Lectures on torsion-free sheaves and their moduli

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February 12, 2007

Introduction

These notes come from the lectures delivered by the author at 25th Autumn School of Algebraic Geometry in Lukecin in 2002 and the lectures delivered by the author at the IMPANGA seminar in 2004–5. The School lectures were largely based on the book [HL], whereas the IMPANGA lectures were very close to the author's papers [La1], [La2] and [La3].

So the author decide to write notes that contain more of a vision of how the topic could be lectured upon than the actual faithful account of the delivered lectures. This particularly refers to the last lecture that contains generic smoothness of moduli spaces of sheaves on surfaces and the proofs in the lectures were based on the different O'Grady's approach. Since this approach was already published by the author (see [La3]), there was no point in copying it so the author decided to follow the Donaldson's approach whose idea (but not necessarily technical details) is easier to understand.

The notes contain some exercises (that are not very evenly distributed) in which the author put a part of the theory that is either analogous to what is done in the lectures or is too far away to be proven and is quite standard.

Since the paper is relatively short, it was impossible to give full proofs of all the theorems. All the proofs are either provided or can be found in the very incomplete references at the end of the paper. The references contain either books or the references that are not contained in these books (partially because they are too new).

1 Lecture 1. Bogomolov's instability and restriction theorems

- Topological classification of vector bundles
- Semistability and its properties
- Bogomolov's instability theorem
- Restriction theorems

Let X be a smooth complex projective variety. Classification of algebraic vector bundles on X can be divided into two parts: discrete, where we distinguish vector bundles using just topological structure, and continuous, where we study holomorphic/algebraic structures on a given topological vector bundle.

1.1 Topological classification of vector bundles

This part is usually neglected in algebro-geometric papers, as fixing basic topological invariants such as rank and Chern classes distinguish algebraic vector bundles sufficiently well to study change of the algebraic structure. Nevertheless, we will recall a few results on the topological classification refering to [BP] for a nice recent account of this topic.

The set of rank r vector bundles on X is isomorphic to the set $[X, \mathrm{Gr}_r(\mathbb{C}^\infty)]$ of homotopy classes of maps from X to the infinite Grassmannian $\mathrm{Gr}_r(\mathbb{C}^\infty) = BU(r)$. Since $H^*(\mathrm{Gr}_r(\mathbb{C}^\infty))$ is a polynomial ring $\mathbb{Z}[c_1, \dots c_n]$, this allows to define characteristic classes of a vector bundle.

In topology, it is easier to classify vector bundles up to stable equivalence (i.e., up to adding a trivial vector bundle). Stable equivalence classes can be read off the topological Grothendieck ring $K_{\mathrm{top}}(X)$, which is the universal ring associated to all vector bundles on X with direct sum and tensor operations. Again the topological group $K_{\mathrm{top}}(X)$ is isomorphic to homotopy classes $[X, B_{\mathbb{C}} \times \mathbb{Z}]$, where $B_{\mathbb{C}} = \lim_{\to} \mathrm{Gr}_r(\mathbb{C}^{2r})$. The K-ring is easier to compute and for example one can show that

$$K^*_{\mathrm{top}}(\mathbb{CP}^n) \simeq \mathbb{Z}[\xi]/\xi^{n+1}.$$

Nevertheless, a complete topological classification of vector bundles on projective spaces is far from being complete.

It is known that a topological rank $r > n$ vector bundle on n-dimensional X is topologically equivalent to a direct sum of a rank n vector bundle and the trivial vector bundle of rank $(r-n)$. So it is sufficient to classify vector bundles of rank $r \leq n$. As an application of the Atiyah-Singer index theorem, one can show that if E is a topological vector bundle on X then

$$\int_X \mathrm{ch}(E)\,\mathrm{ch}(\xi)\,\mathrm{td}X \in \mathbb{Z}$$

for all classes $\xi \in K_{\mathrm{top}}(X)$. In case $H^*(X, \mathbb{Z})$ has no torsion, then by the Bănică-Putinar result (see [BP]), if Chern classes satisfy the above condition then there exists a unique topological rank $r = n$ vector bundle with such Chern classes.

In case of projective spaces, the above conditions on Chern classes of topological vector bundles can be written quite explicitly, and they are known as *Shwarzenberger conditions*. In case of lower rank $r < n$ on $\mathbb{CP}^n$, there can be several different topological structures for a given rank and collection of Chern classes. The first such example was given by M. Atiyah and E. Rees, who classified topological rank 2 vector bundles on $\mathbb{CP}^3$ and in particular showed that there are exactly two different structures if $c_1 \equiv 0 \pmod 2$. Later, vector bundles on $\mathbb{CP}^n$ were classified for $n \leq 6$, but it is not known which of these vector bundles can be realized as algebraic vector bundles. This is connected to the Hartshorne's conjecture saying that there are no indecomposable rank 2 vector bundles on $\mathbb{CP}^n$ for large n ($n \geq 7$, or maybe even $n \geq 5$ as no examples are known).

In the rest of the paper we will be mainly interested in algebraic vector bundle on surfaces. In this case to each vector bundle E one can associate its rank r, the first Chern class $c_1E \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ and the second Chern class $c_2E \in H^4(X,\mathbb{Z}) \simeq \mathbb{Z}$. These are all topological invariants and all triples (r, c_1, c_2) with $r \geq 2$ can be realized as algebraic vector bundles (R. Schwarzenberger).

1.2 Semistability and its properties

Let us fix topological invariants of a vector bundle. As usual in algebraic geometry, instead of a geometric vector bundle we study the associated locally free coherent sheaf of its sections. In general, we cannot expect that the set of all vector bundles with fixed invariants have a nice structure of algebraic variety. The necessity of restricting to some subset of the set of all vector bundles can be understood by the following standard example.

Example 1.2.1. Consider the set $\{\mathscr{O}(n) \oplus \mathscr{O}(-n)\}$ on $\mathbb{P}^1$. The topological invariants are fixed but we have infinitely many points which cannot form a nice algebraic variety. Moreover, there exists a family of vector bundles $\{E_t\}_{t\in\mathbb{C}}$ such that $E_t \simeq \mathscr{O} \oplus \mathscr{O}$ for $t \neq 0$ but $E_0 \simeq \mathscr{O}(n) \oplus \mathscr{O}(-n)$ (Exercise: construct such a family). So the point corresponding to $\mathscr{O} \oplus \mathscr{O}$ would not be closed in the moduli space.

The natural class of vector bundles which admits a nice moduli space comes, at least in the curve case, from Mumford's Geometric Invariant Theory (GIT). The corresponding vector bundles are called stable (or semistable). However, in higher dimensions if we want to get a projective moduli space then we need to add some non-locally free sheaves at the boundary of the moduli space. So we need to define semistability and stability in the more general context.

Let X be a smooth n-dimensional projective variety defined over an algebraically closed field k and let H be an ample divisor on X. For any rank $r > 0$ torsion free sheaf E we define its *slope* by

$$\mu(E) = \frac{c_1E \cdot H^{n-1}}{r},$$

where c_1E denotes the first Chern class of the line bundle $(\bigwedge^r E)^{**}$. The *Hilbert polynomial* $P(E)$ is defined by $P(E)(k) = \chi(X, E \otimes \mathscr{O}_X(kH))$. The *reduced Hilbert polynomial* is defined as $p(E) = P(E)/r$.

Definition 1.2.2.

1. E is called *slope H-stable* if and only if for all subsheaves $F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$ we have
$$\mu(F) < \mu(E).$$
2. E is called *Gieseker H-stable* if and only if for all proper subsheaves $F \subset E$
$$p(F) < p(E)$$
(i.e., the inequality holds for large values k).

Similarly, one can define *(slope or Gieseker) H-semistability* by changing the strict inequality sign $<$ to $\leq$.

Restricting our attention to semistable sheaves is not very restrictive since each (torsion-free) sheaf has a canonical filtration with semistable quotients.

Let us fix a torsion free sheaf E on X. Consider the set $\{\mu(F) F \subset E\}$. This set has a maximal element $\mu_{\max} = \mu_{\max}(E)$ and the set $\{F : F \subset E,$ such that E/F is torsion-free, $\mu(F) = \mu_{\max}\}$ contains a sheaf E_1 of largest rank. This element is the largest element in this set with respect to the inclusion relation and it is called the *maximal destabilizing subsheaf* of E.

Now we consider the maximal destabilizing subsheaf E_2' in E/E_1 and set $E_2 = p^{-1}E_2'$ for the natural projection $p : E \to E/E_1$. Iterating this process we get the unique filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$$

such that

1. all the quotients $F_i = E_i/E_{i-1}$ are semistable, and
2. $\mu_{\max}(E) = \mu(F_1) > \mu(F_2) > \ldots > \mu(F_m) = \mu_{\min}(E)$.

This filtration is called the *Harder–Narasimhan filtration* of E.

1.2.1 Properties of slope semistability in characteristic 0:

1. If E_1, E_2 are slope H-semistable torsion free sheaves then $E_1 \otimes E_2/$Torsion is also slope H-semistable.
2. If $f : Y \to X$ is a finite map between smooth projective varieties then a torsion free sheaf F is slope H-semistable if and only if f^*F is slope f^*H-semistable.

1.3 Bogomolov's inequality

Now let us consider the following question: which Chern classes can be realized by semistable vector bundles?

The class $\Delta(E) = 2rc_2E - (r-1)c_1^2E$ is called the *discriminant* of E. In the surface case we will not distinguish between $\Delta(E)$ and its degree $\int_X \Delta(E)$.

A partial answer to the above question is given by the following theorem.

THEOREM 1.3.1 (BOGOMOLOV). *Let X be a smooth complex projective surface. Then for any torsion free (slope) H-semistable sheaf E we have*

$$\Delta(E) \geq 0.$$

Proof. We can assume that E is a locally free sheaf. Indeed, for any torsion free slope H-semistable sheaf E on a smooth surface, the sheaf E^{**} is locally free, slope H-semistable and $\Delta(E^{**}) \leq \Delta(E)$.

For simplicity, we also assume that $c_1E = 0$. The general case can be reduced to this one using either the $\mathbb{Q}$-vector bundle $E(-\frac{1}{r}\det E)$ or the vector bundle $\mathscr{E}nd E$.

Now let us note that S^nE is slope H-semistable (this follows from 1.2.1.1). Hence for $C \in |kH|$ and $k > 0$ we have

$$h^0(S^nE(-C)) = 0.$$

Therefore from the short exact sequence

$$0 \to O_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

tensored with S^nE we get

$$h^0(S^nE) \leq h^0(S^nE(-C)) + h^0(S^nE_C) = h^0(S^nE_C).$$

Considering $Y = \mathbb{P}(E_C) \to C$ we see that

$$h^0(S^nE_C) = h^0(Y, \mathscr{O}_{\mathbb{P}(E_C)}(n))$$

by the projection formula. Since $\dim Y = r$ there exists a constant C such that

$$h^0(S^nE) \leq h^0(Y, \mathscr{O}_{\mathbb{P}(E_C)}(n)) \leq C \cdot n^r$$

for all $n > 0$. Similarly, using restriction to $C \in |kH|$ for large k, one can see that there exists a constant C' such that

$$h^2(S^nE) = h^0(S^nE^* \otimes K_X) \leq C' \cdot n^r$$

for all $n > 0$.

Therefore

$$\chi(X, S^nE) \leq h^0(S^nE) + h^2(S^nE) \leq (C + C')n^r.$$

But by Exercise 1.5.10 we have

$$\chi(X, S^nE) = -\frac{\Delta(E)}{2r} \frac{n^{r+1}}{(r+1)!} + O(n^r),$$

so $\Delta(E) \geq 0$. □

The above proof of Bogomolov's theorem follows quite closely Y. Miyaoka's proof from [Mi].

PROPOSITION 1.3.2 ([LA1]). *Let X be a smooth complex projective surface and let H be an ample divisor on X. Then for any rank r torsion free sheaf E we have*

$$H^2 \cdot \Delta(E) + r^2(\mu_{\max} - \mu)(\mu - \mu_{\min}) \geq 0.$$

Proof. Let $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ be the Harder–Narasimhan filtration. Set $F_i = E_i/E_{i-1}$, $r_i = \operatorname{rk} F_i$, $\mu_i = \mu(F_i)$. Then by Bogomolov's inequality and the Hodge index theorem

$$\begin{aligned}\frac{\Delta(E)}{r} &= \sum \frac{\Delta(F_i)}{r_i} - \frac{1}{r}\sum_{i<j} r_i r_j \left(\frac{c_1 F_i}{r_i} - \frac{c_1 F_j}{r_j}\right)^2 \\ &\geq -\frac{1}{rH^2}\sum_{i<j} r_i r_j \left(\left(\frac{c_1 F_i}{r_i} - \frac{c_1 F_j}{r_j}\right) H\right)^2 = -\frac{1}{rH^2}\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2.\end{aligned}$$

(The Hodge index theorem says that $(DH)^2 \geq D^2 \cdot H^2$ for any divisor D.) Now the proposition follows from the following lemma:

LEMMA 1.3.3. *Let r_i be positive real numbers and $\mu_1 > \mu_2 > \ldots > \mu_m$ real numbers. Set $r = \sum r_i$ and $r\mu = \sum r_i \mu_i$. Then*

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 \leq r^2 (\mu_1 - \mu)(\mu - \mu_m).$$

Proof. Let us note that

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j)^2 = r \left(\sum_{i=1}^{m-1} \left(\sum_{j \leq i} r_j (\mu_j - \mu) \right) (\mu_i - \mu_{i+1}) \right).$$

Using $\sum_{j\leq i} r_j \mu_j \leq (\sum_{j\leq i} r_j)\mu_1$ and simplifying yields the required inequality. □

□

1.4 Restriction theorems

As an application of the above proposition we get the following effective restriction theorem (see [La1]). A weaker version of this theorem was proved by F.A. Bogomolov (see [HL, Theorem 7.3.5]).

THEOREM 1.4.1. *Let E be a rank $r \geq 2$ vector bundle on a smooth complex projective surface X. Assume that E is slope H-stable. Let $D \in |kH|$ be a smooth curve. If*

$$k \geq \frac{r-1}{r}\Delta(E) + 1$$

then E_D is stable.

Proof. Assume that E_D is not stable. Then there exists a subsheaf $S \subset E_D$ such that $\mu(S) \geq \mu(E_D)$. Let us take the maximal subsheaf with this property. In this case the quotient $T = (E_D)/S$ is a vector bundle on D (it is sufficient to check that T is torsion free; if it has a torsion then the kernel S' of $E_D \to T/\,$Torsion contains S and $\mu(S') \geq \mu(S)$).

Let G be the kernel of the composition $E \to E|_D \to T$. The sheaf G is called an *elementary transformation* of E along T. Set $\rho = \mathrm{rk} S$. Then $\mathrm{rk} E = \mathrm{rk} G = r$ and $\mathrm{rk} T = r - \rho$ (as a vector bundle on D).

Computing $\Delta(G)$ (use Exercise 1.5.11) we get

$$\Delta(G) = \Delta(E) - \rho(r-\rho)D^2 + 2(r \deg_D T - (r-\rho)Dc_1(E)).$$

By assumption $\mu(T) \leq \mu(E|_D)$, so

$$\Delta(G) \leq \Delta(E) - \rho(r-\rho)D^2.$$

Using the stability of E we get

$$\mu_{\max}(G) - \mu(G) = \mu_{\max}(G) - \mu(E) + \frac{r-\rho}{r}DH \leq \frac{r-\rho}{r}kH^2 - \frac{1}{r(r-1)}.$$

Note that we have two short exact sequences:

$$0 \to G \to E \to T \to 0$$

and

$$0 \to E(-D) \to G \to S \to 0.$$

In particular $G^* \subset (E(-D))^*$ and

$$\begin{aligned} \mu(G) - \mu_{\min}(G) &= \mu(E(-D)) - \mu_{\min}(G) + \frac{\rho}{r}DH \\ &= \mu_{\max}(G^*) - \mu((E(-D))^*) + \frac{\rho}{r}DH \leq \frac{\rho}{r}kH^2 - \frac{1}{r(r-1)}. \end{aligned}$$

Hence, applying Proposition 1.3.2 to G we obtain

$$\begin{aligned} 0 &\leq H^2\Delta(G) + r^2(\mu_{\max}(G) - \mu(G))(\mu(G) - \mu_{\min}(G)) \\ &\leq H^2\Delta(E) - \rho(r-\rho)(H^2)^2k^2 + r^2\left(\frac{r-\rho}{r}kH^2 - \frac{1}{r(r-1)}\right)\left(\frac{\rho}{r}kH^2 - \frac{1}{r(r-1)}\right). \end{aligned}$$

Therefore

$$\frac{rH^2}{r-1}k \leq H^2 \cdot \Delta(E) + \frac{1}{(r-1)^2},$$

which contradicts our assumption on k. □

Remark. Note that if E is torsion free then the restriction E_D is also torsion free for a general divisor D in a base point free linear system (see [HL, Corollary 1.1.14] for a precise statement).

As a corollary to Theorem 1.4.1 we get an effective restriction theorem for semistable sheaves.

COROLLARY 1.4.2. *Let E be a torsion free sheaf of rank $r \geq 2$. Assume that E is slope H-semistable. Let D be a general curve of a base point free linear system $|kH|$. If*

$$k \geq \frac{r-1}{r}\Delta(E)+1$$

then E_D is semistable.

Proof. Let $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ be the Jordan–Hölder filtration of E (i.e., such a filtration that all the quotients are slope H-stable torsion free sheaves; compare with the Harder–Narasimhan filtration). Set $F_i = E_i/E_{i-1}$ and $r_i = \mathrm{rk}\, F_i$. Let $D \in |kH|$ be any smooth curve such that all the sheaves $(F_i)_D$ have no torsion. Then the corollary follows from Theorem and the following inequality

$$\frac{\Delta(E)}{r} \geq \sum \frac{\Delta(F_i)}{r_i}$$

(cf. the proof of Proposition 1.3.2). □

1.5 Semistability in positive characteristic

In positive characteristic, to obtain analogues of properties 1.2.1 we need a notion of strong semistability. Let X be defined over a characteristic p field and let $F : X \to X$ be the absolute Frobenius morphism, obtained as identity on topological spaces and raising to p-th power on sections of $\mathcal{O}_X$.

We say that a sheaf E is *strongly slope H-semistable*, if for any integer k, the pull back $(F^k)^*E$ is slope H-semistable.

S. Ramanan and A. Ramanathan showed that in positive characteristic a tensor product of strongly slope H-semistable sheaves is strongly slope H-semistable. It is easy to see that Property 1.2.1.2 also holds for strongly slope semistable sheaves.

Theorem 1.3.1 still holds with a similar proof as before. The only difference is in showing that $h^2(S^nE) = O(n^r)$, because in general $(S^nE)^*$ is no longer isomorphic to $S^n(E^*)$ for sheaves with trivial determinant. In this case one can still prove it using twice Serre's duality:

$$h^2(X,S^nE) = h^0(X,(S^nE)^* \otimes K_X) \leq h^0(C,(S^nE)^* \otimes K_X \otimes \mathcal{O}_C) = h^1(C,S^nE \otimes \mathcal{O}_C(C))$$

and Exercise 1.5.1. Alternatively, one can replace symmetric powers of E by Frobenius pull backs and suitably change the computation.

This analogue of Bogomolov's inequality and generalizations of Proposition 1.3.2 and Corollary 1.4.2 imply boundedness of H-semistable sheaves on surfaces (see Lecture 3).

Remaining properties of semistability, some of which were used before, are put into the following exercises.

EXERCISE 1.5.1. Let X be a projective scheme of dimension d over an algebraically closed field k. Let F be a coherent sheaf on X. Then for any line bundle L on X we have

$$h^i(X,L^{\otimes n} \otimes F) = O(n^d)$$

(Hint: use Grothendieck's method of dévissage).

EXERCISE 1.5.2. Use Bogomolov's theorem to prove the Kodaira vanishing theorem on surfaces: if L is an ample line bundle on a smooth complex projective surface then $H^1(X, K_X + L) = 0$ (Hint: suppose otherwise and use Serre's duality $H^1(K_X + L) = (\mathrm{Ext}^1(L, \mathscr{O}_X))^*$ to construct a vector bundle which violates Bogomolov's inequality).

EXERCISE 1.5.3. M. Raynaud constructed a smooth projective surface X over an algebraically closed field of characteristic p and an ample line bundle L on X such that $H^1(X, L^{-1}) \neq 0$. Use this and Exercise 1.5.2 to construct a counterexample to Bogomolov's theorem for slope semistable sheaves in characteristic p.

EXERCISE 1.5.4. Let X be a smooth projective surface X over an algebraically closed field of characteristic p. One can show that although Bogomolov's inequality fails there exists some $\alpha(r, X, H)$ depending only on r, X and H such that $\Delta(E) \geq \alpha(r, X, H)$ any slope H-semistable torsion free sheaf E. Use this and Exercise 1.5.3 to construct a counterexample to Property 1.2.1.2 in characteristic p.

EXERCISE 1.5.5. Any slope semistable torsion free sheaf E on $\mathbb{P}^2$ is strongly slope semistable. Use this to show that in arbitrary characteristic Theorem 1.4.1 holds on $\mathbb{P}^2$.

EXERCISE 1.5.6. Show that $\Delta(T_{\mathbb{P}^2}) = 3$ and for all smooth curves $C \in |\mathscr{O}_{\mathbb{P}^2}(k)|$ with $k \leq 2$, the restriction E_C is not stable. Theorem 1.4.1 ensures that E_C is stable for any smooth curve $C \in |\mathscr{O}_{\mathbb{P}^2}(k)|$ if $k \geq 3$ (by Exercise 1.5.5 this holds in any characteristic).

EXERCISE 1.5.7. Let $F : X \to X$ be the absolute Frobenius morphism. Use Exercise 1.5.4 and the fact that $F^*E \subset S^pE$ for any vector bundle E to show that 1.2.1.1 also fails in characteristic p.

EXERCISE 1.5.8. Show that both Properties 1.2.1 fail if slope semistability is replaced by Gieseker semistability.

EXERCISE 1.5.9. Show that both Properties 1.2.1 fail if slope semistability is replaced by slope stability.

Partial solution: Let E be a slope H-stable with degree zero. Then E^* is also slope H-stable. If 1.2.1.1 holds for stability then $E \otimes E^*$ is slope stable. But $\mathscr{O}_X$ is a direct summand in $\mathscr{E}nd E = E \otimes E^*$ (at least in characteristic 0), a contradiction.

EXERCISE 1.5.10. Let E be a vector bundle on a smooth surface X. Use the Leray–Hirsch theorem and the Riemann–Roch formula on $\mathbb{P}(E)$ to prove that

$$\chi(X, S^nE) = -\frac{\Delta(E)}{2r}\frac{n^{r+1}}{(r+1)!} + O(n^r).$$

EXERCISE 1.5.11. Let E be a rank r vector bundle on a smooth surface X. Let G be an elementary transformation of E along a vector bundle T on a smooth curve D. Show that G is a rank r vector bundle on X and we have

$$c_1 G = c_1 E - \operatorname{rk} T \cdot D$$

and

$$c_2 G = c_2 E + \deg_D T - \operatorname{rk} T (D \cdot c_1 E) + \frac{\operatorname{rk} T (\operatorname{rk} T - 1)}{2} D^2.$$

(Hints: The first equality follows since T is trivial outside finitely many points of D, $c_1 T = c_1(\mathscr{O}_D^{\operatorname{rk} T})$ and from

$$0 \to \mathscr{O}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0.$$

The second equality follows from the Riemann–Roch formula.)

2 Lecture 2

- Moduli functors and moduli spaces: definition and examples
- Geometric invariant theory
- Moduli space of semistable sheaves

2.1 Moduli functors and moduli spaces

We are interested in providing the set of isomorphism classes of vector bundles on a fixed variety with a natural scheme structure. To explain what "natural" means we need a few notions from category theory. They will also be useful in describing some well-known parameter spaces like the Hilbert scheme, Grothendieck's Quot-scheme, etc.

Let $\mathscr{C}$ be a category (e.g., the category Sch/S of S-schemes of finite type) and let $\mathscr{M} : \mathscr{C} \to \mathrm{Sets}$ be a contravariant functor, called in the following a *moduli functor*. For an object X of $\mathscr{C}$ elements of $\mathscr{M}(X)$ will be called *families*. By $h_X : \mathscr{C} \to \mathrm{Sets}$ we will denote the functor of points of X defined by $h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X)$.

Definition 2.1.1. (1) $\mathscr{M}$ is *corepresented* by an object M of $\mathscr{C}$ if there is a natural transformation $\alpha : \mathscr{M} \to h_M$ such that for any natural transformation $\beta : \mathscr{M} \to h_N$ there exists a unique morphism $\varphi : M \to N$ such that $\beta = h_\varphi \alpha$.

(2) $\mathscr{M}$ is *represented* by an object M of $\mathscr{C}$ if it is isomorphic to the functor h_M.

Let $\mathscr{M}$ is a moduli functor. If the functor $\mathscr{M}$ is corepresented by M then we say that M is a *moduli space* for $\mathscr{M}$. If $\mathscr{M}$ is represented by M (or more precisely by a natural transformation $\mathscr{M} \to h_M$) then we say that M is a *fine moduli space* for the functor $\mathscr{M}$.

A moduli space M for $\mathscr{M}$ is fine if and only if there exists a *universal family* $U \in \mathscr{M}(M)$ such that the natural transformation $h_M \to \mathscr{M}$, given on $\mathrm{Hom}_{\mathscr{C}}(Y, M) \to \mathscr{M}(Y)$ via $g \to g^* U$, is an isomorphism.

We show a few basic moduli spaces in a growing order of generality.

2.1.1 Hilbert scheme.

Let $\mathscr{H}ilb(X/k)$ be a moduli functor $\mathrm{Sch}/k \to \mathrm{Sets}$ defined by

$$\mathscr{H}ilb(X/k)(S) = \{ \text{ subschemes } Z \subset S \times X, \text{ flat and proper over } S\}.$$

Let X be a projective scheme with an ample line bundle $\mathscr{O}_X(1)$. Let Z be a subscheme of X. We define the Hilbert polynomial of Z as

$$P(Z)(m) := \chi(Z, \mathscr{O}_Z \otimes \mathscr{O}_X(1)^{\otimes m}).$$

Let now $Z \overset{i}{\hookrightarrow} S \times X \xrightarrow{p} S$ and let $f = p \circ i$. Let $Z_s := f^{-1}(s)$ be the fiber of f over $s \in S$. Then we define a subfunctor of $\mathscr{H}ilb(X/k)$ by

$$\mathscr{H}ilb_P(X/k)(S) = \{ \text{ subschemes } Z \subset S \times X, \text{ flat and proper over } S \text{ with } P(Z_s) = P\}.$$

THEOREM 2.1.2. *The functor $\mathscr{H}ilb_P(X/_k)$ is representable by a projective k-scheme. The fine moduli space of this functor is called the* Hilbert scheme *and denoted by* $\mathrm{Hilb}_P(X/k)$.

EXERCISE 2.1.3. Let P be a constant polynomial equal to n. Then $\mathrm{Hilb}_P(X/k)$ is the Hilbert scheme of n points in X. Show that if X is a curve then $\mathrm{Hilb}_P(X/k) = S^nX$. What is the universal object in this case?

2.1.2 Grothendieck's Quot scheme

Let $f : X \to S$ be a projective morphism of Noetherian schemes and let $\mathscr{O}_X(1)$ be an f-ample line bundle on X. Let E be an S-flat coherent sheaf on X. We want to define a scheme which parametrizes pairs (s, F) consisting of a point $s \in S$ and a coherent quotient sheaf F of $E_s = E_{X_s}$ with fixed Hilbert polynomial P. To this end let us first define the functor $\mathscr{Q}uot : \mathrm{Sch}/S \to \mathrm{Sets}$, which to each S-scheme T associates the set of all T-flat quotients

$$E_T = \mathscr{O}_T \otimes E \to F$$

of coherent sheaves on $Y = T \times_S X$ such that for all $t \in T$ the sheaf $F_t = (F)_{Y_t}$ on the fiber $Y_t = \mathrm{Spec}\, k(t) \times_T Y$ of $Y \to T$ has Hilbert polynomial P.

THEOREM 2.1.4. *The functor $\mathscr{Q}uot$ is represented by a projective S-scheme* $\mathrm{Quot}_{X/S}(E;P) \to S$. *This scheme is called the* Quot-scheme.

Obviously, the Quot-scheme generalizes the classical Grassmann variety. In the special case when $E = \mathscr{O}_X$ the Quot scheme gives the Hilbert scheme parametrizing S-flat subschemes of the scheme X with given Hilbert polynomial P.

2.1.3 Flag scheme

Let E be an S-flat coherent sheaf on X and let P_i, $i = 1, \ldots, k$, be some fixed polynomials.

Consider the functor $\mathscr{F}lag : \mathrm{Sch}/S \to \mathrm{Sets}$, which to each S-scheme T associates the set of all flags

$$0 \subset F_1 \subset \ldots \subset F_k = E_T = \mathscr{O}_T \otimes E$$

of coherent subsheaves on $Y = T \times_S X$ such that

1. the factors $\mathrm{gr}_i = F_i / F_{i-1}$ of this filtration are T-flat, and
2. for all $t \in T$ the sheaf $\mathrm{gr}_{i,t} = \mathrm{gr}_i |_{Y_t}$ has Hilbert polynomial P_i.

THEOREM 2.1.5. *The functor $\mathscr{F}lag$ is represented by a projective S-scheme* $\mathrm{Flag}_{X/S}(E; P_1, \ldots, P_k) \to S$*, called the* flag scheme.

EXERCISE 2.1.6. Construct the flag scheme $\mathrm{Flag}_{X/S}(E; P_1, \ldots, P_k) \to S$ using existence of the Quot-schemes.

2.2 Geometric invariant theory (GIT)

Example 2.2.1. *(A categorical quotient)* Let a k-group G acts on a k-scheme X. Then for any k-scheme T the T-points of G, i.e., $h_G(T)$, also form a group which naturally acts on $h_X(T)$. So we can consider the functor $h_X / h_G : \mathrm{Sch}/k \to \mathrm{Sets}$, which to any k-scheme T associates the set of $h_G(T)$-orbits of $h_X(T)$. If this functor is corepresented by a scheme Y, then Y is called a *categorical quotient* of X by G and denoted by X/G.

Categorical quotients do not need to exist in general. But they exist if G is a reductive group acting on an affine scheme.

Let X be a k-scheme with the action σ of a k-group G:

$$\begin{array}{ll} G \times X & \xrightarrow{\sigma} X \\ p_2 \downarrow & \\ \quad X & \end{array}$$

Definition 2.2.2. $\mathscr{F}$ is *G-linearized* if there exists an isomorphism

$$\Phi : \sigma^* \mathscr{F} \to p_2^* \mathscr{F}$$

satisfying the cocycle condition.

From now on we assume that X is a projective k-scheme and G is a reductive group (e.g., $G = \mathrm{GL}(n)$). Let L be an ample G-linearized line bundle on X.

We define *(semi)stable points* of the polarized scheme (X, L) as follows:

Definition 2.2.3. 1. $X^{ss}(L) := \{x \in X : \exists n \; \exists s \in H^0(X, nL)^G |\; s(x) \neq 0\}$.

2. $X^s(L) := \{x \in X : x \in X^{ss}(L), G_x$ is finite, and $Gx \subset X^{ss}(L)$ is closed$\}$.

THEOREM 2.2.4. *1. There exists a categorical quotient $X^{ss}(L) \to X^{ss}(L)/G$. k-points of the quotient $X^{ss}(L)/G$ correspond to closed G-orbits in $X^{ss}(L)$ (but not all G-orbits need to be closed).*

2. There exists a categorical quotient $X^s(L) \to X^s(L)/G$. The fibers of this map are closed G-orbits.

To determine $X^{s(s)}(L)$ we will use the following theorem:

THEOREM 2.2.5. (the Hilbert-Mumford criterion) *A point $x \in X$ is semistable (stable) if and only if for all non-trivial one-parameter subgroups $\lambda : \mathbb{G}_m \to G$ we have*

$$\mu^L(x,\lambda) \geq 0 \quad (\mu^L(x,\lambda) > 0, \textit{ respectively}).$$

To define $\mu^L(x,\lambda)$ let us draw a diagram:

$$\begin{array}{rcll} \mathbb{A}^1 \setminus \{0\} = \mathbb{G}_m & \xrightarrow{\lambda} & G & \qquad\qquad g \\ \downarrow & & \downarrow \sigma & \qquad\qquad \downarrow \\ \mathbb{A}^1 & \xrightarrow{f} & X & \qquad\qquad gx = \sigma(g,x) \\ 0 & \to & f(0) = \lim_{t\to 0} \lambda(t)x & \end{array}$$

Let Φ be the linearization of L. This linearization induces on the fibres of L maps

$$\Phi : L(f(gx)) \to L(f(x)).$$

Clearly, $f(0)$ is a fixed point of the action of $\mathbb{G}_m$, so $\mathbb{G}_m$ acts on the fibre $L(F(0))$. Let r be the weight of this action. Then we set

$$\mu^L(x,\lambda) := -r.$$

2.3 Moduli space of semistable sheaves

In these lectures we are interested in the following moduli functor.

Let $(X, \mathscr{O}_X(1))$ be a smooth polarized projective scheme over an algebraically closed field k. For a k-scheme S let p and q denote the projections of $S \times_k X$ to S and X, respectively. Let us define an equivalence relation $\sim$ on S-flat families of sheaves on X by $F_1 \sim F_2$ if and only if there exists a line bundle $\mathscr{L}$ on S such that $F_1 \simeq F_2 \otimes p^*\mathscr{L}$. Let us note that fibrewise the families F_1 and F_2 define the same family of sheaves on X. Therefore it is natural to introduce the following moduli functor $\mathscr{M}_P : \mathrm{Sch}/k \to \mathrm{Sets}$ (and $\mathscr{M}_P^s : \mathrm{Sch}/k \to \mathrm{Sets}$), which sends a k-scheme S to the set of isomorphism classes of S-flat families of Gieseker semistable (respectively, stable) sheaves on X with Hilbert polynomial P modulo the relation $\sim$.

THEOREM 2.3.1. *There exists a moduli scheme M_P for the functor $\mathscr{M}_P$. It is a projective k-scheme of finite type and it contains an open subscheme M_P^s, which is the moduli scheme for $\mathscr{M}_P^s$.*

2.3.1 Geometric meaning of points of the moduli space of sheaves

Let E be a Gieseker semistable torsion free sheaf. Then either it is Gieseker stable or there exists a proper subsheaf $E_1 \subset E$ such that $p(E_1) = p(E)$. By passing to a smaller subsheaf if necessary we can assume that E_1 is Gieseker stable. Then the quotient E/E_1 is Gieseker semistable with $p(E/E_1) = p(E)$.

By induction on the rank we can therefore construct a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_m = E$ in which all the quotients E_i/E_{i-1} are Gieseker stable and $p(E_i/E_{i-1}) = p(E)$. Such a filtration is called a *Jordan–Hölder filtration* of E (but unlike the Harder–Narasimhan filtration it is not unique).

Let us set $\mathrm{gr}_{JH} E = \bigoplus E_i/E_{i-1}$.

EXERCISE 2.3.2. Show that $\mathrm{gr}_{JH} E$ does not depend on the choice of Jordan–Hölder filtration.

Definition 2.3.3. Two Gieseker semistable sheaves E and E' with $p(E) = p(E')$ are called *S-equivalent* if $\mathrm{gr}_{JH} E \simeq \mathrm{gr}_{JH} E'$.

An extension of a sheaf F by a sheaf G is an exact sequence of sheaves $0 \to G \to E \to F \to 0$. The set of all extensions is parametrized by a k-vector space $\mathrm{Ext}^1(F,G)$. Moreover, there exists the *universal extension* $\mathscr{E}$, i.e., such a family of sheaves on $\mathrm{Ext}^1(F,G) \times X$ that $\mathscr{E}_\eta = \mathscr{E}|_{\{\eta\}\times X}$ is the extension defined by $\eta \in \mathrm{Ext}^1(F,G)$.

If we have a Gieseker semistable sheaf E and $E_1 \subsetneq E$ is Gieseker semistable subsheaf with the same reduced Hilbert polynomial then the universal extension induces a morphism $\mathrm{Ext}^1(E/E_1, E_1) \to M$.

On the line $\mathbb{A}^1$ defined by $\eta = [E] \in \mathrm{Ext}^1(E/E_1, E_1)$ the universal extension $\mathscr{E}$ satisfies $\mathscr{E}_{t\eta} \simeq E$ for $t \neq 0$ and $\mathscr{E}_0 \simeq E_1 \oplus E/E_1$. Therefore the map $\mathbb{A}^1 \to M$ is constant (since M is separable) which shows that E defines the same point in M as $E_1 \oplus E/E_1$. Similarly, one can show that E defines the same point as $\mathrm{gr}_{JH} E$.

One can also show that two Gieseker semistable sheaves which are not S-equivalent define different points in M_P.

COROLLARY 2.3.4. *M_P parametrizes S-equivalence classes of Gieseker semistable sheaves. The subscheme M_P^s parametrizes isomorphism classes of Gieseker stable sheaves.*

2.3.2 Relative moduli spaces of pure sheaves

There exists a more general version of this theorem which will also be useful in the following. Before formulating this more general theorem we need to generalize the notion of Gieseker semistability and stability to the so called pure sheaves.

A sheaf E is called *pure* if it is torsion free on its scheme-theoretical support. Equivalently, for any subsheaf $F \subset E$ the dimension of the support of F is equal to the dimension of the support of E (this is denoted by $\dim E$).

The Hilbert polynomial of any coherent sheaf E can be written as a sum

$$P(E)(k) = \chi(X, E(k)) = \sum_{i=0}^{\dim E} \alpha_i(E) \frac{k^i}{i!}.$$

Then we define the *reduced Hilbert polynomial* $p(E) = \frac{P(E)}{\alpha_{\dim E}(E)}$. As in Definition 1.2.2 a pure sheaf E is called *Gieseker semistable* (*stable*) if for any proper subsheaf $F \subset E$ we have $p(F) \leq p(E)$ ($p(F) < p(E)$, respectively).

Let $f : X \to S$ be a projective morphism of k-schemes of finite type with geometrically connected fibers and let $\mathscr{O}_X(1)$ be an f-ample line bundle. Let P be a fixed polynomial.

As before we define the moduli functor $\mathscr{M}_{X/S}(P) : \mathrm{Sch}/S \to \mathrm{Sets}$ as

$$(\mathscr{M}_{X/S}(P))(T) = \left\{ \begin{array}{l} \text{equivalence classes of families of pure Gieseker} \\ \text{semistable sheaves on the fibres of } T \times_S X \to T \\ \text{which are } T\text{-flat and have Hilbert polynomial } P \end{array} \right\}.$$

THEOREM 2.3.5. *There exists a projective S-scheme $M_P(X/S)$ of finite type over S, which is the moduli space for the functor $\mathscr{M}_P(X/S)$. Moreover, there is an open scheme $M_P^s(X/S) \subset M_P(X/S)$ which is the moduli space for the subfunctor of families of geometrically Gieseker stable sheaves.*

2.3.3 Picard scheme

Let $X \to T$ be a flat projective morphism of k-schemes of finite type. Assume that geometric fibres of this morphism are varieties. The *Picard scheme* $\mathrm{Pic}^P_{X/T}$ parametrizes line bundles with fixed Hilbert polynomial P on fibres of $X \to T$. In general, it is a quasi-projective scheme but it is not a fine moduli scheme.

However, if $X \to T$ has a section then it is a fine moduli scheme. In this case there exists a universal family of line bundles on $X \times_T \mathrm{Pic}^P_{X/T}$. This family is called the *Poincare line bundle* and denoted by $\mathscr{P}$.

If X is a smooth projective variety over an algebraically closed field k then $\mathrm{Pic}^P_X = \mathrm{Pic}^P_{X/k}$ is a projective scheme. If P is the Hilbert polynomial of $\mathscr{O}_X$ then Pic^P_X is a group scheme with $1 = [\mathscr{O}_X]$ and tensor product as a group action. If $\mathit{char}\, k = 0$ then by Cartier's theorem any group scheme is smooth, so in particular Pic^P_X is smooth.

Finally let us note that for any two polynomials P_1 and P_2 representing line bundles $\mathscr{L}_1$ and $\mathscr{L}_2$ the corresponding Picard schemes are isomorphic. An isomorphism between $\mathrm{Pic}^{P_1}_X$ and $\mathrm{Pic}^{P_2}_X$ is given, e.g., by tensoring with $\mathscr{L}_2 \otimes \mathscr{L}_1^*$.

EXERCISE 2.3.6. Prove that the Picard scheme $\mathrm{Pic}^P_{X/T}$ is an open subset of the moduli space $M_P(X/T)$.

EXERCISE 2.3.7. Let X be a smooth projective k-variety and let P be the Hilbert polynomial of a line bundle. Prove that the whole moduli space $M_P(X/k)$ of torsion-free sheaves represents a suitably defined Picard functor. In particular, in this case $\mathrm{Pic}^P_{X/k}$ is a projective scheme.

3 Lecture 3

- Boundedness of semistable sheaves
- Construction of the moduli space of semistable sheaves

3.1 Boundedness of semistable sheaves

Corollary 1.4.2 implies boundedness of the family of semistable sheaves on complex surfaces. More precisely, there exists a scheme S of finite type over $\mathbb{C}$ and an S-flat sheaf F on $S \times X$ such that the set $\{F_s\}_{s\in S}$, where $F_s = F \otimes \mathscr{O}_{\{s\}\times X}$, contains all slope H-semistable sheaves with fixed topological invariants.

This can be proved in a few steps. Let us first recall the Castelnuovo-Mumford criterion:

THEOREM 3.1.1. *Let X be a smooth projective variety with $\mathscr{O}_X(1)$ very ample. Let E be a coherent sheaf on X. If $h^i(X, E(-i)) = 0$ for all $i > 0$ then*

1. *E is globally generated, and*
2. *$h^i(X, E(m-i)) = 0$ for all $i > 0$ and all $m \geq 0$.*

In the first step we need to prove boundedness on curves:

Boundedness for curves: Let C be a smooth curve. Let E be a sheaf on C with $\mathrm{rk}\,E = r$. Then

$$\mu_{\min}(E) > \mu(\omega_C(-m)) = \deg \omega_C - m \deg \mathscr{O}_C(1)$$

So if $m > \frac{\deg K_C - \mu_{\min} E}{\deg \mathscr{O}_C(1)}$ then $h^1(C, E(m)) = \mathrm{Hom}(E, \omega_C(-m)) = 0$. By Theorem 3.1.1 $E(m+1)$ is globally generated. Hence for all sheaves E with fixed Hilbert polynomial P and $\mu_{\min}(E)$ bounded from below we can find m_0 such that all $E(m_0)$ are quotients of $\mathscr{O}_X^{P(m_0)}$. So they form a bounded family as follows, e.g., from existence of Quot-schemes (see Theorem 2.1.4). In particular semistable vector bundles of fixed degree and rank form a bounded family.

Boundedness for surfaces:

For simplicity we will consider only stable vector bundles (locally free sheaves). Then Theorem 1.4.1 implies that for all stable locally free sheaves E on a complex surface X, with fixed Hilbert polynomial (or rank, c_1EH and $\Delta(E)$), there exists a fixed curve C such that the restriction E_C is stable. Then it is easy to see, e.g., applying Theorem 3.1.1 and Serre's vanishing theorem that there exists m_1 such that $E(m_1)$ is globally generated and we can conclude as before.

More generally, we have the following theorem due to Maruyama in the characteristic zero case and the author in general:

THEOREM 3.1.2. *(Boundedness Theorem) The family of slope semistable sheaves with fixed numerical data is bounded. This means that for a fixed polynomial P there exists a scheme S of finite type over k and an S-flat coherent sheaf $\mathscr{F}$ on $X \times S$, such that the set $\{\mathscr{F}_s\}_{s\in S}$ contains isomorphism classes of all slope semistable sheaves with Hilbert polynomial P.*

This implies that the moduli space of Gieseker semistable sheaves on X is a projective scheme of finite type.

For the construction of the moduli space we also need a good bound on the number of section of a sheaf:

THEOREM 3.1.3. *Let X be a smooth n-dimensional projective variety with a very ample line bundle H. Then for any rank r torsion free sheaf E on X we have*

$$h^0(X,E) \leq \begin{cases} rH^n \binom{\frac{\mu_{\max}(E)}{H^n}+\ln(r+1)+n}{n} & \text{if } \mu_{\max}(E) \geq 0, \\ 0 & \text{if } \mu_{\max}(E) < 0. \end{cases}$$

In characteristic zero the above theorem was proven by C. Simpson and J. Le Potier (see [HL, Theorem 3.3.1]), using the Grauert-Mülich restriction theorem. In positive characteristic this proof does no longer work and one needs to use Bogomolov's inequality (see [La2]).

Using Theorem 3.1.3 one can prove the following characterization of semistable sheaves among torsion free sheaves:

THEOREM 3.1.4. *Let P be a fixed polynomial and let m be a sufficiently large integer. Then for a torsion free sheaf with Hilbert polynomial P the following conditions are equivalent:*

1. *F is Gieseker semistable*

2. *$h^0(F(m)) = P(m)$ and for all subsheaves F' of F with rank $0 < r' = \operatorname{rk} F' < r = \operatorname{rk} F$ we have*
$$h^0(F'(m)) \leq \frac{r'P(m)}{r}$$
with equality if and only if $P(F') = P(F)$.

EXERCISE 3.1.5. Prove that the following implications hold:

$$\begin{array}{ccc} E \text{ is slope stable} & \Rightarrow & E \text{ is slope semistable} \\ \Downarrow & & \Uparrow \\ E \text{ is Gieseker stable} & \Rightarrow & E \text{ is Gieseker semistable} \end{array}$$

3.2 Construction of the moduli space M_P.

By Theorem 3.1.2 and Exercise 3.1.5 the family of all Gieseker semistable sheaves with fixed Hilbert polynomial P is bounded. This implies that there exists an integer m_0 such that for all $m \geq m_0$ and for all Gieseker semistable sheaves E with Hilbert polynomial P

1. $E(m)$ is globally generated,

2. $H^i(X, E(m)) = 0$ for $i > 0$.

In particular, these conditions imply that $h^0(E(m)) = \chi(X, E(m)) = P(m)$.

Set $V = k^{\oplus P(m)}$ and $\mathscr{H} = V \otimes \mathscr{O}_X(-m)$. Consider the Quot-scheme $Q = \mathrm{Quot}(\mathscr{H}; P)$.

Let R be a subset of Q parameterizing all quotients $[\mathscr{H} \to E]$ such that E is Gieseker semistable and the induced map $V = H^0(X, \mathscr{H}(m)) \to H^0(X, E(m))$ is an isomorphism.

For any point $[\mathscr{H} \to E] \in R$ and any isomorphism $V \to V$ we get a point in R corresponding to the composition

$$V \otimes \mathscr{O}_X(-m) \to V \otimes \mathscr{O}_X(-m) \to E.$$

Therefore $\mathrm{GL}(V)$ acts on R and, at least set theoretically, the quotient of R by $\mathrm{GL}(V)$ parameterizes Gieseker semistable sheaves on X. To get a scheme structure on this quotient we need to use Theorem 2.2.4.

To construct a GIT quotient first we need to find a $\mathrm{GL}(V)$-linearized polarization $\mathscr{L}$ on R. There are two different constructions of polarizations and both of them are used in the description of line bundles on the moduli space. One of them is due to C. Simpson and the other is due to D. Gieseker, and although it is less natural it was constructed much earlier. The problem with Gieseker's construction is that it does not easily generalize to pure sheaves.

In Simpson's construction we consider

$$\mathscr{L}_l := \det(p_*(\tilde{F} \otimes q^* \mathscr{O}_X(l))),$$

where $\tilde{F}$ is the universal quotient sheaf on the Quot-scheme. For large l this line bundle is very ample on Q. The Quot-scheme is constructed as a subscheme of certain Grassmannian and $\mathscr{L}_l$ comes from the Plücker embedding of this Grassmannian into a projective space.

In Gieseker's construction the constructed line bundle is not ample on Q but only on R.

To any family F of sheaves on a smooth projective variety X/k parametrized by S one can associate the family $\det F$ of determinant line bundles. This induces a natural transformation of functors $\mathscr{M}_P$ and $\mathscr{P}ic_X$. Since the functor $\mathscr{P}ic_X$ is represented by $\mathrm{Pic}\, X$ we get a map $\det : S \to \mathrm{Pic}\, X$ such that $\det F \simeq \det^* \mathscr{P} \otimes p^* \mathscr{L}$ for a certain line bundle $\mathscr{L}$ on S.

If we make this construction for the universal quotient $V \otimes q^* \mathscr{O}_X(-m) \to \tilde{F}$ on $Q \times X$ then we get a line bundle $\mathscr{A} = \mathscr{L}_{\overline{R}}$ on the closure of R in Q.

This line bundle $\mathscr{A}$ is also a quotient of $\det^*(\bigwedge^r V \otimes p^* \mathscr{P}^*(-rm))$ and the quotient map induces a morphism of schemes

$$\zeta : \overline{R} \to \mathbb{P}_{\mathrm{Pic}\, X}(\det^*(\bigwedge^r V \otimes p^* \mathscr{P}^*(-rm)))$$

over $\mathrm{Pic}\, X$. One can easily see that $\zeta|_R$ is injective, so $\mathscr{A} = \zeta^* \mathscr{O}_{\mathbb{P}}(1)$ is det-ample.

Once we constructed a $\mathrm{GL}(V)$-linearized line bundle $\mathscr{L}_l$ on $\overline{R}$ one has to prove that the points of R correspond to semistable points of $\mathrm{GL}(V)$ action on $(\overline{R}, \mathscr{L}_l)$ and the points of R^s (corresponding to those points $[\mathscr{H} \to E] \in R$ for which E is Gieseker H-stable) correspond to stable points of $\mathrm{GL}(V)$ action on $(\overline{R}, \mathscr{L}_l)$. Sice the center of $\mathrm{GL}(V)$ acts on Q trivially, we need only to find (semi)stable points for the $\mathrm{SL}(V)$-action. This is the content of Theorem 3.2.2.

Once we know it by Theorem 2.2.4 there exists a good quotient $\pi : R \to M$ and M is the moduli space of Gieseker H-semistable sheaves. Moreover, the restriction $\pi|_{R^s} : R^s \to M^s$ is a geometric quotient (i.e., $\mathrm{GL}(V)$-orbits of stable sheaves are closed).

By the properties of a GIT quotient, the points of M correspond to the closed orbits of $\mathrm{GL}(V)$-action on R. Therefore it is sufficient to identify such orbits. We used a slightly different method of identifying points of the moduli space in 2.3.1.

For further use let us note the following theorem, which is not needed for the construction of the moduli space:

THEOREM 3.2.1. *$R^s \to M^s$ is a principal $\mathrm{PGL}(V)$-bundle, locally trivial in the étale topology.*

This follows from the fact that the scheme-theoretic stabilizer of the $\mathrm{GL}(V)$-action at every point of R^s is equal to $\mathrm{Aut}\, E = k^*$.

THEOREM 3.2.2. *The set $\overline{R}^{ss}(\mathscr{L}_l)$ of semistable points of the action of $SL(V)$ on $(\overline{R}, (\mathscr{L}_l)_{\overline{R}})$ is equal to R.*

Proof. Let us take a point

$$[\rho : V \otimes \mathscr{O}(-m) \to F] \in Q$$

and a 1-parameter subgroup

$$\lambda : \mathbb{G}_m \to \mathrm{SL}(V).$$

We want to find $[\overline{\rho}] := \lim_{t\to 0} \lambda(t)[\rho]$ and the weight of the induced $\mathbb{G}_m$-action on $\mathscr{L}_l([\rho])$.

To this end let us decompose V into weight-spaces for the induced $\mathbb{G}_m$-action: $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where $\mathbb{G}_m$ acts on V_n with weight n. Let us set $V_{\leq n} = \bigoplus_{m\leq n} V_m$, $F_{\leq n} = \rho(V_{\leq n} \otimes \mathscr{O}(-m))$ and $F_n := F_{\leq n}/F_{\leq n-1}$. Then

$$[\overline{\rho}] = [\bigoplus V_n \to \bigoplus F_n].$$

Now we look for the weight of the action of $\mathbb{G}_m$ on the fiber $\mathscr{L}_l([\overline{\rho}])$.

As $\mathbb{G}_m$ acts on F_n with weight n, it acts on $H^i(F_n(l))$ with the same weight n. Since the fiber $\mathscr{L}_l([\overline{\rho}]) = \det H^*((\bigoplus F_n)(l))$, $\mathbb{G}_m$ acts on $\mathscr{L}_l([\overline{\rho}])$ with weight $\sum_{n\in\mathbb{Z}} nP(F_n, l)$. So

$$\mu^{\mathscr{L}_l}([\rho], \lambda) = -\sum_{n\in\mathbb{Z}} nP(F_n, l).$$

Let us take a subspace $V' \subset V$. It gives the filtration together with weights defined up to a multiple, so it is associated to a certain 1-parameter subgroup of $\mathrm{SL}(V)$. Let us set $F' = \rho(V' \otimes \mathscr{O}(-m))$ and

$$\Theta_F(V') = \dim V \cdot P(F', l) - \dim V' \cdot P(F, l).$$

Using the Abel transformation we can rewrite the above expression for $\mu^{\mathscr{L}_l}([\rho], \lambda)$ as

$$\mu^{\mathscr{L}_l}([\rho], \lambda) = \frac{1}{\dim V} \sum_{n\in\mathbb{Z}} \Theta_F(V_{\leq n}).$$

So we get the following corollary:

COROLLARY 3.2.3. *A point $[\rho : \mathscr{H} \to F] \in Q$ is* $\mathrm{SL}(V)$*-semistable if and only if for all $V' \subset V$ we have $\Theta_F(V') \geq 0$.*

Unfortunately, usually one is not able to prove that $\mathrm{SL}(V)$-semistable points of Q are in R. The main problem is to prove that an $\mathrm{SL}(V)$-semistable point of Q corresponds to a torsion free sheaf (or it is a limit of such points). So we need to restrict to $\overline{R}$. Then Theorem 3.1.4 the above corollary imply the following finishing the construction:

COROLLARY 3.2.4. *If $l \gg 0$ then $[\rho : \mathscr{H} \to F] \in \overline{R}$ is* $\mathrm{SL}(V)$*-semistable if and only if F is Gieseker semistable.*

□

4 Lecture 4

• Line bundles on moduli spaces
• Strange duality

4.1 Line bundles on moduli spaces

4.1.1 Grothendieck's K and K^0 groups of sheaves on varieties

Let X be an n-dimensional Noetherian scheme. The *Grothendieck group* $K(X)$ of coherent sheaves on X is the quotient of the free abelian group generated by the coherent sheaves on X by the subgroup generated by $[F]-[F']-[F'']$, where F, F' and F'' are coherent sheaves in the exact sequence

$$0 \to F' \to F \to F'' \to 0.$$

The Grothendieck group $K^0(X)$ of locally free sheaves is defined in the same way but using only locally free sheaves and short exact sequences of locally free sheaves.

If X is smooth then $K(X) \simeq K^0(X)$ (see Exercise 4.2.5) has structure of a commutative ring with $1 = [\mathscr{O}_X]$ and $[F_1]\cdot[F_2] := [F_1 \otimes F_2]$ for locally free sheaves F_1 and F_2.

In this case we can introduce a quadratic form $\chi : K(X) \times K(X) \to \mathbb{Z}$. This is defined by

$$(a,b) \to \chi(a\cdot b) = \int_X \mathrm{ch}(a)\,\mathrm{ch}(b)\,\mathrm{td}(X)$$

for $a, b \in K(X)$, where $\mathrm{ch}(x)$ is the Chern character of x and $\mathrm{td}(X)$ is the Todd class of the tangent bundle of X.

4.1.2 Line bundles defined by families of sheaves

A projective morphism $f : X \to S$ induces a homomorphism $f_! : K(X) \to K(S)$ defined by

$$f_!([F]) = \sum_{i \geq 0} (-1)^i [R^i f_* F].$$

PROPOSITION 4.1.1. *If $f : X \to S$ is a smooth projective morphism, S is a k-scheme of finite type and E is an S-flat coherent sheaf on X then $[E] \in K^0(X)$ and $f_!([E]) \in K^0(S)$. More precisely, E has a locally free resolution $E_\bullet$ such that all sheaves $R^i f_* E_j$ are locally free.*

Let p and q denote natural projections of $S \times X$ to S and X, respectively. The above proposition implies that if E is an S-flat family of sheaves on a smooth projective variety X then $[E] \in K^0(S \times X)$ and we have a map $p_! : K^0(S \times X) \to K^0(S)$. Therefore the following definition makes sense:

Definition 4.1.2. For each such family E we define the homomorphism $\lambda_E : K(X) \to \operatorname{Pic} S$ by

$$\lambda_E(u) = \det p_!(q^* u \cdot [E]).$$

In this way we constructed line bundles on a scheme parametrizing a given family of sheaves on X.

4.1.3 Line bundles on moduli spaces

We want to make a similar construction as above to construct line bundles on the moduli space.

If the moduli space M_P is fine then there exists the universal sheaf $\mathscr{E}$ on $M \times X$ and $\lambda_{\mathscr{E}}$ produces line bundles on M_P for each class $u \in K(X)$. In general M_P is not fine and we cannot expect existence of a homomorphism $\lambda : K(X) \to \operatorname{Pic} M_P$. However, such a homomorphism exists if we restrict to some subspace of $K(X)$.

Let c be a class in $K(X)$. Then we can define the moduli space $M(c)$ of Gieseker H-semistable sheaves of class c. This is a well defined open and closed subscheme of M_P for P defined by $P(m) = \chi(X, c \cdot [\mathscr{O}_X(mH)])$. We also set $M^s(c) = M_P^s \cap M(c)$. We will produce line bundles on $M(c)$ and $M^s(c)$.

Set

$$K_c = c^\perp = \{u \in K(X) : \chi(c \cdot u) = 0\}$$

and

$$K_{c,H} = c^\perp \cap \{1, h, h^2, \ldots, h^{\dim X}\}^{\perp\perp}.$$

THEOREM 4.1.3. *(1) There exists a group homomorphism $\lambda^s : K_c \to \operatorname{Pic} M^s(c)$ such that for any S-flat family $\mathscr{E}$ of Gieseker H-stable sheaves of class c on X if $\Phi_{\mathscr{E}} : S \to M^s(c)$ denotes the classifying morphism then $\Phi_{\mathscr{E}}^*(\lambda^s(u)) = \lambda_{\mathscr{E}}(u)$ for $u \in K_c$.*

(2) ($\operatorname{char} k = 0$) There exists a group homomorphism $\lambda : K_{c,H} \to \operatorname{Pic} M(c)$ such that for any S-flat family $\mathscr{E}$ of Gieseker H-semistable sheaves of class c on X if $\Phi_{\mathscr{E}} : S \to M(c)$ denotes the classifying morphism then $\Phi_{\mathscr{E}}^*(\lambda(u)) = \lambda_{\mathscr{E}}(u)$ for $u \in K_{c,H}$.

(3) ($\operatorname{char} k = 0$) For any $u \in K_{c,H}$ the restriction of $\lambda(u)$ to $M^s(c)$ gives $\lambda^s(u)$.

Proof. We will prove only the first part of the theorem. The rest can be proved in a similar way.

Let $R^s(c) \subset R$ be an open subset of R used in the construction of the moduli space, corresponding to stable sheaves of class c. Let $u \in K(X)$ and set $\mathscr{L} = \lambda_{\tilde{F}}(u)$, where $\tilde{F}$ is the universal quotient sheaf on $R^s(c) \times X$. $\mathscr{L}$ has a natural $\mathrm{GL}(V)$ linearization coming from $\tilde{F}$.

We want to check that if $u \in K_c$ then $\mathscr{L}$ descends to a line bundle on $M^s(c)$. To check it, it is sufficient to check that for any point $[\rho] = [\rho : \mathscr{H} \to E] \in R^s(c)$ the stabilizer of $\mathrm{GL}(V)$ action at $[\rho]$ acts trivially on the fibre $\mathscr{L}([\rho])$.

The stabilizer of $\mathrm{GL}(V)$ at ρ is equal to the image of a natural homomorphism $\mathrm{Aut}\, E \to \mathrm{GL}(V)$ sending φ to $H^0(\rho(m))^{-1} \circ H^0(\varphi(m)) \circ H^0(\rho(m))$ (this is well defined since by our assumption $H^0(\rho(m))$ is an isomorphism).

So we need to understand the action of $\mathrm{Aut}\, E \simeq k^*$ on $\mathscr{L}([\rho])$. Since higher direct images commute with base change we have $\mathscr{L}([\rho]) \simeq \det H^\bullet(X, E \otimes u)$. Therefore $A \in \mathrm{Aut}\, E$ acts by $A^{\chi(c \cdot u)}$ and this action is trivial if $u \in K_c$. ☐

4.2 Strange duality

In the remaining part of this lecture (apart from the exercises) we assume that the base field has characteristic zero.

4.2.1 The strange duality morphism

Let us take two classes c and c^* in $K(X)$ such that $c \in K_{c^*,H}$ and $c^* \in K_{c,H}$ (in particular $\chi(c \cdot c^*) = 0$).

Then we can define $\mathscr{D}_{c,c^*} = \lambda_c(c^*)$ and $\mathscr{D}_{c^*,c} = \lambda_{c^*}(c)$. Let p_1 and p_2 denote the projections of $M(c) \times M(c^*)$ onto the first and the second factor, respectively. Let us take a line bundle $\mathscr{D} = p_1^* \mathscr{D}_{c,c^*} \otimes p_2^* \mathscr{D}_{c^*,c}$.

We will assume that X is either a curve or a surface. We will also make some mild assumptions on the classes c and c^*, e.g., we assume that the rank of c is positive and semistable sheaves of class c^* are pure of dimension $(\dim X - 1)$. Then the line bundle $\mathscr{D}$ has the following universal property:

PROPOSITION 4.2.1 (LE POTIER, [LP2]). *For all S-flat families $\mathscr{F}$ of Gieseker H-semistable sheaves of class c and $\mathscr{G}$ of Gieseker H-semistable sheaves of class c^* let $\Phi_{\mathscr{F}} : S \to M(c)$ and $\Phi_{\mathscr{G}} : S \to M(c^*)$ be the corresponding classifying morphisms. Then for*

$$\Phi = (\Phi_{\mathscr{F}}, \Phi_{\mathscr{G}}) : S \to M(c) \times M(c^*)$$

we have $\Phi^* \mathscr{D} = \det(p_!(\mathscr{F} \otimes \mathscr{G}))$.

If we now assume that $H^2(F \otimes G) = 0$ for all Gieseker H-semistable sheaves F of class c and G of class c^* then we can construct a canonical section $\sigma_{c,c^*} \in H^0(M(c) \times M(c^*), \mathscr{D})$ such that its zero set is equal to $\{([F],[G]) : H^1(F \otimes G) \neq 0\}$.

This section gives an element of $H^0(M(c),\mathscr{D}_{c,c^*})\otimes H^0(M(c^*),\mathscr{D}_{c^*,c})$ so we have a linear map

$$D_{c,c^*}: H^0(M(c^*),\mathscr{D}_{c^*,c})^* \to H^0(M(c),\mathscr{D}_{c,c^*}),$$

which is called the strange duality map.

CONJECTURE 4.2.2 (STRANGE DUALITY CONJECTURE). *Whenever defined and non-zero the map D_{c,c^*} is an isomorphism.*

Geometric interpretation of this conjecture is the following. We have a rational map

$$\Psi : M(c) \dashrightarrow \mathbb{P}(H^0(M(c^*),\mathscr{D}_{c^*,c}))$$

which sends $[F]\in M(c)$ to the divisor $\{[G]\in M(c^*) : H^1(F\otimes G)\neq 0\}$. One can check that $\Psi^*\mathscr{O}(1)=\mathscr{D}_{c,c^*}$. The Strange Duality Conjecture asks if Ψ is a morphism and if the image of Ψ is not contained in any hyperplane.

4.2.2 Strange duality on curves

Let C be a smooth projective curve. We have $K(C)=\mathbb{Z}\oplus \operatorname{Pic} C$, where the first factor corresponds to the rank and the second to the determinant. We also have a numerical K-group $K(C)_{num}=\mathbb{Z}\oplus\mathbb{Z}$, where the first factor corresponds to the rank and the second one to the degree of the determinant.

Let us take $c=[\mathscr{O}_C^r]\in K(C)$ and $c^*=(1,g-1)\in K(C)_{num}$. One can easily check that $\chi(c\cdot c^*)=0$ and assumptions needed to define the strange duality map are satisfied.

In this case $\mathscr{D}_{c,c^*}$ is the generator of $\operatorname{Pic} M(c)$ and $M(c^*)=J^{g-1}$ is the Jacobian parametrizing line bundles of degree $g-1$.

Since we have the Poincare line bundle $\mathscr{P}$ on J^{g-1} (see 2.3.3) we get

$$\mathscr{D}_{c^*,c}=\det(\pi_!(\mathscr{P}\otimes c))=r\det(\pi_!\mathscr{P})=r\Theta,$$

where Θ is the theta divisor defined by $\Theta=\{L\in J^{g-1}: h^0(C,L)\neq 0\}$.

The strange duality map $\Psi: M \dashrightarrow \mathbb{P}(H^0(J^{g-1},r\Theta))$ sends $[E]$ to $\{L\in J^{g-1}: h^0(C,E\otimes L)\neq 0\}$ and the strange duality implies that

$$h^0(M(c),\mathscr{D}_{c,c^*})=h^0(J^{g-1},r\Theta).$$

This is a special case of the so called *Verlinde formula* computing the number of sections of tensor powers of $\mathscr{D}_{c,c^*}$ on $M(c)$.

THEOREM 4.2.3. *Strange duality conjecture holds for curves.*

This theorem was first proven for generic curves by P. Belkale in [Be] and then for all curves by A. Marian and M. Oprea in [MO]. Unfortunately, the proof uses the Verlinde formula, so we do not get a new proof of the Verlinde formula.

4.2.3 Strange duality on the projective plane

Let us consider the projective plane $\mathbb{P}_2$. Let c be the class of a rank 2 vector bundle with $c_1 = 0$ and $c_2 = n$ and let c^* be the class of $\mathscr{O}_L(-1)$ for some line L.

Then $\chi(c \cdot c^*) = 0$ and one can check that all our assumptions used in the definition of the strange duality map are satisfied.

Let us note that $M(c^*)$ is the dual $\mathbb{P}_2$ (it parametrizes lines in $\mathbb{P}_2$) and $M(c)$ parametrizes semistable sheaves of rank 2 on $\mathbb{P}_2$ with $c_1 = 0$ and $c_2 = n$.

By the Grauert–Müllich theorem (false if $\operatorname{char} k > 0$) the restriction of any such semistable sheaf to a general line is trivial. The lines on which the restriction is non-trivial (the so called *jumping lines*) form a curve of degree n in $\mathbb{P}_2^*$ (this fact is called *Barth's theorem*).

In this case Ψ is defined everywhere on $M(c)$ and it is known as *the Barth morphism*. However, even in this case the strange duality conjecture is known only for small n:

THEOREM 4.2.4 (DANILA, SEE [DA]). *The strange duality map is an isomorphism for* $2 \le n \le 19$.

EXERCISE 4.2.5. Let X be a smooth variety over an algebraically closed field k. Then there exists an isomorphism $K^0(X) \simeq K(X)$ (see [Ha, Chapter 3, Exercise 6.9]).

EXERCISE 4.2.6. Show that for any coherent sheaves F_1 and F_2 we have

$$[F_1] \cdot [F_2] = \sum_i (-1)^i [\mathscr{T}or_i^{\mathscr{O}_X}(F_1, F_2)]$$

EXERCISE 4.2.7. Let us set $[F]^* = \sum (-1)^i [\mathscr{E}xt^i(F, \mathscr{O}_X)]$ for a coherent sheaf F (note that if F is locally free then $[F]^* = [F^*]$). Show that

$$[F_1] \cdot [F_2]^* = \sum (-1)^i [\mathscr{E}xt^i(F_2, F_1)].$$

EXERCISE 4.2.8. For any two coherent sheaves F_1 and F_2 let us set

$$\chi(F_1, F_2) = \sum (-1)^i \dim \operatorname{Ext}^i(F_1, F_2).$$

Show that $\chi(F_1, F_2) = \chi(X, [F_2] \cdot [F_1]^*)$.

EXERCISE 4.2.9. Let E and F be torsion free sheaves on a smooth surface X. Set $r_E = \operatorname{rk} E$, $r_F = \operatorname{rk} F$ and $\xi_{E,F} = \frac{c_1 F}{r_F} - \frac{c_1 E}{r_E}$. Show that

$$\chi(E, F) = -\left(r_E \frac{\Delta(F)}{2r_F} + r_F \frac{\Delta(E)}{2r_E} \right) + r_E r_F \left(\frac{1}{2}\xi_{E,F}^2 - \frac{1}{2}\xi_{E,F} K_X + \chi(\mathscr{O}_X) \right).$$

5 Lecture 5

- Deformation theory of Quot schemes and local structure of moduli spaces
- *e*-stability
- Examples of unobstructed moduli spaces

This lecture starts with a local study of moduli spaces.

5.1 Deformation theory of Quot schemes and local structure of moduli spaces

THEOREM 5.1.1 (SEE [HL], PROPOSITION 2.2.8). *Let X be a projective scheme over k, $\mathscr{H}$ a coherent sheaf and P a Hilbert polynomial. Take a k-rational point $[\sigma : \mathscr{H} \to E] \in \mathrm{Quot}(\mathscr{H}; P)$ and set $K = \ker \sigma$. Then*

$$\dim \mathrm{Hom}(K,E) - \dim \mathrm{Ext}^1(K,E) \leq \dim_{[\sigma]} \mathrm{Quot}(\mathscr{H}; P) \leq \dim \mathrm{Hom}(K,E).$$

Moreover, if $\mathrm{Ext}^1(K,E) = 0$ then $\mathrm{Quot}(\mathscr{H}; P)$ is smooth at $[\sigma]$.

COROLLARY 5.1.2. *Let E be a Gieseker stable sheaf with Hilbert polynomial P. If $\mathrm{Ext}^2(E,E) = 0$ then M_P^s is smooth at $[E]$ and it has dimension $\dim \mathrm{Ext}^1(E,E)$.*

Proof. Take a point $[\mathscr{H} \to E] \in R^s$ as in the construction of M_P^s. From the short exact sequence

$$0 \to K \to \mathscr{H} \to E \to 0$$

we get

$$\begin{aligned} 0 \ &\to \mathrm{Hom}(E,E) \to \mathrm{Hom}(\mathscr{H},E) \to \mathrm{Hom}(K,E) \to \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^1(\mathscr{H},E) \to \\ &\to \mathrm{Ext}^1(K,E) \to \mathrm{Ext}^2(E,E) \to \mathrm{Ext}^2(\mathscr{H},E). \end{aligned}$$

Now by construction $\mathrm{Ext}^i(\mathscr{H},E) = H^i(E(m))^{\dim V}$ vanishes for $i > 0$ and it has dimension $(\dim V)^2$ for $i = 0$. Since E is stable, $\dim \mathrm{Hom}(E,E) = 1$. Then $\mathrm{Ext}^1(K,E) = \mathrm{Ext}^2(E,E) = 0$ and by the above theorem R^s is smooth at $[\mathscr{H} \to E]$. Moreover, at this point R^s has dimension $\dim \mathrm{Hom}(K,E) = \dim \mathrm{Ext}^1(E,E) + \dim \mathrm{PGL}(V)$.

Since $R^s \to M^s$ is a $\mathrm{PGL}(V)$-principal bundle (see Theorem 3.2.1), by descent R^s is smooth if and only if M^s is smooth. So the above dimension count and Theorem 5.1.1 imply the required assertions. □

In fact, one can prove a more precise version of this theorem. Namely, for any Gieseker stable sheaf E there exists a map

$$\mathrm{Ext}^1(E,E) \supset U \xrightarrow{\Phi} \mathrm{Ext}^2(E,E)$$

well defined on an open neighbourhood of 0 such that the germ of M^s at $[E]$ is the germ of $\Phi^{-1}(0)$ at 0. The map Φ is called *a Kuranishi map* but it is not unique in algebraic category.

Let X be a smooth projective variety. Then we have a map

$$\det : M \to \operatorname{Pic} X$$

sending $[E]$ to $[\det E]$ (it is well defined even at points corresponding to strictly semistable sheaves).

THEOREM 5.1.3. *Let E be a Gieseker stable sheaf. Then after canonical isomorphisms the tangent map $T_{[E]}M \to T_{[\det E]}\operatorname{Pic} X$ can be identified with the trace map*

$$\operatorname{Tr} : \operatorname{Ext}^1(E,E) \to \operatorname{Ext}^1(\det E, \det E) \simeq H^1(\mathscr{O}_X).$$

Moreover, obstructions for $[E]$ to be a smooth point of M map to the obstructions for $[\det E]$ to be a smooth point of $\operatorname{Pic} X$ *via the trace map*

$$\operatorname{Tr} : \operatorname{Ext}^2(E,E) \to \operatorname{Ext}^2(\det E, \det E) \simeq H^2(\mathscr{O}_X).$$

Let us denote the kernel of the trace map

$$\operatorname{Tr} : \operatorname{Ext}^i(E,E) \to \operatorname{Ext}^i(\det E, \det E) \simeq H^i(\mathscr{O}_X)$$

by $\operatorname{Ext}^i(E,E)_0$.

If the characteristic of the base field is 0 then by Cartier's theorem any group scheme is smooth, so $\operatorname{Pic} X$ is smooth. In this case, if E is a Gieseker stable sheaf then M^s is smooth at $[E]$ if $\operatorname{Ext}^2(E,E)_0 = 0$.

More precisely, if we set $M(\mathscr{L}) = \det^{-1}([\mathscr{L}])$ (this definition needs alteration in positive characteristic) then the germ of $M^s(\mathscr{L})$ at any point $[E]$ is the germ at 0 of $\Psi^{-1}(0)$ for some map

$$\operatorname{Ext}^1(E,E)_0 \supset U \xrightarrow{\Psi} \operatorname{Ext}^2(E,E)_0$$

defined on an open neighbourhood U of 0.

5.2 e-stability

Let X be a smooth projective n-dimensional variety and let H be an ample divisor on X. Let us set $|H| = \sqrt[n]{H^n}$.

Definition 5.2.1. Let $e \geq 0$ be a real number. A torsion-free sheaf E on X is called *e-(semi)stable* if for every subsheaf $E' \subset E$ of rank $r' < r = \operatorname{rk} E$ we have

$$\mu(E')\,(\leq)\,\mu(E) - \frac{e|H|}{r'}.$$

Note that any e-(semi)stable sheaf is slope (semi)stable.

LEMMA 5.2.2. *Let E be a rank r torsion free sheaf on X. Assume that there exists a non-zero $s \in \operatorname{Hom}(E, E \otimes \mathscr{A})$ with trivial determinant.*

1. *If $c_1\mathscr{A} \cdot H < 0$ then E is not slope semistable.*
2. *E is not e-stable for all*

$$e \geq \frac{(r-1)[c_1\mathscr{A} \cdot H]_+}{2|H|}$$

Proof. Since $\det s = 0$, the subsheaves $\ker s \subset E$ and $\operatorname{im} s \subset E \otimes \mathscr{A}$ are non-trivial of ranks $k < r$ and $r - k < r$, respectively. Hence, if E is e-stable then

$$\mu(\ker s) < \mu(E) - \frac{e|H|}{k}$$

and

$$\mu(\operatorname{im} s) < \mu(E) + c_1\mathscr{A} \cdot H - \frac{e|H|}{r-k}.$$

Then

$$r\mu(E) = k\mu(\ker s) + (r-k)\mu(\operatorname{im} s) < r\mu(E) + (r-k)c_1\mathscr{A} \cdot H - 2e|H|$$

and hence

$$e < \frac{(r-k)c_1\mathscr{A} \cdot H}{2|H|} \leq \frac{(r-1)[c_1\mathscr{A} \cdot H]_+}{2|H|},$$

which proves the second part of the lemma. The first one can be proven in the same way. $\square$

COROLLARY 5.2.3. *Assume that $h^0(E, E \otimes \mathscr{A}) > h^0(\mathscr{A}^r)$.*

1. *If $c_1\mathscr{A} \cdot H < 0$ then E is not slope semistable.*
2. *If $c_1\mathscr{A} \cdot H \geq 0$ then E is not e-stable for all*

$$e \geq \frac{(r-1)c_1\mathscr{A} \cdot H}{2|H|}$$

Proof. Consider the (non-linear) map $\varphi : \operatorname{Hom}(E, E \otimes \mathscr{A}) \to H^0(\mathscr{A}^r)$ given by the determinant. Since $\varphi^{-1}(0) \neq \emptyset$, the fiber $\varphi^{-1}(0)$ is at least 1-dimensional. Hence there exists a non-zero $s \in \operatorname{Hom}(E, E \otimes \mathscr{A})$ with trivial determinant and we can apply the previous lemma. $\square$

The above corollary generalizes a well known fact that a slope stable sheaf is simple (we get this for $\mathscr{A} = \mathscr{O}_X$).

If E is a torsion free slope H-semistable sheaf which is not e-stable for some $e \geq 0$ then we have a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_k = E$$

such that all quotients $F_i = E_i/E_{i-1}$ are slope semistable,

$$\mu(E_1) > \mu(E) - \frac{e|H|}{\operatorname{rk} E_1}$$

and

$$\mu(F_2) \geq \ldots \geq \mu(F_k).$$

Such a filtration can be constructed first by taking e-destabilizing subsheaf of E with torsion free quotient E/E_1 and then lifting the Harder–Narasimhan filtration of this quotient to E.

Let X be a smooth projective surface. In the following we always fix the determinant $\mathscr{L}$. Our aim is to bound the dimension of the locus $R(e)$ of e-unstable sheaves in $R_{\mathscr{L}}$ and the corresponding locus in $M(r, \mathscr{L}, c_2)$.

Note that for a point $[\rho : \mathscr{H} \to E] \in R(e)$ we have a filtration

$$0 \subset \mathscr{H}_0 \subset \mathscr{H}_1 \subset \ldots \subset \mathscr{H}_k = \mathscr{H}$$

such that $\mathscr{H}_0 = \ker \rho$ and $\mathscr{H}_i/\mathscr{H}_{i-1} = F_i$ for $i \geq 0$.

Set $P_i = P(F_i)$. Then $[\rho : \mathscr{H} \to E]$ lies in the image of the map

$$Y = \operatorname{Flag}(\mathscr{H}; P_0, P_1, \ldots, P_k) \to \operatorname{Quot}(\mathscr{H}; P)$$

obtained by forgetting all of the flag except for $\mathscr{H}_0 \to \mathscr{H}$. There exists only a finite number of such Flag-schemes which are non-empty. So in order to bound the dimension of $R(e)$ it is sufficient to bound the dimension of Y.

To do this one can use a version of Theorem 5.1.1 for Flag-schemes. The only thing we need to know about this is that the dimension of $\operatorname{Flag}(\mathscr{H}; P_0, P_1, \ldots, P_k)$ at $[\mathscr{H}_\bullet]$ is less or equal to the dimension of a certain Ext-group:

$$\dim \operatorname{Ext}^0_+(\mathscr{H}_\bullet, \mathscr{H}_\bullet) \leq h^0(\mathscr{H}om(\mathscr{H}, \mathscr{H})) - 1 + \sum_{i \leq j} \dim \operatorname{Ext}^1(F_j, F_i).$$

Now the dimension of $\operatorname{Ext}^1(F_j, F_i)$ can be computed using the Euler characteristic $\chi(F_j, F_i)$ (which can be computed by the Riemann–Roch formula; see Exercise 4.2.9) and the dimensions of $\operatorname{Hom}(F_j, F_i)$ and $\operatorname{Hom}(F_i, F_j \otimes \omega_X) \simeq (\operatorname{Ext}^2(F_j, F_i))^*$ (see Theorem 5.3.2). These dimensions can be bounded using Theorem 3.1.3. After some computation O'Grady used this idea to prove the following theorem.

THEOREM 5.2.4 (O'GRADY).

$$\dim R(e) \leq \left(1 - \frac{1}{2r}\right)\Delta + c_1 e^2 + c_2 e + c_3 + (\dim V)^2,$$

where c_1, c_2 and c_3 are some explicit functions depending only on r, X and H.

Let $M(e)$ denote the locus of e-unstable sheaves in the moduli space $M(r, \mathscr{L}, c_2)$.

COROLLARY 5.2.5.

$$\dim M(e) \leq \left(1 - \frac{1}{2r}\right)\Delta + c_1 e^2 + c_2 e + c_3 + r^2.$$

Proof. By the construction $\pi : R(e) \to M(e) = R(e)/\,\mathrm{GL}(V)$, so that

$$\dim M(e) \leq \dim R(e) - \dim \pi^{-1}([E])$$

for some $[E] \in M(e)$. But the stabilizer of $\mathrm{GL}(V)$ action at $[\mathscr{H} \to E]$ is equal to $\mathrm{Aut}\,E$, so

$$\dim \pi^{-1}([E]) = \dim \mathrm{GL}(V) - \dim \mathrm{Aut}(E) \geq (h^0(\mathscr{H}))^2 - r^2$$

and the asserted inequality follows from the above theorem. □

5.3 Examples of unobstructed moduli spaces.

THEOREM 5.3.1 (MARUYAMA). *Let X be a smooth projective surface and let H be an ample divisor such that $HK_X < 0$. Then for any slope H-semistable torsion free sheaf E we have* $\mathrm{Ext}^2(E,E) = 0$. *In particular, any moduli space M_P^s is smooth.*

Proof. We will use the following useful version of the Serre duality theorem:

THEOREM 5.3.2. *Let X be a smooth n-dimensional projective variety. Then for any two coherent sheaves F and G we have*

$$\mathrm{Ext}^p(F,G) \simeq (\mathrm{Ext}^{n-p}(G, F \otimes \omega_X))^*.$$

The above theorem implies that $\mathrm{Ext}^2(E,E) \simeq (\mathrm{Hom}(E, E \otimes \omega_X))^*$. But $h^0(\omega_X^{\otimes r}) = 0$ since $K_X H < 0$. Hence if $\mathrm{Ext}^2(E,E) \neq 0$ then $h^0(E, E \otimes \omega_X) > h^0(\omega_X^{\otimes r})$ and E is not slope semistable by Lemma 5.2.3, a contradiction. □

In fact, we can also deal with another case:

PROPOSITION 5.3.3. *Let X be a smooth projective surface with $\omega_X \simeq \mathscr{O}_X$. Then the moduli space of Gieseker stable sheaves $M^s(\mathscr{L})$ with fixed determinant $\mathscr{L}$ is smooth. In particular, if the characteristic of the base field does not divide r then the moduli space $M^s(\mathscr{L})$ has dimension* $\Delta(E) - (r^2 - 1)\chi(\mathscr{O}_X)$. *Moreover, if the Picard scheme of X is reduced then M_P^s is smooth.*

Proof. To prove smoothness it is sufficient to prove that for any Gieseker H-stable sheaf E the kernel of the trace map $\mathrm{Ext}^2(E,E) \to H^2(\mathscr{O}_X)$ is zero. Since the trace map is dual to the map $H^0(\omega_X) \to \mathrm{Hom}(E, E \otimes \omega_X)$ induced by the diagonal embedding, it suffices to show that the cokernel of this map is trivial. But this is obvious as Gieseker stable sheaves are simple.

The second part follows from the easy dimension count:

$$\dim \mathrm{Ext}^1(E,E)_0 = -\chi(E,E) + \chi(\mathscr{O}_X) = \Delta(E) - (r^2 - 1)\chi(\mathscr{O}_X).$$

□

Note that in positive characteristic there exist smooth projective surfaces with $\omega_X \simeq \mathscr{O}_X$, whose Picard scheme is non-reduced.

If X is a smooth projective surface with K_X numerically trivial, e.g., if X is an Enriques surface (i.e., such that $h^0(\omega_X) = 0$, but $\omega^2 \simeq \mathscr{O}_X$), then the moduli space $M^s(\mathscr{L})$ need not be smooth. Here we show when it happens refering to [Kim] for a more precise description of the moduli space in this case.

PROPOSITION 5.3.4 ([KIM]). *Let X be an Enriques surface. Then the moduli space of slope stable rank r torsion free sheaves $M^{\mu}(\mathscr{L})$ with fixed determinant $\mathscr{L}$ is smooth if r is odd and it is singular precisely at points corresponding to E such that $E \simeq E \otimes \omega_X$ if r is even.*

Proof. Let E be a rank r slope stable sheaf on X. We have $\mathrm{Ext}^2(E,E)_0 = \mathrm{Ext}^2(E,E) \simeq (\mathrm{Hom}(E, E\otimes \omega_X))^*$. If there exists a non-zero $s \in \mathrm{Hom}(E, E\otimes\omega_X)$ then $\det s \neq 0$ by Lemma 5.2.2. Since K_X is numerically trivial, this implies that s gives rise to an isomorphism $E \simeq E\otimes\omega_X$ (E and $E\otimes\omega_X$ have equal Hilbert polynomials). In particular, comparing determinants we get $\omega_X^r \simeq \mathscr{O}_X$, so the rank r must be even. It is easy to see that at such points the moduli space is singular as the dimension of the tangent space is larger than the actual dimension. $\square$

There are many examples where the moduli space $M^s(\mathscr{L})$ is singular for an Enriques surface. We should note that in [Kim] the author's statement is different to ours and he distinguishes some cases where $E \simeq E\otimes\omega_X$ but $[E]$ is a smooth point of the moduli space. This happens because he forgets about the scheme structure and looks only at the underlying reduced scheme which is not a moduli space in the functorial sense of Definition 2.1.1.

EXERCISE 5.3.5. Assume that r and $c_1\mathscr{L}\cdot H$ are relatively prime. Show that every rank r slope H-semistable sheaf with determinant $\mathscr{L}$ is slope H-stable. In particular, in this case the moduli spaces $M^{\mu}(\mathscr{L})$ and $M^s(\mathscr{L})$ are projective.

6 Lecture 6

• Spectral covers
• An algebraic interlude on local deformations
• Generic smoothness of moduli spaces of sheaves on surfaces

6.1 Spectral covers

Let N be a line bundle on a smooth projective variety X. Let $\pi : \mathbb{V}(N^*) \to X$ be the corresponding geometric bundle, i.e., $\mathbb{V}(N^*) = \mathrm{Spec} \bigoplus_{r\geq 0}(N^*)^r$.

Let us recall that the *F-theoretic support* of a sheaf supported on a divisor is the scheme corresponding to its 0th Fitting ideal. This should not be mistaken for a *scheme theoretic support* which is defined by the annihilator ideal sheaf.

A sheaf F on a regular scheme Z is called *Cohen–Macaulay* if at every point z in the scheme-theoretic support Y of F, the depth of F at z is equal to the codimension of z in Y. We will use the fact that if $\pi : Y \to X$ is a finite morphism onto a regular scheme X then F is Cohen–Macaulay if and only if $\pi_* F$ is locally free. A Cohen–Macaulay sheaf supported on a divisor in Z can be thought of as a pure sheaf of rank 1 on its F-theoretic support.

PROPOSITION 6.1.1. *There exists a canonical bijection between*

1. *Isomorphism classes of Cohen–Macaulay (or pure) sheaves L on $\mathbb{V}(N^*)$, whose F-theoretic support $Y \subset \mathbb{V}(N^*)$ is a proper degree r cover of X,*

2. *Isomorphism classes of pairs (E,s), where E is a rank r locally free (torsion free, respectively) sheaf on X and $s : E \to E \otimes N$ is a homomorphism.*

The cover $Y \to X$ corresponding to (E,s) is called a spectral cover.

Proof. If we are given L as in 1, then we set $E = \pi_* L$ and we recover s from the $\pi_* \mathscr{O}_{\mathbb{V}(N^*)}$-module structure on $\pi_* L$:

$$\pi_* \mathscr{O}_{\mathbb{V}(N^*)} \otimes \pi_* L \to \pi_* L.$$

To recover L from (E,s) we need to define a section $x \in H^0(\mathbb{V}(N^*), \pi^* N) = H^0(X, \pi_* \mathscr{O}_{\mathbb{V}(N^*)} \otimes N)$, as the section corresponding to the constant section 1 of $\mathscr{O}_X$ in $\pi_* \mathscr{O}_{\mathbb{V}(N^*)} \otimes N = N \oplus \mathscr{O}_X \oplus N^{-1} \oplus \ldots$.

Then we define L by the following exact sequence

$$\pi^* E \xrightarrow{x \mathrm{Id}_{\pi^* E} - \pi^* s} \pi^* E \otimes \pi^* N \to L \otimes \pi^* N \to 0,$$

i.e., $L = coker(x \mathrm{Id}_{\pi^* E} - \pi^* s) \otimes \pi^*(N^*)$. By definition the F-theoretic support Y is given by vanishing of the determinant $\det(x \mathrm{Id}_{\pi^* E} - \pi^* s)$. $\square$

The above proposition is a generalization of [BNR, Proposition 3.6] to non-integral spectral covers in higher dimension.

Let (E,s) be a pair consisting of a rank r torsion free sheaf E and a homomorphism $s : E \to E \otimes N$. Let $a_i \in H^0(X, N^i)$, $i = 0, 1, \ldots, r$ be elementary symmetric function in eigenvalues of s (they are easily defined on the open subset where E is locally free and then uniquely extended to the whole X). Then by the Cayley–Hamilton theorem we have

$$\sum_{i=0}^{r} a_i s^{r-i} = 0$$

as a homomorphism $E \to E \otimes N^r$, whereas the spectral cover is given by

$$\sum_{i=0}^{r} a_i x^i s^{r-i} = 0$$

6.2 An algebraic interlude on local deformations

Let $S = k[[x_1, \ldots, x_l]]$. Let us choose $0 \neq f \in S$ and set $R = S/(f)$. Let M be a Cohen–Macaulay R-module. As an S-module it has for some integer n a resolution

$$0 \to S^n \xrightarrow{\varphi} S^n \to M \to 0.$$

Then there exists $\psi : S^n \to S^n$ such that $\varphi\psi = \psi\varphi = f \cdot \mathrm{Id}_{S^n}$. This is called a *matrix factorization* of f. After reducing modulo ideal (f) we get an infinite periodic free resolution of M:

$$\ldots \to R^n \xrightarrow{\overline{\psi}} R^n \xrightarrow{\overline{\varphi}} R^n \xrightarrow{\overline{\psi}} R^n \xrightarrow{\overline{\varphi}} R^n \to M \to 0.$$

Let N denote the image of $\overline{\varphi}$. Then we have two short exact sequences:

$$0 \to N \to R^n \to M \to 0$$

and

$$0 \to M \to R^n \to N \to 0.$$

From the first sequence we get an exact sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(M,R^n) \to \mathrm{Hom}_R(M,M) = R,$$

so $\mathrm{Hom}_R(M,N)$ is a reflexive R-module.

Using the second sequence we see that there is an exact sequence

$$\mathrm{Hom}_R(M,N) \to \mathrm{Ext}^1_R(M,M) \to \mathrm{Ext}^1_R(M,R^n) = 0.$$

LEMMA 6.2.1. *Let $J = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_l})$. Then $J\,\mathrm{Ext}^1_R(M,M) = 0$.*

Proof. Then

$$\frac{\partial \varphi}{\partial x_i}\psi + \varphi\frac{\partial \psi}{\partial x_i} = \frac{\partial f}{\partial x_i}\mathrm{Id}_{S^n},$$

so multiplication of the resolution of M by $\frac{\partial f}{\partial x_i}$ is homotopical to the zero map. This easily implies the required assertion. □

Using the above lemma we see that there exists a surjection

$$\mathrm{Hom}_R(M,N)/J\mathrm{Hom}_R(M,N) \to \mathrm{Ext}^1_R(M,M) \to 0.$$

Assume that R is an isolated singularity. Then $\mathrm{Ext}^1_R(M,M)$ is a module of finite length bounded from the above by

$$l(\mathrm{Hom}_R(M,N)/J\mathrm{Hom}_R(M,N)),$$

where $l(S)$ denotes the length of S. Let us also assume that the number of variables $l = 3$ and $f = x_3^2 - g(x_1, x_2)$. Then $J' = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})$ is an ideal generated by a system of parameters of

R. Since $\mathrm{Hom}_R(M,N)$ is reflexive over a normal surface singularity, it is Cohen–Macaulay. Therefore we can apply [BH, Corollary 4.6.11] and we get

$$l(\mathrm{Hom}_R(M,N)/J\mathrm{Hom}_R(M,N)) \leq l(\mathrm{Hom}_R(M,N)/J'\mathrm{Hom}_R(M,N)) = \mathrm{rk}M \cdot \mathrm{rk}N \cdot l(R/J').$$

But $R/J' \simeq (k[[x_1,x_2]]/(\frac{\partial g}{\partial x_1},\frac{\partial g}{\partial x_2}))[x_3]/(x_3^2-g)$, so

$$l(\mathrm{Ext}^1_R(M,M)) \leq 2\,\mathrm{rk}M \cdot (n-\mathrm{rk}M)\cdot \mu_0,$$

where μ_0 is the Milnor number of the plane singularity $g(x_1,x_2)=0$ (at 0).

The above arguments prove in particular the following proposition:

PROPOSITION 6.2.2. *Let $(E, s: E \to E\otimes N)$ be as in Proposition 6.1.1. Assume that X is a surface, Y is normal and* $\mathrm{rk}\,E = 2$. *Then L from Proposition 6.1.1 is a rank* 1 *reflexive sheaf on Y and at a point $y \in Y$*

$$h^0(Y, \mathcal{E}xt^1(L,L)_y) \leq 2\mu_x,$$

where μ_x is the Milnor number of the branching curve $(\det s = 0)$ *at $x = \pi(y)$.*

EXERCISE 6.2.3. Assume that $\mathit{char}\,k \neq 2$. Prove that if $C := (xy=0) \subset \mathbb{A}^2_k$ is a nodal curve then on the double cover of $\mathbb{A}^2$ branched along C there exist precisely one rank 1 reflexive sheaf L which is not locally free (cf. [Ha, II, Example 6.5.2]). Prove that $\mathrm{Ext}^1(L,L)=k$.

6.3 Generic smoothness of moduli spaces of sheaves on surfaces

Let $M = M^{\mu}_{\mathrm{lf}}(2,L,c_2)$ be the moduli space of slope stable locally free rank 2 sheaves with determinant L and second Chern class c_2.

The following theorem was proven (over $\mathbb{C}$) by S. Donaldson in the rank 2 case and then generalized to higher rank (and strengthened) by D. Gieseker, J. Li and K. O'Grady (all in $\mathit{char}\,k=0$). The author generalized this to higher characteristic (also strenghening a part of the theorem in $\mathit{char}\,k=0$). We will try to imitate the original Donaldson's proof. Although it gives weaker results, its idea is much simpler and more straightforward. Assumption on the characteristic is added mostly for our convenience (otherewise some covers become non-separable and the dimension of the moduli space is not so easy to compute).

THEOREM 6.3.1. *Assume $\mathit{char}\,k \neq 2$. If $c_2 \gg 0$ then M is generically smooth, i.e., smooth at a generic point of each irreducible component. Moreover, the dimension of each irreducible component is equal to $\Delta - 3\chi(\mathcal{O}_X)$ (and for even larger c_2 there is only one irreducible component).*

Proof. If E is a Gieseker stable locally free sheaf then by Theorem 5.1.3 we see that

$$T_{[E]}M \simeq \mathrm{Ext}^1(E,E)_0 \simeq H^1(\mathrm{End}_0 E)$$

and obstructions for M to be smooth lie in

$$\mathrm{Ext}^2(E,E)_0 \simeq H^2(\mathrm{End}_0 E) \simeq (H^0(\mathrm{End}_0 E \otimes \omega_X))^*.$$

Here we use the fact that the trace map splits as the characteristic is not equal to 2. So by the Riemann–Roch theorem M has at $[E]$ dimension at least

$$h^1(\mathrm{End}_0 E) - h^2(\mathrm{End}_0 E) = \Delta - 3\chi(\mathscr{O}_X).$$

To prove the theorem (except for the last part in the brackets which will be skipped), it is sufficient to show that for $c_2 \gg 0$ the closed subset $S = \{[E] \in M : H^0(\mathrm{End}_0 E \otimes \omega_X) \neq 0\}$ of M has dimension smaller than $\Delta - 3\chi(\mathscr{O}_X)$. This follows from Theorem 6.3.2. $\square$

THEOREM 6.3.2. *Let $\mathscr{A}$ be a line bundle on X and let*

$$S_{\mathscr{A}} = \{[E] \in M : H^0(\mathrm{End}_0 E \otimes \mathscr{A}) \neq 0\}.$$

If $c_1\mathscr{A} \cdot H < 0$ then $S_{\mathscr{A}} = \emptyset$. If $c_1\mathscr{A} \cdot H \geq 0$ and $c_2 \gg 0$ then

$$\dim S_{\mathscr{A}} < \Delta - 3\chi(\mathscr{O}_X).$$

Proof. Since the first part is clear (see Lemma 5.2.2) we can assume that $c_1\mathscr{A} \cdot H \geq 0$.

Let $|\mathscr{A}| \to |\mathscr{A}^2|$ be a map sending divisor D to $2D$. We denote the image by $2|\mathscr{A}|$. We will prove the theorem under a simplyfying assumption that all divisors in $|\mathscr{A}^2| - 2|\mathscr{A}|$ are reduced (this is satisfied, e.g., if $\mathrm{Pic}\, X$ is generated by $\mathscr{A}$).

We decompose $S_{\mathscr{A}}$ into two locally closed subsets:

$$S'_{\mathscr{A}} = \{[E] \in M : \text{ there exists a non-zero } s \in H^0(\mathrm{End}_0 E \otimes \mathscr{A}) \text{ such that } \det s = 0 \text{ or } \det s \in 2|\mathscr{A}|\}$$

and

$$S''_{\mathscr{A}} = \{[E] \in M : \text{ for every non-zero } s \in H^0(\mathrm{End}_0 E \otimes \mathscr{A}) \text{ we have } \det s \in |\mathscr{A}^2| - 2|\mathscr{A}|\}.$$

First let us note that $S'_{\mathscr{A}} \subset M(\frac{c_1\mathscr{A} \cdot H}{2|H|})$. This follows by Lemma 5.2.2 since if $E \in S'_{\mathscr{A}}$ then there exists $s \in H^0(\mathrm{End}_0 E \otimes \mathscr{A})$ such that either $\det s = 0$ or there exists $t \in H^0(\mathscr{A})$ such that $\det s + t^2 = 0$ and then for $s' = s - t \cdot \mathrm{Id}_E$ (which is non-zero as $\mathrm{Tr}\, s = 0$) we have $\det s' = 0$. Hence $\dim S'_{\mathscr{A}}$ can be bounded using Corollary 5.2.5.

To bound the dimension of $S''_{\mathscr{A}}$ we will use the moduli space M' of pairs (E, s) such that $E \in S''_{\mathscr{A}}$ and $0 \neq s \in H^0(\mathrm{End}_0 E \otimes \mathscr{A})$. This moduli space exists and it can be constructed as a locally closed subscheme in the moduli space of Gieseker stable pure sheaves on a compactification $\mathbb{P}(\mathscr{O} \oplus \mathscr{A}^*)$ of $\mathbb{V}(\mathscr{A}^*)$ (with respect to polarization $\pi^* H + aZ$, where Z is the complement on $\mathbb{V}(\mathscr{A}^*)$ and a is a small positive rational number). This is a special case of the moduli space of (generalized) Higgs bundles on X.

Note that we have a surjection $M' \to S''_{\mathscr{A}}$ so it suffices to bound the dimension of M'. There exists a partial Hitchin's map $M' \to |\mathscr{A}^2|$ sending (E, s) to the determinant of s. By Proposition 6.1.1 the fiber over $B \in |\mathscr{A}^2|$ is a subset of the moduli space M''_B of reflexive rank 1 sheaves on the double cover Y of X branched along B. The cover $p : Y \to X$ is constructed as $\mathrm{Spec}(\mathscr{O}_X \oplus \mathscr{A}^*)$, where the $\mathscr{O}_X$-algebra structure on $\mathscr{O}_X \oplus \mathscr{A}^*$ is given by $\mathscr{A}^* \otimes \mathscr{A}^* \to \mathscr{O}_X$ coming from multiplication by the chosen equation of B. By assumption B is reduced, so the surface Y is normal (it is S_2 as a hypersurface in a smooth 3-fold and it has only isolated singular points lying over singular points of B).

The dimension of M''_B at a rank 1 reflexive sheaf L is bounded from the above by $\dim \mathrm{Ext}^1(L,L)$. Using the local to global Ext spectral sequence we get

$$\dim \mathrm{Ext}^1(L,L) \le h^0(\mathscr{E}xt^1_Y(L,L)) + h^1(\mathscr{H}om_Y(L,L)).$$

Note that $\mathscr{H}om_Y(L,L) \simeq \mathscr{O}_Y$, so $h^1(\mathscr{H}om_Y(L,L)) = h^1(p_*\mathscr{O}_Y) = h^1(\mathscr{O}_X) + h^1(\mathscr{A}^*)$ and we need to bound only local contributions. In our case this can be done using Proposition 6.2.2, which implies that

$$h^0(\mathscr{E}xt^1_Y(L,L)) \le 2 \sum_{x \in \mathrm{Sing}\, B} \mu_x = 2(K_X+B)B + 2e_{\mathrm{top}}(B),$$

where $\mathrm{Sing}\, B$ denotes the set of singular points of B and $e_{\mathrm{top}}(B)$ is the "topological" Euler characteristic of B (it also makes sense in positive characteristic and it is bounded from the above by twice the number of irreducible components of B). Summing up, we get

$$\dim S''_{\mathscr{A}} < h^0(\mathscr{A}^2) + h^1(\mathscr{O}_X) + h^1(\mathscr{A}^*) + 2(K_X+B)B + 2e_{\mathrm{top}}(B),$$

where the right hand side does not depend on c_2. $\square$

One should note that K. Zuo also used Donaldson's idea to give a proof of Donaldson's Theorem 6.3.2 (in the rank 2 case) using some results of R. Friedman. However, his proof is incomplete as he "forgets" to consider the case when there exists $s \in H^0(\mathrm{End}_0 E \otimes \mathscr{A})$ such that $\det s \notin 2|\mathscr{A}|$ but the corresponding spectral cover is reducible. This happens if the curve $\det s = 0$ decomposes $D_+ + D_-$ for some divisors $D_\pm \in |\mathscr{A} \pm \tau|$, where τ is a 2-torsion. Nowadays, Theorem 6.3.2 is usually proven using O'Grady's approach which is based on a completely different idea than spectral covers.

Acknowledgements.

I would like to thank Professor P. Pragacz without whose persistence these notes would have not been written. My thanks also go to H. Tutaj-Gasińska who wrote one version of these notes (and which unfortunately were not used by the author...). A part of these notes (Section 2 of Lecture 5 and Lecture 6) was written while the author's stay in the Max Planck Institute für Mathematik.

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