A Gysin formula for Hall-Littlewood polynomials

Piotr Pragacz
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland
P.Pragacz@impan.pl

To Bill Fulton on his 75th birthday

Abstract

We give a formula for pushing forward the classes of Hall-Littlewood polynomials in Grassmann bundles, generalizing Gysin formulas for Schur $S$- and $P$-functions.

Let $E \rightarrow X$ be a vector bundle of rank $n$ over a nonsingular variety $X$ over an algebraically closed field. Denote by $\pi : G^q(E) \rightarrow X$ the Grassmann bundle parametrizing rank $q$ quotients of $E$. Let $\pi_* : A(G^q(E)) \rightarrow A(X)$ be the homomorphism of the Chow groups of algebraic cycles modulo rational equivalence, induced by pushing-forward cycles (see [3, Chap. 1]). There exists an analogous map of cohomology groups. A goal of this note is to give a formula (see Theorem 7) for the image via $\pi_*$ of Hall-Littlewood classes from the Grassmann bundle.

Hall-Littlewood polynomials appeared implicitly in Hall’s study [5] of the combinatorial lattice structure of finite abelian $p$-groups, and explicitly in the work of Littlewood on some problems of representation theory [8]. A detailed account of the theory of Hall-Littlewood functions is given in [9].

The formula in Theorem 7 generalizes some Gysin formulas for Schur $S$- and $P$-functions. In particular, it generalizes the formula in [11, Prop. 1.3(ii)], and provides an explanation of its intriguing coefficient. We refer to [4] for general information about the appearance of Schur $S$- and $Q$-functions in cohomological studies of algebraic varieties.

Let $t$ be an indeterminate. The main formula will be located in $A(X)[t]$, or in the extension $H^*(X, \mathbb{Z})[t]$ of the cohomology ring for a complex variety $X$. Let $\tau_E : Fl(E) \rightarrow X$ be the flag bundle parametrizing flags of quotients of $E$ of ranks $n, n-1, \ldots, 1$. Suppose that $x_1, \ldots, x_n$ is a sequence of the Chern roots of $E$. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we define

$$R_\lambda(E; t) = (\tau_E)_* \left( x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i < j} (x_i - tx_j) \right), \quad (1)$$

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where \((\tau_E)_*\) acts on each coefficient of the polynomial in \(t\) separately. (The same convention will be used for other flag bundles.)

The Grassmann bundle \(\pi : G^q(E) \to X\) is endowed with the tautological exact sequence of vector bundles

\[
0 \to S \to \pi^*E \to Q \to 0,
\]

where \(\text{rank}(Q) = q\). Let \(r = n - q\) be the rank of \(S\). Suppose that \(x_1, \ldots, x_q\) are the Chern roots of \(Q\) and \(x_{q+1}, \ldots, x_n\) are the ones of \(S\).

**Proposition 1.** For sequences \(\lambda = (\lambda_1, \ldots, \lambda_q)\) and \(\mu = (\mu_1, \ldots, \mu_r)\) of non-negative integers, we have

\[
\pi_*(R_\lambda(Q; t)R_\mu(S; t) \prod_{i \leq q < j} (x_i - tx_j)) = R_{\lambda \mu}(E; t),
\]

where \(\lambda \mu = (\lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_r)\) is the juxtaposition of \(\lambda\) and \(\mu\).

**Proof.** Consider a commutative diagram

\[
\begin{array}{ccc}
\text{Fl}(Q) & \times_{G^q(E)} & \text{Fl}(S) \\
\tau_Q \times \tau_S \downarrow & & \downarrow \tau = \tau_E \\
G^q(E) & \pi & X
\end{array}
\]

It follows that

\[
\pi_*(\tau_Q \times \tau_S)_* = \tau_*.
\]

Using Eq.(1) for \(Q\) and \(S\) and Eq.(2), we obtain

\[
\pi_*(R_\lambda(Q; t)R_\mu(S; t) \prod_{i \leq q < j} (x_i - tx_j)) = \pi_*(\tau_Q)_* (x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \leq q} (x_i - tx_j)) = \pi_*(\tau_Q \times \tau_S)_* (x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \leq q} (x_i - tx_j)) = \tau_*(x_1^{\lambda_1} \cdots x_q^{\lambda_q} x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{i < j} (x_i - tx_j)) = R_{\lambda \mu}(E; t).
\]

In the argument above, we have used the following equality:

\[
\prod_{i < j \leq q} (x_i - tx_j) \prod_{q < i < j} (x_i - tx_j) \prod_{i \leq q < j} (x_i - tx_j) = \prod_{i < j} (x_i - tx_j).
\]

We now set

\[
v_m(t) = \prod_{i=1}^m \frac{1 - t^i}{1 - t} = (1 + t)(1 + t + t^2) \cdots (1 + t + \cdots + t^{m-1}).
\]
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Consider the maximal subsets $I_1, \ldots, I_d$ in $\{1, \ldots, n\}$, where the sequence $\lambda$ is constant. Let $m_1, \ldots, m_d$ be the cardinalities of $I_1, \ldots, I_d$. So we have $m_1 + \cdots + m_d = n$. We set

$$v_\lambda(t) = \prod_{i=1}^{d} v_{m_i}(t).$$

(4)

Let $S_n$ be the symmetric group of permutations of $\{1, \ldots, n\}$. We define a subgroup $S_n^\lambda$ of $S_n$ as the stabilizer of $\lambda$. Of course,

$$S_n^\lambda = \prod_{i=1}^{d} S_{m_i}.$$  

Finally, we associate to a sequence $\lambda$ a $(d - 1)$-step flag bundle (with steps of lengths $m_i$)

$$\eta_\lambda : Fl_\lambda(E) \to X,$$

parametrizing flags of quotients of $E$ of ranks

$$n - m_d, n - m_d - m_{d-1}, \ldots, n - m_d - m_{d-1} - \cdots - m_2.$$  

(5)

**Example 2.** Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition (see [9, I,1,Ex.9]) with $k \leq n$. Let $\lambda = \nu \cdot n^{k - k}$ be the sequence $\nu$ with $n-k$ zeros added at the end. Then $d = k + 1$, $(m_1, \ldots, m_d) = (1^k, n - k)$, $v_\lambda(t) = v_{n-k}(t)$, $S_n^\lambda = (S_1)^k \times S_{n-k}$, and $\eta_\lambda : Fl_\lambda(E) \to X$ is the flag bundle, often denoted by $E^k$, parametrizing quotients of $E$ of ranks $k, k - 1, \ldots, 1$.

If $\lambda = (a^n b_{n-p})$, then $d = 2$, $(m_1, m_2) = (p, n - p)$, $v_\lambda(t) = v_p(t) v_{n-p}(t)$, $S_n^\lambda = S_p \times S_{n-p}$, and $\eta_\lambda$ is here the Grassmann bundle $\pi : G^p(E) \to X$.

We shall now need some results from [9, III]. Let $y_1, \ldots, y_n$ and $t$ be independent indeterminates. We record the following equation from [9, III, (1.4)]:

**Lemma 3.** We have

$$\sum_{w \in S_n} w \left( \prod_{i<j} \frac{y_i - ty_j}{y_i - y_j} \right) = v_n(t).$$

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we define

$$R_\lambda(y_1, \ldots, y_n; t) = \sum_{w \in S_n} w \left( y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j} \frac{y_i - ty_j}{y_i - y_j} \right)$$

Arguing as in [9, III (1.5)], we show with the help of Lemma 3 the following result.

**Proposition 4.** The polynomial $v_\lambda(t)$ divides $R_\lambda(y_1, \ldots, y_n; t)$, and we have

$$R_\lambda(y_1, \ldots, y_n; t) = v_\lambda(t) \sum_{w \in S_n / S_n^\lambda} w \left( y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j, \lambda_i \neq \lambda_j} \frac{y_i - ty_j}{y_i - y_j} \right).$$
Let us invoke the following description of the Gysin map for the flag bundle $F l_\lambda(E) \to X$ with the help of a symmetrizing operator. Recall that $A(F l_\lambda(E))$ as an $A(X)$-module is generated by $S_n^\lambda$-invariant polynomials in the Chern roots of $E$ (see [1, Thm 5.5]). We define for an $S_n^\lambda$-invariant polynomial $f = f(y_1, \ldots, y_n)$,

$$
\partial_\lambda(f) = \sum_{w \in S_n/S_n^\lambda} w(\frac{f(y_1, \ldots, y_n)}{\prod_{i<j, \lambda_i \neq \lambda_j} (y_i - y_j)}).
$$

The following result is a particular case of [2, Prop. 2.1] (in the situation of Corollary 6, the result was shown already in [10, Sect. 2]).

**Proposition 5.** With the above notation, we have

$$
(\eta_\lambda)_*(f(x_1, \ldots, x_n)) = ((\partial_\lambda f)(y_1, \ldots, y_n))(x_1, \ldots, x_n).
$$

It follows from Propositions 4 and 5 that

$$
R_\lambda(E; t) = v_\lambda(t)(\eta_\lambda)_*(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j, \lambda_i \neq \lambda_j} (x_i - tx_j)),
$$

where $x_1, \ldots, x_n$ are the Chern roots of $E$.

Let $\lambda$ be a sequence of nonnegative integers. Extending [9, III, 2], we set

$$
P_\lambda(E; t) = \frac{1}{v_\lambda(t)} R_\lambda(E; t). \tag{6}
$$

It follows from Proposition 4 that $P_\lambda(E; t)$ is a polynomial in the Chern classes of $E$ and $t$.

Let us record the following particular case.

**Corollary 6.** Let $\nu$ be a strict partition with length $k \leq n$. Set $\lambda = \nu \circ \iota^{n-k}$. We have

$$
P_\lambda(E; t) = (\tau_E^\nu)_*(x_1^{\nu_1} \cdots x_n^{\nu_n} \prod_{i<j, i \leq k} (x_i - tx_j)).
$$

As a consequence of Propositions 1 and 4, using Eq.(6), we obtain the following result.

**Theorem 7.** Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be sequences of nonnegative integers. Then we have

$$
\pi_* \left( \prod_{i \leq q < j} (x_i - tx_j) P_{\lambda}(Q; t) P_{\mu}(S; t) \right) = \frac{v_\lambda(t)}{v_\lambda(t)v_\mu(t)} P_{\lambda\mu}(E; t).
$$

We first consider the specialization $t = 0$.

**Example 8.** We recall Schur $S$-functions. Let $s_i(E)$ denotes the $i$th complete symmetric function in the roots $x_1, \ldots, x_n$, given by

$$
\sum_{i \geq 0} s_i(E) = \prod_{j=1}^n \frac{1}{1 - x_j}.
$$
Given a partition λ = (λ₁ ≥ ... ≥ λₙ ≥ 0), we set

\[ s_\lambda(E) = \left| s_{\lambda_{i-j}}(E) \right|_{1 \leq i, j \leq n}. \]

(See also [9, I, 3].) Translating the Jacobi-Trudi formula (loc. cit.) to the Gysin map for \( \tau_E : Fl(E) \to X \) (see, e.g. [11, Sect. 4]), we have

\[ s_\lambda(E) = (\tau_E)_*(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}). \]

We see that \( P_\lambda(E; t) = s_\lambda(E) \) for \( t = 0 \). Under this specialization, the theorem becomes

\[ \pi_\ast((x_1 \cdots x_q)s_\lambda(Q)s_\mu(S)) = \pi_\ast(s_{\lambda+r, \ldots, \lambda+r}(Q)s_\mu(S)) = s_{\lambda+r}(E), \]

a result obtained originally in [7, Prop. p. 196] and [6, Prop. 1].

If a sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is not a partition, then \( s_\lambda(E) \) is either 0 or \( \pm s_\mu(E) \) for some partition \( \mu \). One can rearrange \( \lambda \) by a sequence of operations \( \ldots, i, j, \ldots \mapsto \ldots, j-1, i+1, \ldots \) applied to pairs of successive integers. Either one arrives at a sequence of the form \( \ldots, i, i+1, \ldots \), in which case \( s_\lambda(E) = 0 \), or one arrives in \( d \) steps at a partition \( \mu \), and then \( s_\lambda(E) = (-1)^d s_\mu(E) \).

**Corollary 9.** Let \( \nu \) and \( \sigma \) be strict partitions of lengths \( k \leq q \) and \( h \leq r \). It follows from Eq. (3) that

\[ \frac{v_{\nu_0^q-k, \sigma_0^r-k}(t)}{v_{\nu_0^q-k}(t)v_{\sigma_0^r-k}(t)} = \left[ \frac{n-k-h}{q-h} \right] (t) \cdot (1+t)^e, \]

the Gaussian polynomial times \((1+t)^e\) where \( e \) is the number of common parts of \( \nu \) and \( \sigma \). Thus the theorem applied to the sequences \( \lambda = \nu_0^q-k \) and \( \mu = \sigma_0^r-h \) yields the following equation:

\[ \pi_\ast\left( \prod_{i \leq q < j} (x_i - tx_j)P_\nu(Q; t)P_\sigma(S; t) \right) = \left[ \frac{n-k-h}{q-h} \right] (t) \cdot (1+t)^e \cdot P_{\lambda+r}(E; t). \quad (7) \]

We need the following property of Gaussian polynomials, which should be known but we know no precise reference.

**Lemma 10.** At \( t = -1 \), the Gaussian polynomial

\[ \begin{bmatrix} a+b \\ a \end{bmatrix} (t) \]

specializes to zero if \( ab \) is odd and to the binomial coefficient

\[ \left( \frac{(a+b)/2}{\lfloor a/2 \rfloor} \right) \]

otherwise.

**Proof.** We have

\[ \begin{bmatrix} a+b \\ a \end{bmatrix} (t) = \frac{(1-t)(1-t^2) \cdots (1-t^{a+b})}{(1-t) \cdots (1-t^a)(1-t) \cdots (1-t^b)}. \]
Since \( t = -1 \) is a zero with multiplicity 1 of the factor \((1 - t^d)\) for even \( d \), and a zero with multiplicity 0 for odd \( d \), the order of the rational function \( \frac{a+b}{a} \) at \( t = -1 \) is equal to
\[
\left\lfloor \frac{(a + b)/2}{a/2} \right\rfloor (t^2) .
\]
The order (8) is equal to 1 when \( a \) and \( b \) are odd, and 0 otherwise. In the former case, we get the claimed vanishing, and in the latter one, the product of the factors with even exponents is equal to
\[
\left\lfloor \frac{a+b}{a/2} \right\rfloor (1) \text{ which is the binomial coefficient}
\]
This is the requested value since the remaining factors with an odd exponent give 2 in the numerator and the same number in the denominator.

The assertions of the lemma follow. \( \Box \)

We now consider the specialization \( t = -1 \).

**Example 11.** Consider Schur \( P \)-functions \( P_\lambda(E) = P_\lambda \) (or \( P_\lambda(y_1, \ldots, y_n) = P_\lambda \)) defined as follows. For a strict partition \( \lambda = (\lambda_1 > \ldots > \lambda_k > 0) \) with odd \( k \),
\[
P_\lambda = P_{\lambda_1} P_{\lambda_2} \ldots P_{\lambda_k} - P_{\lambda_2} P_{\lambda_1, \lambda_3, \ldots} + \cdots + P_{\lambda_k} P_{\lambda_1, \ldots, \lambda_{k-1}} ,
\]
and with even \( k \),
\[
P_\lambda = P_{\lambda_1, \lambda_2} P_{\lambda_3, \ldots} - P_{\lambda_1, \lambda_2} P_{\lambda_3, \ldots} + \cdots + P_{\lambda_1, \lambda_2} P_{\lambda_3, \ldots} .
\]
Here, \( P_\lambda = \sum s_\mu \), the sum over all hook partitions \( \mu \) of \( \lambda \), and for positive \( i > j \) we set
\[
P_{i,j} = P_i P_j + 2 \sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j} .
\]
(See also [9, III, 8].) It was shown in [12, p. 225] that for a strict partition \( \lambda \) of length \( k \),
\[
P_\lambda(y_1, \ldots, y_n) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w \left( y_1^{\lambda_1} \cdots y_n^{\lambda_k} \prod_{i<j, i \leq k} (y_i + y_j) \right)
\]
(see also [9, III, 8]). This implies
\[
P_\lambda(E) = (r_{E, S}^E)^* \left( x_1^{\lambda_1} \cdots x_k^{\lambda_k} \prod_{i<j, i \leq k} (x_i + x_j) \right) .
\]
By Corollary 6, we see that \( P_\lambda(E) = P_\lambda(E; t) \) for \( t = -1 \).

We now use the notation from Corollary 9. Specializing \( t = -1 \) in Eq.(7), we get by Lemma 10
\[
\pi_* (c_{\nu \sigma}(Q \otimes S) P_{\nu}(Q) P_{\sigma}(S)) = d_{\nu \sigma} P_{\nu \sigma}(E) ,
\]
where $d_{\nu,\sigma} = 0$ if $(q - k)(r - h)$ is odd and
\[
d_{\nu,\sigma} = (-1)^{(q-k)h} \left( \frac{[(n - k - h)/2]}{[(q - k)/2]} \right)
\]
otherwise. This result was obtained originally in [11, Prop. 1.3(ii)] in a different way. The present approach gives an explanation of the intriguing coefficient $d_{\nu,\sigma}$.

Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is not a strict partition. If there are repetitions of elements in $\lambda$, then $P_\lambda$ is zero; if not then $P_\lambda = (-1)^l P_\mu$, where $l$ is the length of the permutation which rearranges $(\lambda_1, \ldots, \lambda_k)$ into the corresponding strict partition $\mu$.

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References


