 0$. After changing coordinates, the singularities are given locally by $xy - zt = 0$, hence these are 60 ordinary double points. Let us now describe the small resolution X of X'. Consider the variety Y given in $\mathbb{P}(1,1,1,1,1,4)$ with coordinates $(x_0, x_1, \dots, x_4, u)$ by the equations

$$f(x_1, \dots, x_4, u) + x_0 u$$
$$\text{and}$$
$$g(x_1, \dots, x_4, u) + x_0 c(x_1, \dots, x_4).$$

The threefold Y has only one singular point at $(1,0,0,0,0,0)$ being a isolated Gorenstein singularity. Indeed, since

$$u_1 = g(x_1, \dots, x_4, u) + x_0 c(x_1, \dots, x_4)$$

is a generic quartic, we can change coordinates to $(x_0, \dots, x_4, u_1)$ and see that Y is a quintic in $\mathbb{P}^4$ with one singular point, that has tangent cone given by $c(x_1, \dots, x_4)$. Performing a weighted blow up $Y \subset$

$\mathbb{P}(1,1,1,1,1,4)$ in the point $(1,0,\dots,0)$ with weights $1,1,1,1,4$, we obtain its resolution of singularities $Z \subset \mathbb{P}(1,1,1,1,1,4)\times\mathbb{P}(1,1,1,1,4)$. Observe that Y contains 60 lines, given by the equations

$$f(x_1,\dots,x_4,u)=g(x_1,\dots,x_4,u)=c(x_1,\dots,x_4)=u=0,$$

passing through the point $(1,0,0,0,0,0)\in Y$. These lines become disjoint on Z. Now, X is the image of the natural projection $Z\to\mathbb{P}(1,1,1,1,4)$ and $Z\to X$ is a small resolution that contracts exactly the considered 60 lines. Finally, the rank of the Picard group of Y is 1, thus the Picard group of Z, that is the blow up of Y in one point, has rank 2 (see [Har, Prop. 2.6 II]). $\square$

Using a different language, in the above proof we flopped the exceptional curves of the blowing up of $D'\subset X'$, and obtained a Calabi–Yau threefold containing a del Pezzo surface D of degree 3. The primitive contraction of type II with D as exceptional divisor has image being a normal quintic in $\mathbb{P}^4$. Observe that we found also in the proof the quintic equation of the image.

Del Pezzo surfaces of degrees 4 and 5 have already been embedded into Calabi–Yau threefolds of degrees 5 and 9 respectively in Section 5 from [KK]. Observe however that using the method described above we can find an exact description of the images also in these cases. It remains to find appropriate embeddings for del Pezzo surfaces of degree ≥ 6.

Remark 2.2. Observe that by analogy, an eventual Calabi–Yau threefold corresponding to a del Pezzo surfaces of degrees 1 would need to have degree 0.

Remark 2.3. The descriptions from Table 1 of the anti-canonical images of del Pezzo surfaces of degrees $i\leq 5$ were discussed in a more general context in [KK1]. To see that in the remaining cases the considered equations describe del Pezzo surfaces, it is enough to show that these equations define smooth surfaces of degree i in $\mathbb{P}^i$ (see the Theorem of Nagata [N]).

3. Del Pezzo of degree 6

Let $D'\subset\mathbb{P}^6$ be an anti-canonically embedded del Pezzo surface of degree 6. It can be proved that D' is defined by the 2×2 minors of a generic 3×3 matrix M of linear forms in $\mathbb{P}^6$.

We shall embed D' into a nodal Calabi–Yau threefold defined by the 6×6 Pfaffians of a 7×7 skew-symmetric matrix. First, since each $n\times n$ matrix is a sum of a symmetric and an skew-symmetric matrix, we can write $M = S + A$ as such a sum. Let B be the extra-symmetric (i.e. skew-symmetric and symmetric with respect to the second diagonal) 6×6 matrix

$$\begin{pmatrix} A & S^r \\ -S^r & ((A^r)^r)^t \end{pmatrix}$$

where $.^r$ denotes the rotatation of the matrix by 90 degrees and $.^t$ is the transposition. Observe that the set of 2×2 minors of M and the set of 4×4 Pfaffians of B generate the same ideal (the ideal of D'). Let C denote the 8×8 extra-symmetric matrix

$$\left(\begin{array}{c|ccc|c} 0 & t_1 & \dots & t_6 & t_7 \\ -t_1 & & & & t_6 \\ \vdots & & B & & \vdots \\ -t_6 & & & & t_1 \\ -t_7 & -t_6 & \dots & -t_1 & 0 \end{array}\right)$$

where $t_1, .., t_7$ are linear forms in $\mathbb{P}^6$. Let C_1 be the skew-symmetric matrix obtained from C by deleting the last row and the last column. Let X' be a threefold defined by 6×6 Pfaffians of C_1 for a generic choice of $t_1, ..., t_6$. To see that X' contains D', we use the fact that Pfaffians can be expanded along any of their rows. In this way each 6×6 Pfaffian of C_1 can be seen as an element of the ideal generated by the 4×4 Pfaffians of B.

Observe moreover that the 6×6 Pfaffians of C define a smooth surface G' of degree 20 in $\mathbb{P}^6$, that is contained in X'. Indeed, since C can be represented in the form

$$C = \begin{pmatrix} \tilde{A} & \tilde{S}^r \\ -\tilde{S}^r & (((\tilde{A})^r)^r)^t \end{pmatrix}$$

where $\tilde{A}$ is a skew-symmetric and $\tilde{S}$ a symmetric 4×4 matrix, it follows that G' is also defined by the 3×3 minors of the 4×4 matrix $\tilde{A} + \tilde{S}$ (observe that in the upper left corner of this matrix we have t_7). From the Porteous formula (see [F]) we obtain that G' has degree 20. Moreover, since $\tilde{A} + \tilde{S}$ is a generic 4×4 matrix with linear forms, G' is smooth.

Now, using Singular [GPS], we compute (for some $t_1, ..., t_6$) that G' and D' have exactly 20 points in common (the radical of the ideal $\mathcal{I}_{G'} + \mathcal{I}_{D'}$ has degree 20). Moreover, the ideal $\mathcal{I}_{G'} + \mathcal{I}_{D'}$ has degree 20. This means that the ideal $\mathcal{I}_{G'} + \mathcal{I}_{D'}$ is radical. Hence we obtain two

Weil divisor cutting transversally in each singular point of X'. This suggest the following theorem.

Theorem 3.1. *The threefold $X' \subset \mathbb{P}^6$ is a nodal Calabi–Yau threefold with 20 nodes.*

Proof. Let $Y \subset \mathbb{P}^7$ (with coordinates $(x_0, \dots, x_7)$) be a scheme defined by the vanishing of the 3×3 minors of the 4×4 matrix F obtained from the matrix $\tilde{A}+\tilde{S}$ (with linear forms depending of 7 variables $x_1, \dots, x_7$) by replacing the form t_7 in its upper left corner by the remaining free variable x_0. Notice that the singular locus of the determinantal variety in $\mathbb{P}^{15}$ parameterizing 4×4 matrices of rank ≤ 3 is the locus of matrices of rank ≤ 2. It follows now from the Bertini theorem that Y is smooth outside the point $P = (1, 0, \dots, 0)$. By the discussion in [Ha, p. 257] we conclude that the singularity of Y has tangent cone being a cone over a del Pezzo surface of degree 6 determined by the vanishing of the 2×2 minors of the lower right 3×3 sub-matrix of F. Moreover, from general properties of determinantal varieties the singularity of Y can be resolved by blowing up P (see Lemma 3.9).

Let us consider the projection $\psi\colon \mathbb{P}^7 \supset Y - P \to \mathbb{P}^6$ with center at the point P. The map ψ can be naturally extended to $\tilde{\psi}\colon Y_P \to \mathbb{P}^6$, where Y_P is the blow up of $Y \subset \mathbb{P}^7$ in the point P. Denote by D the exceptional divisor of this blow up. It is a del Pezzo surface of degree 6 (that is isomorphic to D').

Lemma 3.2. *The morphism $\tilde{\psi}$ is birational, surjective onto X' and injective outside the sum of 20 lines contained in Y_P, which are contracted to 20 points.*

Proof. Let us start with the proof that $\tilde{\psi}$ is birational. To do this it is enough to prove that $\psi|_U$ is birational for some open subset U of Y Choose a codimension 2 linear space $L \subset \mathbb{P}^7$ that does not contain P. Let H be the hyperplane spanned by L and P. Let U be the set $Y_L = Y \setminus H$. It is a subset of $\mathbb{C}^7 = \mathbb{C}^6 \times \mathbb{C}$ with coordinates chosen in such a way that the first $\mathbb{C}^6$ parameterizes lines passing through the point P and the last coordinate parameterizes hyperplanes containing L. Recall moreover that varying the upper left linear form t_7 in the matrix $\tilde{A} + \tilde{S}$ we obtain a family $G'_{t_7} \subset X'$ of surfaces. This permits us to find the following description of Y_L

$$Y_L = \{(p, x) \in \mathbb{C}^6 \times \mathbb{C}\colon p \in G'_{l_x}\},$$

where l_x is the linear form defining the hyperplane corresponding to x. In this notation $\psi|_{Y_L}$ is given by the projection onto the $\mathbb{C}^6$. To study the behavior of this projection consider the variety.

$$\mathcal{Y} = \{(p, l) \in \mathbb{P}^6 \times \mathbb{C}^7 \colon p \in G'_l\}.$$

We claim that for a fixed point $p \in X' - D'$, the set of linear forms $t_7 \in \mathbb{C}^7$ such that $p \in G_{t_7}$ is a hyperplane. Indeed, the determinant of the matrix B in the point p is zero. It follows that performing linear operations on rows and columns of B we can assume that the only nonzero entries in $B(p)$ are contained in the 4×4 sub-matrix obtained by deleting the first and the last rows and columns. We obtain that all 6×6 Pfaffians of C except one vanish in p. Thus, this means that we have exactly one linear condition on the value $t_7(x)$. Using the fact that Y_L is a restriction of $\mathcal{Y}$ obtained by choosing a line in $\mathbb{C}^7$ we deduce that for a generic point p of $X' \setminus D'$ there is exactly one x such that $G'_{l_x} \ni P$ (the point of intersection of the hyperplane with the line). It follows that $\tilde{\psi}$ is birational onto X' and injective over $X' \setminus D'$.

The same argument shows that the image $\tilde{\psi}(Y_P \setminus D)$ is contained in the sum of all surfaces of the form G'_l. In particular

$$\tilde{\psi}(Y_P \setminus D) \subset (X' \setminus D') \cup \bigcup_{l \in \mathbb{C}^7} D' \cap G'_l .$$

The set of all intersection points of surfaces G'_l with D' is defined (set theoretically) by the vanishing on D' of the 3×3 minors of the matrix $T = \tilde{A} + \tilde{S}$ with $t_7 = 0$. We continue by proving the following claim.
Claim The 3×3 minors of T defines on D' a set of 20 points.
Observe that if for a 3×3 matrix with complex entries the determinant is 0, the upper left entry is 0, the minor obtained by deleting the first row and the first column is 0 and if the minor obtained by deleting the first row an the last column is nonzero, then the minor obtained by deleting the last row an the first column is zero. After some investigation this implies that for a given x the 3×3 minors of T and the 2×2 minors of the matrix obtained by deleting the first row and the first column all vanish in x if and only if either all 2×2 minors of the 3×4 matrix obtained from T by deleting the first row vanish in x, or all 2×2 minors of the 3×4 matrix obtained from T by deleting the first column vanish in x. From the Porteous formula we conclude that T defines on D' two disjoint sets of 10 points. The claim follows.

The above claim implies in particular that $\tilde{\psi}(D) = D'$. Hence, since we know that both D and D' are del Pezzo surfaces of degree 6, we deduce that $\tilde{\psi}|_D$ is an isomorphism onto D'. It remains to identify the set of curves contracted by $\tilde{\psi}$. The only curves that might be contracted by the considered morphism are curves that map to one of the 20 intersection points of surfaces G'_l with D'. An easy calculation shows that the fibers over these points are one dimensional (the fiber of $\mathcal{Y} \to \mathbb{P}^6$ under these points are $\mathbb{C}^7$). $\square$

The above Lemma implies that $\tilde{\psi}$ is a small resolution of X', provided that X' is normal.

Lemma 3.3. *The variety X' is normal.*

Proof. Since X' is of the expected codimension, we obtain by general properties of Pfaffian subschemes that its singularities are Gorenstein. We claim that the singularities of X' are isolated. Indeed, from the fact that the determinantal variety in $\mathbb{P}^{14}$ determined by the 6×6 Pfaffians of the 7×7 skew-symmetric matrix with the lower right 6×6 submatrix being extra-symmetric has singular locus of dimension 8 and degree 20 (calculation of the Jacobian ideal with Singular [GPS]) and since X' is a linear section of this variety, it follows from the Bertini theorem that the singularities of X' are isolated, Gorenstein, and in consequence normal. $\square$

To conclude, observe that through each singular point of X' there passes a family of smooth Weil divisors G'_{t_7} cutting transversally each other in this singular point. Indeed, it is enough to prove that there exists a linear form l such that G'_l and D' meet transversally. We easily find such an l using Singular or by calculation with a given example. Next, we compute the partial derivatives of the expands of the determinants defining G'_l along its first rows to see that elements of the considered family meet transversely. The theorem is now a direct consequence of the following Lemma. $\square$

Lemma 3.4. *Let X' be a normal threefold with only isolated Gorenstein threefold singularities having a small resolution. Suppose that there exists seven Weil divisors such that each two of them meet transversally in these singular points, then X' is a nodal threefold.*

Proof. Let us choose P a singular point of X'. We shall first prove that the singularity of X' in the point P is of type cA_1 (i.e it can be described locally analytically by the equation $x^2+y^2+z^2+tf(x,y,z,t)=0$). To do this take a generic hyperplane section H through a singular point P of X'. We need to prove that P is a singularity of type A_1 on this hyperplane section.

From [R4, Cor. 1.12] we know that the hyperplane section of P is a rational double point (an ADE singularity). Suppose that it is a singularity of type A_k, where $k \geq 2$. The traces on H of four of our Weil divisors W_1, W_2, W_3, and W_4 are four curves cutting each other transversally. A singularity of type A_k can be described as the double covering of $\mathbb{C}^2$ branched along the curve $x^2 + y^k = 0$. The image of W_1, W_2, W_3, and W_4 by this covering are four curves C_1, C_2, C_3, C_4 on $\mathbb{C}^2$ passing through $(0,0)$. Observe that we can choose at least two of them (say C_1 and C_2) not to be tangent to the line $x = 0$. This follows from the fact that $x = 0$ is the only line passing through $(0,0)$ that splits in the double covering. We thus obtain a contradiction since the Weil divisors mapping to C_1 and C_2 cannot cut transversally. The case of singularities of type D_n an E_n can be treated similarly, and makes use of all seven divisors.

Since P is a singularity of type cA_1 it can be described by the equation $x^2 + y^2 + z^2 + t^{2n}$. Consider its projective tangent cone that is a quadric of rank ≥ 3 that contain two disjoint lines (corresponding to the tangent of our Weil divisors). It follows that this quadric has rank 4, so P is an ordinary double point. □

Remark 3.5. We can consider the morphism $\phi\colon X' \setminus D' \to \mathbb{P}^7$ that associates to a point $x \in X' \setminus D'$ the hyperplane of linear form l in $\mathbb{C}^7$ such that $x \in G'_l$. Observe that ϕ is inverse to ψ.

Let us see the above results in a different language. Blowing up D' we resolve the singularities of X'. Flopping the exceptional divisors we obtain a smooth Calabi–Yau threefold X. Denote by D and G the strict transforms to X of D' and G' respectively.

Proposition 3.6. *The linear system $|G|$ defines a birational morphism $\pi\colon X \to \tilde{X}$ with D as exceptional locus into a normal variety described by the vanishing of the 3×3 minors of the 4×4 matrix F (see the proof of Theorem 3.1).*

Proof. First, since each rational curve contracted by $\tilde{\psi}\colon Y_P \to X'$ cuts D'' with multiplicity 1, one obtains that $\tilde{\psi}$ is exactly the small resolution $X \to X'$. Furthermore, from general properties of determinantal varieties Y is projectively normal (see [MS]). We argue as in the proof of [K, Thm. 2.3] to show that $\tilde{X}$ is normal. It follows, that the morphism $Y_P \to Y$ is given by the linear system $|G|$. □

Remark 3.7. We compute with Singular using the method described in [GP, Rem. 4.1] that $h^{1,2}(X) = 32$. From the fact that X' has 20 ordinary double points, we obtain $h^{1,1}(X) = 3$.

In order to see more geometrically the Hodge numbers of X and Y, let us change once more the point of view. Let us first analyze the Kähler–Mori cone of a smooth Calabi–Yau threefold Y defined by the 3×3 minors of a 4×4 matrix of generic linear forms in $\mathbb{P}^7$. Denote by S the secant variety of $\mathbb{P}^3 \times \mathbb{P}^3$ embedded by the Segre embedding in $\mathbb{P}^{15}$. It is well known (see [Ha]) that the dimension of S is 11, the degree 20 and that $\mathbb{P}^3 \times \mathbb{P}^3 \subset S$ is the singular locus.

Theorem 3.8. *The threefold Z in $\mathbb{P}^7$ defined by the 3×3 minors of a generic 4×4 matrix is a Calabi–Yau threefold with Picard group of rank 2. Moreover, the two faces of the Kähler–Mori cone give small contractions into nodal complete intersections (with 56 nodes) of a quadric and a quartic in $\mathbb{P}^5$.*

Proof. From Proposition 2.3 in the Appendix we obtain $h^{1,1}(Z) = 2$. The threefold Z can be seen as the intersection of S (the secant variety) with a generic 7-dimensional linear subspace W of $\mathbb{P}^{15}$. This intersection is a smooth Calabi–Yau threefold (this follows from Proposition 3.6). The Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^3$ in $\mathbb{P}^{15}$ is covered by two families of 3-dimensional linear spaces (the images of $\{x\} \times \mathbb{P}^3$ and $\mathbb{P}^3 \times \{x\}$). Let L_1 and L_2 be two linear spaces from one of this families and B one from the other (L_i cuts B in one point l_i for $i = 1, 2$). The join of L_1 and L_2 is a 7-dimensional linear subspace that contains all those linear spaces from the family of L_1 that correspond to points on the line l_1l_2. We obtain a map $\pi_1 \colon Z \to G(2, 4)$ such that the image of a point g of Z is the line l_1l_2 on B, where the linear space spanned by L_1 and L_2 cuts W in g. Indeed, since the codimension of 7-dimensional spaces that cut W is 4 (see [F]) the map π_1 is a birational morphism that contracts a finite number of disjoint lines (these are found by Schubert cycles calculations).

We claim that the image of π_1 is normal. For this it is enough to show that the singularities appear only when a line is contracted to a point. This shall follow from an explicit local description of the map π_1. Recall first (see [Ha, ex.14.16]) that $S \subset \mathbb{P}^{15}$ can be seen as the set of matrices of rank ≤ 2 (where $\mathbb{P}^{15}$ is the set of all matrices). A point $P \in Z$ corresponds to a linear map $A_P \colon \mathbb{C}^4 \to \mathbb{C}^4$ with 2-dimensional image I_P and 2-dimensional kernel K_P. The map π_1 can be seen as a map that associates to a point P the line $I_P \subset \mathbb{P}^3$. Consider the map

$$\Omega \colon \mathcal{M} \ni (x^i_j)_{i,j} \longrightarrow [(x^1_1, ..., x^1_4), (x^2_1, ..., x^2_4)] \in G(2, 4),$$

defined on the subset $\mathcal{M}$ of the set of 4×4 matrices that is an appropriate neighborhood of a point P lying outside a contracted line.

Observe that locally on Z near P the morphisms π_1 and Ω are equal (after an appropriate change of coordinates). Moreover, if G is a 3-dimensional linear space, then $\Omega|_G$ is an isomorphism as soon as it is injective. Thus, it follows that Ω restricted to the tangent space T_P to Z at P is an isomorphism (this is the linear subspace of maps in W carrying the kernel of A_P into the image of A_P). Indeed, since the

7-dimensional linear space L_1L_2 is the set (outside $\mathbb{P}^3 \times \mathbb{P}^3$) of points consisting of matrices that have common image, and since W meets L_1L_2 in one point, we deduce that $\Omega|_{T_P}$ is injective.

Since $K_Z = 0$ and the image of π_1 is normal, we obtain that this image is a Calabi–Yau threefold with Gorenstein terminal singularities (cDV singularities). Moreover, since the Grassmannian $G(2,4)$ is a quadric in $\mathbb{P}^5$ and the image is normal, we can use Klein theorem [Har, Ex. 6.5] and conclude that this image is isomorphic to a complete intersection $X_{2,4} \subset \mathbb{P}^5$ of a quadric and a quartic in $\mathbb{P}^5$.

Let, us show that $X_{2,4}$ is nodal. We claim that the generic hyperplane section W of Z containing a fixed contracted line C is smooth. Indeed, observe that it is enough to choose W such that it does not contain any tangent space to Z in points of the curve $C \subset Z$. Such choice can be done since the considered tangent spaces to Z induce a curve in the Grassmannian $G(2,6)$ (of dimension 3 linear spaces containing C). We can choose for dimension reason W such that the induced 9-dimensional $G(2,5) \subset G(2,6)$ (such a family separates points in $G(2,6)$) is disjoint from the induced curve. We conclude that the normal bundle of $C \subset Z$ has subbundle $\mathcal{O}(-1)$, and we can argue as in the proof of [K, Thm. 2.1].

To compute the number of nodes, we find the difference between the Euler characteristics of Z and a smooth complete intersection of a quadric and a quartic. We obtain that the difference is 112, which gives 56 nodes. $\square$

We next study determinantal varieties using the Grassmann blow up (see [CM]). Let us describe its properties.

Let A be an $n \times n$ matrix with coordinates as entries. Let M_3 (resp. M_2) be the variety given in $\mathbb{P}^{n^2-1}$ by the vanishing of all 3×3 (resp. 2×2) minors of the matrix A. It is a well known fact that M_2 is the image of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ by its Segre embedding and that M_3 is the secant variety of M_2. It follows that M_2 is the singular locus of M_3. There are two natural resolutions of singularities of M_3. The first one is the blowing up of the singular locus M_2. We will denote this blowing up by $\pi\colon X \to M_3$. The exceptional locus of π is a $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ bundle over M_2. The second resolution is given by the formula

$$\tau\colon M_3 \times G(n-2,n) \supset Y = \{(A,\Lambda)\colon A|_\Lambda = 0\} \longrightarrow M_3,$$

where τ is the projection onto the first part. The exceptional locus of τ is a $\mathbb{P}^{n-1}$ bundle.

Lemma 3.9. *There is a commutative diagram.*

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & \swarrow & \\ M_3 & & \end{array}$$

where the morphism σ restricted to the exceptional set over each point of M_2 is the projection onto one of the variables of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Proof. From the definition of blowing up, we can view X as a subset of $\mathbb{P}^{n^2-1} \times \mathbb{P}^{\left(\frac{n(n-1)}{2}\right)^2-1}$. Using the identification $\mathbb{P}(M(n,n)) = \mathbb{P}^{n^2-1}$, we can describe X as the set of points of the form (B, C), where B corresponds to a matrix from M_3 and C is the $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix of the 2×2 minors of B. The morphism π is then the projection onto the first component of the product. Consider now the morphism.

$$\tilde{\tau}\colon M_3 \times G(n-2, n) \supset \tilde{Y} = \left\{(A, \Lambda)\colon A^{\mathrm{T}}|_{\Lambda} = 0\right\} \longrightarrow M_k.$$

We claim that the fiber product $Z = Y \times_{M_3} \tilde{Y}$ is isomorphic to X. In order to prove this, we view Z as a subset of

$$M_3 \times G(n-2, n) \times G(n-2, n)$$

in its turn embedded in $\mathbb{P}^{n^2-1} \times \mathbb{P}^{\left(\frac{n(n-1)}{2}\right)^2-1}$ by the composition of the Plücker embeddings of each Grassmannian and the Segre embedding. The morphism from Z to M_3 is then given by the natural projection. It is now enough to prove that the traces of Z and X on the set $M_3 \setminus M_2 \times \mathbb{P}^{\left(\frac{n(n-1)}{2}\right)^2-1}$ coincide, i.e. that for each matrix M of rank 2 the point

$$\mathrm{Segre}(\mathrm{Plücker}(\ker(M)), \mathrm{Plücker}(\ker(M^{\mathrm{T}})))$$

has coordinates being the 2×2 minors of M. The latter is checked by direct computation after a suitable choice of generators of the kernels. This ends the proof of the claim.

It follows that X and Z are the closures of the same set, hence are equal. $\square$

Proposition 3.10. *The rank of the Picard group of X is 3 and the morphism $\pi\colon X \to Y$ contracts a two dimensional space of curves on the Kähler–Mori cone of X. Moreover, the Hodge number $h^{1,2}$ of a generic Calabi–Yau threefold defined by 3×3 minors of a 4×4 generic matrix is* 34.

Proof. First by Proposition 2.3 from the Appendix, we obtain that the Chow group $A^1(S)$ of the secant variety S of $\mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15}$ is $\mathbb{Z} \oplus \mathbb{Z}$. The blowing up $\pi\colon \tilde{S} \to S$ of $\mathbb{P}^3 \times \mathbb{P}^3 \subset S$ factorizes through the Grassmann blow up [Ha, p. 206] that gives a resolution of S. Denote by E its exceptional set. Since S is regular in codimension 1, we obtain from [Har, Prop. 2.6 II] that the rank of the Picard group of $\tilde{S}$ is 3. From the Grothendieck–Lefshetz theorem applied (several times) to the pull-back to $\tilde{S}$ of the system of hyperplanes that pass through a fixed point of $\mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15}$ (this system is very ample), we obtain that the Picard group of X has rank 3.

In order to compute the Hodge numbers of the generic element of the smoothing family of Y, we shall use [Nm, Thm. 10]. We need a description of the Kähler–Mori cone of X. The hyperplane W defining Y passes through exactly one point Q from $\mathbb{P}^3 \times \mathbb{P}^3$ (Q corresponds to the point $(1, 0, \ldots, 0)$ in $Y \subset \mathbb{P}^7$). The incidence correspondence

$$\mathcal{C} = \{(A, \Lambda) \colon A|_\Lambda = 0\} \subset Y \times G(2, 4)$$

(where a point of Y corresponds to a matrix A that gives a linear map $\mathbb{C}^4 \to \mathbb{C}^4$) gives a partial resolution $\rho_1 \colon \mathcal{C} \to Y$ of Y such that $\mathcal{C}$ is normal (since its image Y is normal). Observe that the exceptional locus of this Grassmann resolution is isomorphic to $\mathbb{P}^2$. From Lemma 3.9 the blowing up $X \to Y$ factorizes through ρ_1. We hence obtain a morphism $\theta_1 \colon X \to \mathcal{C}$ that maps the exceptional del Pezzo surface D to $\mathbb{P}^2$. It follows that $\theta_i|_D$ for $i = 1, 2$ is a blowing down of three rational curves. Since $K_X = 0$ these curves map to three terminal singularities. Since they are contained in a smooth surface D, their normal bundle is $\mathcal{O}(-1) \otimes \mathcal{O}(-1)$ (see the proof of [K, Thm. 2.1])). It follows that these singularities are three ordinary double points (see [R4, Rem. 5.13(b)]), thus $\mathcal{C}$ is a nodal Calabi–Yau threefold.

We see that the image of the restriction map

$$Pic(X) \otimes \mathbb{C} \to Pic(D) \otimes \mathbb{C}$$

has dimension 2. Moreover, the *Case 4* with $r = 2$ in Theorem 10 from [Nm] holds. We deduce that the image of the natural map of Kuranishi spaces $Def(Y) \to Def(Y, P)$ coincides with the one dimensional smoothing component S_1 of $Def(Y, P)$. Let $D_{1,loc}$ be the sub-functor of $Def(Y, P)$ corresponding to S_1 (see [Nm, Lemm. 11]), then we have a surjection on tangent spaces $T_{Def(T)} \to T_{D_1}$ and we can argue using [G, Thm. 1.9] (as in the proof of Theorem 3.3 [KK]). We obtain that the Hodge number $h^{1,2}$ of a generic Calabi–Yau threefold defined by 3×3 minors of a 4×4 generic matrix is

$$h^{1,2}(X) + dim(S_1) = 32 + 2.$$

Here we use the fact that X' has 20 nodes and that a generic Calabi–Yau threefold defined by Pfaffians of a 7×7 matrix are $h^{1,1} = 1$ and $h^{1,2} = 50$ (see [Rod]). $\square$

Remark 3.11. From the proof of Lemma 3.9, we obtain the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\theta_2} & \mathcal{D} \\ {\scriptstyle\theta_1}\downarrow & & {\scriptstyle\rho_2}\downarrow \\ \mathcal{C} & \xrightarrow{\rho_1} & Y \end{array}$$

where

$$\mathcal{D} = \{(A, \Lambda) \colon A^r|_\Lambda = 0\} \subset Y \times G(2,4),$$

θ_1 and θ_2 are primitive contraction of type III onto nodal Calabi–Yau threefolds $\mathcal{C}$ and $\mathcal{D}$. Observe that $\theta_1 \circ \rho_1$ is then the ordinary blow up.

4. Del Pezzo of degree 7

Let $D' \subset \mathbb{P}^7$ be an anti-canonically embedded del Pezzo surface of degree 7. It is well known that D' can be described by the 2×2 minors of the 3×4 matrix obtained by deleting the last row from a symmetric 4×4 matrix M.

We embed D' into a Calabi–Yau threefold X' defined by the 3×3 minors of the 4×4 matrix

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ & & & \\ & M & & \\ & & & \end{pmatrix}$$

where $s_1, \ldots, s_4$ are generic linear forms on $\mathbb{P}^7$.

To prove that X' is a nodal Calabi–Yau threefold and to compute the number of nodes consider the following 4×5 matrix N

$$\left(\begin{array}{c|cccc} l & s_1 & s_2 & s_3 & s_4 \\ \hline s_1 & & & & \\ s_2 & & M & & \\ s_3 & & & & \end{array}\right)$$

where l is a generic linear forms in $\mathbb{P}^7$ chosen such that this matrix is obtained from a symmetric 5×5 matrix by deleting the last row. The 3×3 minors of the matrix K define a smooth canonically embedded surface $G' \subset X'$ of degree 27. We can compute using Singular that G' and D' have 11 points in common. This suggests the following theorem.

Proposition 4.1. *The threefold $X' \subset \mathbb{P}^7$ is a nodal Calabi–Yau threefold with* 11 *nodes. The blowing up of $D' \subset X'$ is a small resolution. Let $X \to X'$ be the flopping of the exceptional curves of this resolution.*

Then the Calabi–Yau threefold X contains a del Pezzo surface $D \simeq D'$ and has Picard group of rank 3.

Proof. We argue as in the proof of Theorem 3.1. Let $Y \subset \mathbb{P}^8$ be the variety defined by the vanishing of the 3×3 minors of the matrix $\tilde{N}$ obtained by replacing the upper left entry l in the matrix N by the remaining free variable. We prove first that Y has exactly one singular point at $(1, 0, \dots, 0)$, which is resolved by the blowing up $\mathbb{P}^7 \times \mathbb{P}^8 \supset Z \to Y$ of this point. The exceptional divisor of this resolution is isomorphic to D'. Next, we shall see that the projection $Z \to X' \subset \mathbb{P}^7$ is a small resolution such that $Z = X$.

Let T be the determinantal variety in $\mathbb{P}^{13}$ of 4×5 matrices of rank ≤ 2. The incidence variety

$$\mathcal{E} = \{(A, \Lambda) \colon A|_\Lambda = 0\} \subset T \times G(2, 4)$$

gives a partial resolution $\mathcal{E} \to T$. Let us show that $\mathcal{E}$ is smooth. By straightforward computations, we see that the fibres of the projection $\xi \colon \mathcal{E} \to G(2, 4)$ are 4-dimensional projective spaces in $\mathbb{P}^{13}$. Since ξ is flat, we obtain a morphism

$$\chi \colon G(2, 4) \to G(4, 14).$$

We shall show that the image of this morphism is smooth. First it is clearly injective, so the image is generically smooth. Now if Λ_1, Λ_2 are linear surfaces in $\mathbb{C}^4$ containing 0, then we can find an automorphism $P \in GL(4)$ such that $P\Lambda_1 = \Lambda_2$. We obtain an automorphism σ of $G(4, 14)$ induced by the linear map $A \to QAP^{-1}$ between 4×5 partially symmetric matrices, where Q is a 5×5 matrix with $(P^{-1})^t$ in the upper left corner, 1 in the lower right corner, and 0 elsewhere. The automorphism σ maps $\chi(\Lambda_1)$ into $\chi(\Lambda_2)$ and preserves the image of χ. It follows that the image of χ is smooth thus we can argue as in [Ha, p. 205] and show that $\mathcal{E}$ is smooth. We conclude that the blowing up $S \to T$, that factorizes through $\mathcal{E} \to T$ (see Lemma 3.9), gives a resolution of T. It follows that the blowing up $Z \to Y$ in $(1, 0, \dots, 0)$ is a resolution.

To show that X' is nodal, $Z \to X'$ is a small resolution, and that $Z = X$, we argue as in Theorem 3.1, using the fact that T has Gorenstein singularities (see [Co]). Finally, from Propositon 2.3, we deduce as in the proof of Proposition 3.10 that $\rho(X) = 3$. □

Denote by G the strict transform of G' on X.

Proposition 4.2. *The linear system $|G|$ gives a birational morphism $\pi \colon X \to Y$ into a normal variety in $\mathbb{P}^8$ described by the vanishing of the 3×3 minors of a 4×5 partially symmetric matrix $\tilde{N}$ (from the* *proof of Proposition 4.1).The exceptional locus of this morphism is D. Moreover, π factorizes as $X \xrightarrow{\rho} V \xrightarrow{\psi} Y$, where ρ is a small contraction from X into a nodal Calabi–Yau threefold with two nodes.*

Proof. Let H be the pull back to X of the system of hyperplanes in $\mathbb{P}^8$. We claim that $G \in |H + D|$. Indeed, let q be the determinant of a 2×2 minor B of M. Then the quadric $q = 0$ cuts X' along the divisor $D' + S'$. Now, by applying the algorithm computing the quotient of ideals, using an algebra computer system, we show that the 3×3 minor of the matrix K_l containing B' with first row and column of K_l added, determines a cubic that cuts X along $S + G$.

The tangent cone of Y in the point $(1, 0, \dots, 0) \in Y$ is determined by the vanishing of 2×2 minors of the matrix obtained from $\tilde{N}$ by deleting the first row and column. Hence, the exceptional divisor of the blowing up $Z \to Y$ is isomorphic to the del Pezzo surface D' (see [Ha, p. 257]).

We obtain the factorization of π from the composed morphism $S \to \mathcal{E} \to T$ (see the proof of Proposition 4.1). The exceptional divisor of $\psi\colon V \to Y$ is thus isomorphic to $\mathbb{P}^2$. We conclude that V has two nodes (see the proof of Proposition 3.10). ☐

Theorem 4.3. *The threefold R defined in $\mathbb{P}^8$ by a generic partially symmetric 4×5 matrix is a smooth Calabi–Yau threefold with Picard group of rank 2. Moreover, one face of its Kähler–Mori cone gives a contraction with exceptional set being the surface $\mathbb{P}^1 \times \mathbb{P}^1$ (the image is described in Proposition 5.1). The other face gives a small contraction to a nodal Calabi–Yau threefold with 63 nodes, that is a complete intersection of a quadric and a quartic in $\mathbb{P}^5$. Moreover, the Hodge number $h^{1,2}(R) = 26$.*

Proof. From the proof of Proposition 4.1 the threefold K is smooth (the codimension of the singular locus of T is 4). From Proposition 2.3 in the Appendix, we obtain $A^1(T) = \mathbb{Z} \oplus \mathbb{Z}$, so from the Grothendieck–Lefschetz theorem $\rho(R) = 2$.

We compute $h^{1,2}(R)$ using the morphism $\pi\colon X \to Y$. Indeed, from the Proposition 4.1 we find that the image of the restriction map $Pic(X) \to Pic(D)$ is generated by K_D and $E_1 + E_2$ (where E_1 and E_2 are exceptional divisors on D). We obtain, as in [Nm, Thm. 10], that $Def(Y) \to Def(Y, P)$ has image being the one dimensional smoothing component of $(Def(Y, P) \simeq \mathbb{C}[[x_1, x_2]]/(x_1^2, x_1 x_2))$. Now, arguing as in the proof of Proposition 3.10, we obtain $h^{1,2} = 26$.

Set $\mathcal{C} = \{(A, \Lambda)\colon A|_\Lambda = 0\} \subset R \times G(2, 4)$. Since R is smooth, the natural projection $\mathcal{C} \to R$ is an isomorphism. Consider the second natural projection $p_1\colon \mathcal{C} \to G(2,4)$ and the projection $p_2\colon \mathcal{E} \to G(2,4)$ from the proof of Proposition 4.1. Observe that p_1 is obtained from p_2 by choosing a general linear space $\mathbb{P}^7 \subset \mathbb{P}^{13} \supset \mathcal{E}$. Note that the fibers of p_2 are 4-dimensional linear space in $\mathbb{P}^{13}$, forming a 4-dimensional family $\chi(G(2,4)) \subset G(4,14)$. It folloes that p_1 is birational into its image and has a finite number of fibers being lines in $\mathbb{P}^7$. We claim that these lines are contracted to ODP singularities. To see this, we argue as in the proof of Theorem 3.8, showing that we can find a smooth surface in R containing a contracted line. We thus conclude that the image of p_1 is a nodal intersection of a quadric and a quartic in $\mathbb{P}^5$. To find the number of nodes, we compute the difference between the Euler characteristics of R and a smooth complete intersection of a quadric and a quartic in $\mathbb{P}^5$. The contraction corresponding to the second ray is discussed in Proposition 5.1. □

Remark 4.4. Observe that there is another del Pezzo surface of degree 7 contained in X'. Indeed, consider the surface $\tilde{D}$ defined by the 2×2 minors of the 3×4 matrix obtained from N by deleting the first and the last row. Now, we can repeat the above construction for $\tilde{D} \subset X'$, and obtain another (birational to Y) singular Calabi–Yau threefold Y'. This one being determined by the vanishing of the 3×3 minors of a partially symmetric 5×4 matrix.

5. Del Pezzo of degree 8

Let $D \subset \mathbb{P}^8$ be an anti-canonically embedded del Pezzo surface of degree 8. We have two possibilities. The surface D is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{P}^2$ blown up in one point.

5.1. If the surface D is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, it can be described by the vanishing of the 2×2 minors of a symmetric 4×4 matrix M. We can as before embed D into a smooth Calabi–Yau threefold X defined by the vanishing of the 3×3 minors of a 5×4 matrix N. Indeed, we construct N from M by adding one row with generic linear forms in such a way that the matrix N could also be obtained by deleting the first column from a symmetric matrix K_l (where l is a generic linear form in the upper left corner of K_l). The 3×3 minors of K_l define then a smooth (because K_l is generic) surface G (we obtain in fact a family). Denote by K the 5×5 symmetric matrix obtained from K_l by replacing l by the remaining free variable in $\mathbb{P}^9$ (as in the proof of Theorem 3.1).

Proposition 5.1. *The linear system $|G|$ defines a birational morphism with D as exceptional locus, into a normal Calabi–Yau threefold Y in*

$\mathbb{P}^9$ defined by the vanishing of the 3×3 minors of the symmetric 5×5 matrix K (defined above).

Proof. First, we see as in the proof of Proposition 4.2 that $G \in |H+D|$. Next, we argue as in Proposition 4.1. Let $T \subset \mathbb{P}^{14}$ be the variety parametrizing symmetric 5×5 matrices of rank ≤ 2. Consider the variety

$$\mathcal{E} = \{(A, \Lambda)\colon A|_{\Lambda} = 0\} \subset T \times G(3,5).$$

Observe that $\mathcal{E}$ is smooth. Now, the blow up of T along the locus of matrices of rank 1 has $\mathbb{P}^3 \subset \mathbb{P}^9$ (defined by the 2×2 minors of a symmetric 4×4 matrix) as exceptional fibers, whereas the projection $\alpha\colon \mathcal{E} \to T$ has $G(3,4) \simeq \mathbb{P}^3$ as exceptional fibers. So arguing as in Lemma 3.9, we obtain that $\mathcal{E} \to T$ is the blowing up of the singular locus of T.

Observe that Y is obtained by taking a general linear section of dimension 9 passing through a singular point of $T \subset \mathbb{P}^{14}$. Since X is smooth, the projection

$$\mathcal{E} \supset \alpha^{-1}(Y) \to G(2,5) \subset \mathbb{P}^9$$

is an isomorphism onto X. Thus the blow up $\alpha^{-1}(Y) \to Y$ is the morphism given by $|G|$. □

Theorem 5.2. *The threefold defined in $\mathbb{P}^9$ by a generic symmetric 5×5 matrix is a smooth Calabi–Yau threefold with Picard group of rank 1 (i.e. $h^{1,1} = 1$). Moreover, its Hodge number $h^{1,2} = 27$.*

Proof. First, X can be seen as a linear section of the natural embedding in $\mathbb{P}^{14}$ of the quotient of $\mathbb{P}^4 \times \mathbb{P}^4$ by the involution $(x, y) \to (y, x)$. Since the Picard group of $\mathbb{P}^4 \times \mathbb{P}^4$ is $\mathbb{Z} \oplus \mathbb{Z}$ and the involution transforms one generator into the other, we obtain that the Picard group of the quotient is 1. Now, from the Grothendieck–Lefshetz theorem, we deduce that $\rho(X) = 1$. The above fact is also proved in a more general context in Proposition 2.3 from the Appendix. To compute the Hodge number $h^{1,2}$, we argue as in Proposition 3.10. □

Remark 5.3. We shall describe a natural relation between the threefold X and a quintic in $\mathbb{P}^4$, that closes our cascade. As it was observed in the proof of [GP, Thm. 7.4], the smooth Calabi–Yau threefold X defined by the 3×3 minors of a symmetric 5×5 matrix in $\mathbb{P}^9$ admits an unramified covering being a Calabi–Yau threefold. Indeed, let T be the pre-image to $\mathbb{P}^4 \times \mathbb{P}^4$ of X by the involution. The image of the projection p_1 of T on $\mathbb{P}^4$ can be seen to be a quintic (the determinant of a 5×5 matrix) with 50 nodes (for generic choice of X). The projection $T \to p_1(T)$ is then a primitive contraction of type I.

5.2. If the surface D is isomorphic to $\mathbb{F}_1$ ($\mathbb{P}^2$ blown up in one point), it can be described by the 2×2 minors of a 3×5 matrix R of linear forms in $\mathbb{P}^8$, obtained by joining two symmetric 3×3 matrix with one common column.

$$\left(\begin{array}{cc|c|cc} l_1 & l_2 & l_3 & l_4 & l_5 \\ l_2 & l_6 & l_4 & l_7 & l_8 \\ l_3 & l_4 & l_5 & l_8 & l_9 \end{array}\right)$$

We will call such matrices double-symmetric. Let us embed D into a singular Calabi–Yau threefold X' defined by the 3×3 minors of the 4×5 matrix T obtained by adding one row to the matrix R in such a way that T can be obtained by deleting the last row from a symmetric 5×5 matrix. Denote by $Q_{l_{12}}$ (or Q) the following double-symmetric 4×6 matrix

$$\left(\begin{array}{cc|cc|cc} l_1 & l_2 & l_3 & l_4 & l_5 & l_8 \\ l_2 & l_6 & l_4 & l_7 & l_8 & l_{10} \\ l_3 & l_4 & l_5 & l_8 & l_9 & l_{11} \\ l_4 & l_7 & l_8 & l_{10} & l_{11} & l_{12} \end{array}\right)$$

where l_{10}, l_{11}, l_{12} are generic linear forms in $\mathbb{P}^8$. Let $G_l \subset X'$ be the surface defined by the 2×2 minors of Q_l.

Proposition 5.4. *The threefold Y defined by the 3×3 minors of a generic 4×6 double-symmetric matrix of linear forms in $\mathbb{P}^9$ has 12 isolated singular points analytically isomorphic to cones over $\mathbb{F}_1$. Moreover, the threefold X' has 1 ordinary double point and 11 more singularities described bellow.*

Proof. Consider the variety $T_k \subset \mathbb{P}^{11}$ parametrizing double symmetric 4×6 matrices of rank $\leq k$. Choosing coordinates $l_1, \dots, l_{12}$ in $\mathbb{P}^{11}$, the scheme T_k is defined by the $k+1\times k+1$ minors of Q. We find using Singular that the dimensions of T_2 and T_1 are 5 and 2 (and degrees 12 and 35) respectively. Let us show that $T_2 - T_1$ is smooth and that there exists a linear change of coordinates of $\mathbb{P}^{11}$ preserving 4×6 double symmetric matrices that transforms a given element of T_1 to $(1, 0, \dots, 0)$. In particular the type of singularities on T_2 wll then be the same in all points of T_1. Observe first that the matrix Q can be described in the following form

$$\begin{pmatrix} A & B & C \\ B & C & D \end{pmatrix}$$

where A, B, C, and D are symmetric 2×2 matrices. Consider the following operations preserving double symmetric matrices.

(1) The transformation that change (A,B,C,D) from the matrix Q into

$$(A, sA+B, (s^2+s)A+sB+sC, 2(s^3+s^2)A+(2s^2+s)B+2sC),$$

for chosen $s\in\mathbb{C}$.

(2) The central symmetry.

(3) The operations between symmetric 2×2 matrices

$$\begin{pmatrix} a,b \\ b,c \end{pmatrix} \longrightarrow \begin{pmatrix} a, rb+ta \\ b+sa, rc+sb+s^2a \end{pmatrix},$$

for chosen $t,r\in\mathbb{C}$, performed simultaneously on A,B,C,D.

(4) The operation of exchanging rows 1 with 2 and 3 with 4, composed with the operation exchanging columns 1 with 2, 3 with 4, and 5 with 6.

We claim that the composition group of the above operations acts transitively on T_1. Indeed, the rank 1 double symmetric matrices are exactly those with $aA=bB=cC=dD$ such that $a,b,c,d\in\mathbb{C}$, $ac-b^2=bd-c^2=ad-bc=0$, and such that A,B,C,D have rank 1.

Observe that such $Q\in T_2-T_1$ with $B=0$ and $aA=cC=dD$ of rank 1 have two dimensional orbits of the action of the group. Except in the latter case, we can find operations that transform a matrix $Q\in T_2-T_1$ into a matrix R with A and B of rank 2. Since R has rank 2, we compute that $AC-B^2=AD-BC=BD-C^2=0$. In fact if such equation is satisfied and at least one of the matrices A, B, C, D have rank 2 then R has rank 2. In the above case the orbits of the considered operations are three dimensional.

In a neighborhood of $R\in T_2-T_1$ with A and B of rank 2, we find a natural parametrization (fixing the 6 entries of A and B) and conclude that T_2 is smooth in R. To show the smoothness in the points corresponding to matrices $Q\in T_2-T_1$ with

$$A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A=aC,\ D=dC$$

, where $a\neq 0$ (the case $C=B=0$ is analogical), consider the intersection $T_2\cap H_2\cap H_3\cap H$, where $H_i=\{l_i=0\}$ for $i=2,3$ and $H=\{l_5=d\cdot l_9\}$. We find a local parametrization of this intersection in a neighborhood of Q with complex plane coordinates (x,r) close to $(a,0)$

$$\begin{pmatrix} x & 0 & 0 & xr & 1 & dxr \\ 0 & x^2r^2 & xr & dx^2r^2 & dxr & d^2x^2r^2+xr^2 \\ 0 & xr & 1 & dxr & d & d^2xr+r \\ xr & dx^2r^2 & dxr & d^2x^2r^2+xr^2 & d^2xr+r & d^3x^2r^2+2dxr^2 \end{pmatrix}.$$

Since $T_2 \cap H_2 \cap H_3 \cap H$ is smooth, we obtain that $T_2 - T_1$ is smooth.

We claim that the blow up of $T_1 \subset T_2$ gives a resolution of T_2. Indeed, we know from the descriptions above that T_1 is smooth and the generic three dimensional and transversal to T_1 complete intersection in T_2 has a node as singularity.

We obtain that the singularities of Y are locally isomorphic to a cone over $\mathbb{F}_1$ (the 9-dimensional linear section of T_2 in $(1, 0, \ldots, 0)$ has tangent cone being a cone over $\mathbb{F}_1$).

Using Singular, we compute that for generic l the multiplicity of the intersection of surfaces G_l and that D is 34 and the radical has degree 12. This suggests the following description. Consider the threefold $Y \subset \mathbb{P}^9$ defined by 3×3 minors of Q_x, where x in a new free variable. Blow up $\mathbb{P}^8 \times \mathbb{P}^9 \supset Z \to Y \subset \mathbb{P}^9$ in $(0, \ldots, 0, 1)$. The projection $Z \to \mathbb{P}^8$ contracts 12 lines 11 of which pass through the singular points of Z. We see that the remaining line has normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and is contracted to an ODP. $\square$

Remark 5.5. Observe that if we continue the cascade and embed the del Pezzo surface of degree 9, that is defined by $2 \times 2 \times 2$ minors of a $3 \times 3 \times 3$ matrix with linear forms in $\mathbb{P}^9$ symmetric with respect to three rectangular diagonals containing a chosen main diagonal, into one of the above Calabi–Yau threefolds, we obtain a threefold that cannot be smoothed (his singularity is rigid). The resulting variety is possibly defined by $3 \times 3 \times 3$ minors of a $4 \times 4 \times 4$ matrix symmetric with respect to three rectangular diagonals containing a chosen main diagonal.

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Appendix

A note on the Chow groups of projective determinantal varieties

by Piotr Pragacz

In the present note we shall consider the following types of determinantal varieties.

(i) (generic) Let W,V be two vector spaces over an arbitrary field K with $m = \dim W \geq n = \dim V$. For $r \geq 0$, set

$$D_r = D_r(\varphi) = \{x \in P : \text{rank } \varphi(x) \leq r\}\,, \tag{5.1}$$

where $\varphi : W_P \longrightarrow V_P \otimes \mathcal{O}(1)$ is the canonical morphism on $P = \mathbb{P}(\text{Hom}(W,V))$.

(ii) (symmetric) Take the following specialization of (i): let $m = n$, $W = V^*$, $P = \mathbb{P}(\text{Sym}^2(V))$, and $\varphi : V_P^* \longrightarrow V_P \otimes \mathcal{O}(1)$ be the canonical symmetric morphism on P. Define D_r by (5.1).

(iii) (partially symmetric) Consider the following specialization of (i): let $m > n$, $W^* \twoheadrightarrow V$, $P = \mathbb{P}(W^* \vee V)$ (in the notation of [LP]), and let

$$\varphi : W_P \longrightarrow V_P \otimes \mathcal{O}(1)$$

be the canonical partially symmetric morphism on P. Define D_r by (5.1).

(One can also, for even r, consider the skew-symmetric analogs of (ii) and (iii).)

In all cases (i), (ii) and (iii), we get a sequence of *projective determinantal varieties*

$$\emptyset = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{n-1} \subset D_n = D_{n+1} = \cdots\,.$$

The scheme D_r can be seen as a variety defined in P by the vanishing of $(r+1) \times (r+1)$ minors of a generic $m \times n$ matrix of linear forms. The codimensions of the determinantal varieties D_r in the respective cases are: (i) $(m-r)(n-r)$; (ii) $(n-r)(n-r+1)/2$; (iii) $(m-n)(n-r) + (n-r)(n-r+1)/2$.

In the present note, we compute $A^1(D_r)$ for the above determinantal varieties, getting the answers: $\mathbb{Z} \oplus \mathbb{Z}$ in cases (i) and (iii), and $\mathbb{Z}$ in case (ii). We also discuss generators of the Chow groups $A_*(D_r \setminus D_{r-1})$ in case (i); for $r = 1$ and $r = n-1$, we give some linearly independent generators.

Background. The content of this note was obtained in the late 80's, and has not been written up to now. Due to a recent ask of G.

and M. Kapustka, we have decided to write this material up because it is needed in their research.

In this note, we shall use notation, conventions, and some results from [P], [LP]. In particular, as for what concerns the Chow groups, we shall use notation and conventions from [F].

1. $D_r \setminus D_{r-1}$ AS FIBER BUNDLES

In this section, we follow basically [P].

(i) For $f \in \mathrm{Hom}(W,V)$, we set $K_f = \mathrm{Ker}(f)$, $C_f = \mathrm{Coker}(f)$. When f varies in $D_r \setminus D_{r-1}$, we get the vector bundles K and C of ranks $m-r$ and $n-r$ on $D_r \setminus D_{r-1}$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow G_{m-r}(W) \times G_r(V)$$

given by $f \mapsto (K_f,\ C_f)$. Its fiber is equal to the space of nonsingular $r \times r$ matrices over K. More explicitly, let

$$P' = \mathbb{P}(\mathrm{Hom}(Q_W,R_V)) \longrightarrow G_{m-r}(W) \times G_r(V)\,.$$

The bundle Q_W is the pullback on $G_{m-r}(W) \times G_r(V)$ of the tautological quotient rank r bundle on $G_{m-r}(W)$. Moreover, the bundle R_V is the pullback on $G_{m-r}(W) \times G_r(V)$ of the tautological subbundle on $G_r(V)$.

On P', there is the tautological morphism

$$\varphi' : (Q_W)_{P'} \longrightarrow (R_V)_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have

$$P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}\,. \tag{1.1}$$

(ii) For symmetric $f \in \mathrm{Hom}(V^*, V)$, we have $K_f \cong C_f^*$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow G_r(V)$$

given by $f \mapsto C_f$. Its fiber is equal to the space of nonsingular symmetric $r \times r$ matrices. To be more explicit, let

$$P' = \mathbb{P}(\mathrm{Sym}^2(R)) \longrightarrow G_r(V)\,,$$

where R is the tautological subbundle on $G_r(V)$. On P', there is the tautological symmetric morphism

$$\varphi' : R^*_{P'} \longrightarrow R_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have $P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}$.

(iii) For a partially symmetric $f \in \mathrm{Hom}(W, V)$, we have $K_f^* \twoheadrightarrow C_f$. Let Fl denote the flag variety parametrizing the pairs (A, B), where A

is an $(m-r)$-dimensional quotient of W^*, B is an $(n-r)$-dimensional quotient of V and we have $A \twoheadrightarrow B$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow Fl$$

given by $f \mapsto (K_f^*, C_f)$. Its fiber is equal to the space of nonsingular $r \times r$ symmetric matrices. More explicitly, let

$$P' = \mathbb{P}(\mathrm{Sym}^2(R)) \longrightarrow Fl\,.$$

The bundle R is here the tautological rank r subbundle on Fl. On P', there is the tautological symmetric morphism

$$\varphi' : R_{P'}^* \longrightarrow R_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have $P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}$.

2. Computations of $A^1(D_r)$

Let i' denote the embedding $D_{r-1}(\varphi')) \to P'$.

Lemma 2.1. *In each case (i), (ii) and (iii), we have the following exact sequence of the Chow groups:*

$$A_*(D_{r-1}(\varphi')) \xrightarrow{i'_*} A_*(P') \longrightarrow A_*(D_r \setminus D_{r-1}) \longrightarrow 0\,. \tag{2.1}$$

This follows by combining (1.1) and its analogues with [F], Sect.1.8 applied to the embedding $D_{r-1} \subset D_r$.

With the help of the Schur S- and Q-functions (cf, e.g., [P]), we now record

Lemma 2.2. *In case (i), the image $Im(i'_*)$ is generated by*

$$s_I(Q) - s_I(R \otimes L)\,,$$

where

$$Q = (Q_W)_{P'}, \quad R = (R_V)_{P'}, \quad L = \mathcal{O}_{P'}(1)\,,$$

and I runs over all partitions of positive weight.

In cases (ii) and (iii), by putting M to be the formal square root of L, the image $Im(i'_*)$ is generated by

$$Q_I(R \otimes M)\,,$$

where R denotes the pullback to P' of the corresponding tautological rank r subbundle (on $G_r(V)$ or Fl), and and I runs over all (strict) partitions of positive weight.

This follows from [P], Corollary 3.13 and its symmetric analog established also in [P].

Proposition 2.3. *In cases (i) and (iii), we have $A^1(D_r) \cong \mathbb{Z} \oplus \mathbb{Z}$ for any $r \geq 1$. In case (ii), we have $A^1(D_r) \cong \mathbb{Z}$ for any $r \geq 1$.*

Proof. Since $\operatorname{codim}(D_{r-1}, D_r) \geq 2$ for $r \geq 1$, it suffices to prove the same assertions for $A^1(D_r \setminus D_{r-1})$ instead of $A^1(D_r)$. Set, in all three cases, $h = c_1(L)$.
(i) By Lemmas 2.1 and 2.2 we see that $A^1(D_r \setminus D_{r-1})$ is generated (over $\mathbb{Z}$) by $s_1(Q)$, $s_1(R)$, and h, modulo the following single relation:

$$s_1(Q) = s_1(R \otimes L) = s_1(R) + h\,.$$

Thus the assertion follows.
(ii) We see that $A^1(D_r \setminus D_{r-1})$ is generated by $s_1(R)$ and h, modulo the following single relation:

$$Q_1(R \otimes M) = 2(s_1(R) + s_1(M)) = 2s_1(R) + h = 0\,,$$

which implies the assertion.
(iii) Since Fl is a Grassmann bundle over a Grassmannian, we have

$$A^1(Fl) \cong \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}s_1(R) \oplus \mathbb{Z}x\,,$$

for some x. We see that $A^1(D_r \setminus D_{r-1})$ is generated by $s_1(R)$, x and h, modulo the following single relation:

$$Q_1(R \otimes M) = 2s_1(R) + h = 0\,.$$

Hence the assertion follows. $\square$

Similarly, one shows that the Chow groups D_r (r even) of skew-symmetric and partially skew-symmetric projective determinantal varieties are of rank 1 and 2, respectively.

3. Remarks on other Chow groups of $D_r \setminus D_{r-1}$

We work here in the generic case (i).

Proposition 3.1. *For $r \geq 1$, we have the following inequalities:*

$$\binom{n}{r} \leq \text{rank } A_*(D_r \setminus D_{r-1}) \leq \binom{n}{r}(m - r + 1)\,. \tag{3.1}$$

Proof. To prove the first inequality, we invoke the following exact sequence of the Chow groups (cf. [F], Example 2.6.2):

$$A_k(D_r) \xrightarrow{\cdot\, h} A_{k-1}(D_r) \to A_k(C_{D_r}) \to 0\,, \tag{3.2}$$

where C_{D_r} is the affine cone over D_r. We recall the following result from [P], Proposition 4.2 (recall that we assume $m \geq n$):

$$\text{rank } A_*(C_{D_r}) = \binom{n}{r}\,. \tag{3.3}$$

The equality (3.3), combined with the surjection in the sequence (3.2), implies the first inequality.

To prove the second equality, we show that the elements $s_I(R) \cdot h^j$, where $I \subset (n-r)^r$ and $j = 0, \ldots, m-r$, generate over $\mathbb{Q}$ the Chow group $A^k(D_r \setminus D_{r-1})$, where $k = |I| + j$. It follows from Schubert calculus (cf., e.g. [F], Chap.14) and the surjection in (2.1) that the group $A_*(D_r \setminus D_{r-1})$ is generated by $s_I(Q)$, $I \subset (r)^{m-r}$; $s_J(R)$, $J \subset (n-r)^r$; and powers of the class h. By Lemma 2.2, in $A_*(D_r \setminus D_{r-1})$ we have

$$s_I(Q) = s_I(R \otimes L)\,,$$

and we see that the group $A_*(D_r \setminus D_{r-1})$ is generated by $s_J(R)$ (with $J \subset (n-r)^r$) and powers of the class h.

If $I \not\subseteq (n-r)^r$, then $s_I(Q) = 0$. Thus, invoking the Lascoux formula for the Schur polynomial of the twisted vector bundle (cf., e.g., [F], Ex. A.9.1), we get for such I:

$$0 = \sum_{J \subset I,\ J \subset (n-r)^r} d_{IJ} \cdot s_J(R) \cdot h^{|I|-|J|}\,. \tag{3.4}$$

These relations allow us to express the powers $h^{m-r+1}, h^{m-r+2}, \ldots$ with the help of h^j, $0 \le j \le m-r$, and $s_I(R)$, $I \subset (n-r)^r$. $\square$

Example 3.1. Let $r = 1$. By the proof of Proposition 3.1, we know that $s_i(R) \cdot h^j$, where $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$, generate over $\mathbb{Q}$ the Chow group $A_*(D_1)$. But by the Segre embedding, we have $D_1 \cong \mathbb{P}(W) \times \mathbb{P}(V)$. Since rank $A_*(\mathbb{P}(W) \times \mathbb{P}(V)) = mn$, the displayed elements are, in fact, $\mathbb{Z}$-linearly independent generators of $A_*(D_1)$. This can be also seen from the relations given in the proof of Proposition 3.1.

Example 3.2. Let now $r = n-1$.[1] In this case, C is a line bundle and $G_r(V) \cong \mathbb{P}^{n-1}$. Set $c = c_1(C)$. From the long exact sequence of bundles relating K and $C(h)$ one gets, for $a \ge m-n+2$,

$$\binom{m}{m-a} \cdot h^a - \binom{m}{m-a+1} \cdot h^{a-1} \cdot c = 0\,. \tag{3.5}$$

With the help of (3.5), one can deduce that the elements:

$$h^i \cdot c^j \ \ (0 \le i \le m-n,\ 0 \le j \le n-1), \qquad h^{m-n+1}, \qquad h^{m-n+1} \cdot c$$

are $\mathbb{Q}$-linearly independent generators of $A_*(D_{n-1} \setminus D_{n-2})$.

[1] This computation was done in collaboration with S.A. Strømme.

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