Positivity of Legendrian Thom polynomials

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Abstract

We study Legendrian singularities arising in complex contact geometry. We define a one-parameter family of bases in the ring of Legendrian characteristic classes such that any Legendrian Thom polynomial has nonnegative coefficients when expanded in these bases. The method uses a suitable Lagrange Grassmann bundle on the product of projective spaces. This is an extension of a nonnegativity result for Lagrangian Thom polynomials obtained by the authors previously. For a fixed specialization, other specializations of the parameter lead to upper bounds for the coefficients of the given basis. One gets also upper bounds of the coefficients from the positivity of classical Thom polynomials (of singularities of mappings), obtained previously by the last two authors.

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1 Introduction

The aim of the present paper (which is a continuation of [9], [21] and [18]) is to study the *positivity* of Legendrian Thom polynomials. The pioneering papers [8] of Griffiths and [7] of Fulton and Lazarsfeld investigated numerical positivity related to ample vector bundles in differential and algebraic geometry, respectively. Their various variants are nowadays widely investigated in algebraic geometry. We refer to the monograph [17] for a more detailed account. Also, the recent paper [4] gives a combinatorial interpretation for the coefficients of certain Thom-like polynomials, providing further motivation for studying general positivity phenomena.

Our main goal is to define a certain one-parameter family of bases in the ring of Legendrian characteristic classes. Any Legendrian Thom polynomial has nonnegative coefficients when expanded in any member of this family (Theorems 8 and 10).

The main difference, comparing with the previous papers (in particular, with [9] and [18, Remark 14]), is the definition of *Legendre singularity classes* to which these Thom polynomials are attached. They are introduced as closed algebraic subsets in the space parametrizing *pairs* of Legendrian submanifolds in the contact space (see Lemma 1). Regardless of the technical differences, the main theorem of [18] follows from Theorem 10 by specialization of the parameter t to 0. As in [9], [21] and [18], when we study Thom polynomials of commonly bounded degree, it is enough to deal with finite jets of germs of submanifolds through the origin.

The principal technique involved in the proof of these theorems is *transversality* with respect to some stratification of a Lagrange Grassmann bundle. This is a subject of Section 6. The key technical result is Theorem 5. We show that the intersection (in the jet bundle) of a Legendre singularity class with the preimage of the closure of a stratum of the stratification, is represented by a nonnegative cycle. In fact, for our purposes we need a certain Lagrange Grassmann bundle over the product of two projective n-spaces, see Theorem 8. A concrete form of the resulting one-parameter family of bases follows from degeneracy locus formulas from [13], [16], [20] and a formula of Kazarian.

In Section 8, we examine the parameter of the constructed family of bases, and give a precise proof of a result announced in [18], describing algebraically the basis corresponding to the value of the parameter equal to 1 (see Theorem 11).

Let us fix the value of the parameter. It turns out that the nonnegativity of coefficients of the bases for some other values of the parameter can imply upper bounds of the coefficients in the basis for the given parameter. This is a subject of Section 9.

In Section 10, we show (Proposition 13) that for nonempty stable singularity classes, the corresponding Legendrian (and Lagrangian) Thom polynomials are nonzero. This is an amelioration of the main result of [18]. The proof uses the fact [21] that the Thom polynomials for functions $\mathbb{C}^n \to \mathbb{C}$ are nonzero for nonempty singularity classes. We also show how this last result gives some upper bounds on the coefficients of Legendrian Thom polynomials in the basis from Theorem 11.

In Section 11, we list some examples of Legendrian Thom polynomials expanded in different bases from the family.

In the Appendix, we prove some new positivity result for the intersection coefficients in the Lagrange Grassmann bundle. This result concerns the stratification (14), transverse to the one studied in Section 6. It relies on a "large group" action from [1].

2 Some Legendrian geometry

Fix $n \in \mathbb{N}$. Suppose for the moment that W is a vector space of dimension n, and ξ is a vector space of dimension one. Let

$$V := W \oplus (W^* \otimes \xi) \tag{1}$$

be the standard symplectic space equipped with the twisted symplectic form $\omega \in \Lambda^2 V^* \otimes \xi$. We study the germs at the origin of the Legendrian submanifolds in the standard *contact space*

$$V \oplus \xi = W \oplus (W^* \otimes \xi) \oplus \xi\,,$$

or equivalently the germs of the Lagrangian submanifolds in the symplectic space V. Any Legendrian submanifold in $V \oplus \xi$ is determined by its projection to V and any Lagrangian submanifold in V lifts to $V \oplus \xi$, see [2, Proposition p. 313]. Therefore we will perform all the constructions in the realm of symplectic geometry.

Two Lagrangian submanifolds, if they are in generic position, intersect transversally. The singular relative positions can be divided into Legendrian singularity classes. To classify all the possible relative positions, it suffices to consider only two types of submanifolds:

(i) *linear subspaces*: they are parametrized by Lagrangian Grassmannian denoted by $LG(V, \omega)$;

(ii) *the submanifolds which have the tangent space at the origin equal to* W: they are the graphs of the differentials of the functions $f : W \to \xi$ satisfying $df(0) = 0$ and $d^2 f(0) = 0$, see [18, Lemma 2].

The group of symplectomorphisms of V acts on the pairs of Lagrangian submanifolds. By the following lemma, we can restrict our attention only to the submanifolds of the types (i) and (ii).

Lemma 1 *Any pair of Lagrangian submanifolds is symplectic equivalent to a pair (L_1, L_2) such that L_1 is a linear Lagrangian subspace and the tangent space T_0L_2 is equal to W.*

Proof. The lemma follows essentially from the Darboux Theorem (see [2], Theorem, p. 287). Indeed, it follows from this theorem that any Lagrangian submanifold is symplectomorphic to a linear one (given by vanishing of the p-coordinates in the notation of the theorem). We apply this to the first Lagrangian submanifold, getting a linear L_1'. Applying then an appropriate rotation, we get the tangency condition for L_2, equal to the rotated second Lagrangian submanifold. The image L_1 of L_1' under this rotation is linear. The pair (L_1, L_2) satisfies the assertion of the lemma. $\Box$

Let us fix a suitable large k. We identify two Lagrangian submanifolds if the degree of their tangency at 0 is greater than k. The equivalence class will be called "a k-jet of a submanifold". The k-jets of submanifolds satisfying the condition (ii) are in bijection with elements of the vector space

$$\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi\,.$$

We want to describe a space parametrizing all possible relative positions of Lagrangian submanifolds. A suggestion of Kazarian and Lemma 1 justifies the following definition. We denote by $\mathfrak{J}^k(W,\xi)$ the set of pairs (L_1, L_2) of k-jets of Lagrangian submanifolds of V such that L_1 is a linear space and $T_0L_2 = W$. Let

$$\pi : \mathfrak{J}^k(W,\xi) \to LG(V,\omega) \tag{2}$$

be the projection to the first factor. Clearly, π is a trivial vector bundle with the fiber equal to:

$$\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi\,.$$

Remark 2 In [18], we considered the relative position of the Lagrangian submanifolds with respect to the fixed linear space W^*. We obtained a jet bundle over $LG(V)$ which was not a trivial bundle, and we had to deform that bundle to its linear part. Fixing the tangent space of the nonlinear submanifold and moving the second linear space, is an important simplification comparing with [18].

We are interested in a larger group than just the group of symplectomorphisms, namely the group of *(complex) contact automorphisms* of $V \oplus \xi$. It acts on the pairs of Legendrian submanifolds in $V \oplus \xi$. Again by [2, Proposition p. 313], we obtain an action on the pairs of Lagrangian submanifolds.

In particular, we take into account the automorphisms of V which transport the symplectic form ω to a proportional one. An example of such an automorphism is given by ϕ_t defined by

$$\phi_t(q,p) = (q, tp), \quad \text{where} \quad q \in W,\ p \in W^* \otimes \xi \ \text{ and } \ t \in \mathbb{C}^*.$$

By a *Legendre singularity class* we mean a closed algebraic subset

$$\Sigma \subset \mathfrak{J}^k(\mathbb{C}^n, \mathbb{C}),$$

invariant with respect to holomorphic contactomorphisms of $\mathbb{C}^{2n+1}$. Additionally, we assume that the singularity class Σ is stable with respect to enlarging the dimension of W, as in [21, Sect. 2]. Unfortunately, we do not know any place in the literature where the relation between cohomological stability and the infinitesimal stability in the sense of see [2, Sect. 6] is discussed. This problem is treated in [5, Sect. 7.2] in a different context. It seems to be common knowledge that infinitesimal stability implies cohomological stability. On the other hand, we would like to mention that our main results: Theorems 8 and 10, hold without the stability assumption.

3 The jet bundle $\mathfrak{J}^k(W, \xi)$

The vector space ξ may have no distinguished coordinate. It happens so for example when we deal with a fiber of a vector bundle. In other words we have a nontrivial action of $\mathbb{C}^*$ on $\mathfrak{J}^k(\mathbb{C}^n, \mathbb{C})$. Now we repeat the construction of the space $\mathfrak{J}^k(W, \xi)$ parametrizing the relative positions of two Lagrangian submanifolds, assuming that ξ is a line bundle over some base space. We could have used the universal base $BU(1)$, but it is more convenient to work with bundles defined over various base spaces. Also, it will be useful to assume that W is a (possibly nontrivial) vector bundle.

Let X be a topological space, W a complex rank n vector bundle over X, and ξ a complex line bundle over X. The fibers of W, ξ, V over a point $x \in X$ are denoted by W_x, ξ_x, V_x. Let

$$\tau : LG(V, \omega) \to X \tag{3}$$

denote the Lagrange Grassmann bundle parametrizing Lagrangian linear submanifolds in V_x, $x \in X$. We have a relative version of the map (2):

$$\pi : \mathfrak{J}^k(W, \xi) \to LG(V, \omega). \tag{4}$$

The space $\mathfrak{J}^k(W, \xi)$ fibers over X. It is equal to the pull-back:

$$\mathfrak{J}^k(W, \xi) = \tau^* \left(\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \right). \tag{5}$$

In the following, the pull-backs to $LG(V,\omega)$ of the bundles W, V and ξ will be denoted by the same letters, if no confusion occurs. Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines a cycle

$$\Sigma(W,\xi) \subset \mathfrak{J}^k(W,\xi). \tag{6}$$

We will study the classes[1] defined by the cycles $\Sigma(W,\xi)$.

4 Legendrian characteristic classes

The tautological bundle over $LG(V,\omega)$ is denoted by $R_{W,\xi}$, or by R for short. The symplectic form ω gives an isomorphism

$$V \cong V^* \otimes \xi\,. \tag{7}$$

There is a tautological sequence of vector bundles on $LG(V,\omega)$:

$$0 \to R \to V \to R^* \otimes \xi \to 0\,. \tag{8}$$

Consider the virtual bundle

$$A := W^* \otimes \xi - R_{W,\xi}\,. \tag{9}$$

Using the sequence (8), we get the relation

$$A + A^* \otimes \xi = 0\,. \tag{10}$$

The Chern classes $a_i = c_i(A)$ generate the cohomology

$$H^*(LG(V,\omega),\mathbb{Z}) \cong H^*(\mathfrak{J}^k(W,\xi),\mathbb{Z})$$

as an algebra over $H^*(X,\mathbb{Z})$.

Let us fix an approximation of $BU(1) = \bigcup_{n\in\mathbb{N}} \mathbf{P}^n$, that is, we set $X = \mathbf{P}^n$, $\xi = \mathcal{O}(1)$. Let $W = \mathbf{1}^n$ be the trivial bundle of rank n. Then the cohomology ring

$$H^*(LG(V,\omega),\mathbb{Z}) \cong H^*(\mathfrak{J}^k(W,\xi),\mathbb{Z})$$

is isomorphic to the ring of Legendrian characteristic classes for degrees smaller than or equal to n. The element $[\Sigma(W,\xi)]$ of $H^*(\mathfrak{J}^k(W,\xi),\mathbb{Z})$, is called the *Legendrian Thom polynomial* of Σ, and is often denoted by $\mathcal{T}^\Sigma$. It is written in terms of the generators a_i and $s = c_1(\xi)$ (cf. [9, Sect. 3.4], [10, Sect. 4]).

[1] In this paper, whenever we speak about the classes of algebraic cycles, we always mean their *Poincaré dual classes* in cohomology.

Remark 3 The ring of Legendrian characteristic classes is the quotient of the polynomial ring

$$\mathbb{Z}[a_1, a_2, a_3, \ldots; u]$$

by the relations coming from the identity (10). After inverting 2 and applying the twist by $\xi^{-\frac{1}{2}}$, we obtain the ring of Lagrangian classes extended by one additional free variable t, that is,

$$\mathbb{Z}[\frac{1}{2}][a'_1, a'_2, a'_3, \ldots; t]/\mathcal{I}\,,$$

where $\mathcal{I}$ is generated by the polynomials

$$(a'_i)^2 + 2\sum_{k=1}^{i}(-1)^k a'_{i+k}a'_{i-k}, \qquad i > 0\,.$$

The even Chern classes a'_{2i} are expressed by odd ones and this ring is just the polynomial ring

$$\mathbb{Z}[\frac{1}{2}][a'_1, a'_3, a'_5, \ldots; t]\,.$$

A similar procedure can be applied to the untwisted variables a_i.

5 Cell decompositions of the Grassmann bundle

We describe two "transverse" cell decompositions of the Lagrange Grassmannians.

To begin with, let $\xi, \alpha_1, \alpha_2, \ldots, \alpha_n$ be vector spaces of dimensions equal to one, and let

$$W := \bigoplus_{i=1}^{n} \alpha_i\,, \qquad V := W \oplus (W^* \otimes \xi)\,. \tag{11}$$

We have a twisted symplectic form ω defined on V with values in ξ. The Lagrangian Grassmannian $LG(V, \omega)$ is a homogeneous space with respect to the group action of the symplectic group $Sp(V, \omega) \subset \mathrm{End}(V)$. Fix two "opposite" standard isotropic flags in V:

$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i\,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha^*_{n-i+1} \otimes \xi \tag{12}$$

for $h = 1, 2, \ldots, n$ and consider two subgroups $B^\pm \subset Sp(V, \omega)$ which are the Borel groups preserving the flags $F_\bullet^\pm$. The orbits of $B^\pm$ in $LG(V, \omega)$ form two "opposite" cell $\Omega^\pm$-decompositions $\{\Omega_I(F_\bullet^\pm, \xi)\}$ of $LG(V, \omega)$ (Bruhat decompositions), indexed by strict partitions

$$I \subset \rho := (n, n-1, \ldots, 1)$$

(cf. [19]). The cells of the Ω^+-decomposition are transverse to the cells of the Ω^--decomposition.

We pass now to the relative version of the above decompositions.

The description just presented is functorial with respect to the automorphisms of the lines ξ and α_i's (they form a torus $(\mathbb{C}^*)^{n+1}$). Thus the construction of the cell decompositions can be repeated for bundles ξ and $\{\alpha_i\}_{i=1}^n$ over any base X. We obtain a Lagrange Grassmann bundle

$$\tau : LG(V,\omega) \to X$$

and a group bundle (group scheme over X)

$$Sp(V,\omega) \to X$$

together with two subgroup bundles $B^\pm \to X$. Moreover, $LG(V,\omega)$ admits two (relative) stratifications

$$\{\Omega_I(F_\bullet^\pm,\xi) \to X\}_{\text{strict } I\subset\rho}\,.$$

Assume that $X = G/P$ is a compact manifold, homogeneous with respect to an action of a linear group G. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$ (it is again a Bruhat decomposition). The subsets

$$Z_{I\lambda}^- := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F_\bullet^-,\xi) \tag{13}$$

form an algebraic cell decomposition of $LG(V,\omega)$, called Z^--*decomposition* or *distinguished decomposition* in the following. The classes of their closures give a basis of homology, called Z^--basis. Note that each $Z_{I\lambda}^-$ is transverse to each stratum $\Omega_J(F_\bullet^+,\xi)$, where $J \subset \rho$ is a strict partition.

Similarly, we define the subsets

$$Z_{I\lambda}^+ := \tau^{-1}(\sigma_\lambda) \cap \Omega_I(F_\bullet^+,\xi)\,, \tag{14}$$

which form a Z^+-*decomposition* and give rise to the corresponding Z^+-basis of the cohomology.

Example 4 If $X = \mathbf{P}^1$, $W = \mathbf{1}$, $\xi = \mathcal{O}(d)$ (for $d > 0$), then $LG(V,\omega)$ is the Hirzebruch surface Σ_d which can be presented as the sum of the space of the line bundle ξ and the section at infinity,

$$\Sigma_d = \xi \cup \mathbf{P}^1_\infty\,.$$

Then $\mathbf{P}^1_0$, the zero section of the bundle ξ, is a stratum of the Ω^+-decomposition and the section at infinity $\mathbf{P}^1_\infty$ is a stratum of the Ω^--decomposition. For the cell decomposition of $X = \mathbf{P}^1 = \mathbb{C}\cup\{\infty\}$, we obtain two cell decompositions of Σ_d. The closures of the one-dimensional cells are the following:

$$\left\{\overline{Z_{I\lambda}^+}\right\} = \left\{\tau^{-1}(\infty),\ \mathbf{P}^1_0\right\}, \qquad \left\{\overline{Z_{I\lambda}^-}\right\} = \left\{\tau^{-1}(\infty),\ \mathbf{P}^1_\infty\right\}.$$

Two resulting bases of *cohomology* are mutually dual with respect to the intersection product.

The cycles of Z^+-decomposition have the following property: any effective cycle has a nonnegative intersection number with them. This is not true for the elements of Z^--decomposition: for example, the self-intersection of $\mathbf{P}^1_\infty$ is equal to $-d$.

6 Positivity in the jet bundle

We pass now to a nonnegativity result on the Legendrian Thom polynomials and the Z^--decomposition. We recall that our goal is to study cycles $\Sigma(W,\xi)$ in:

$$\mathfrak{J}^k(W,\xi) = \tau^*\left(\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*)\otimes\xi\right).$$

To abbreviate the notation, we set $\mathfrak{J}^k(W,\xi)=\mathfrak{J}$.

This vector bundle is equipped with a B^+-action, i.e., an appropriate map

$$B^+\times_X \mathfrak{J}\to\mathfrak{J}$$

over X.

A Legendre singularity class Σ defines the subset $\Sigma(W,\xi)\subset\mathfrak{J}$, which is a Zariski-locally trivial fibration over X. Moreover, $\Sigma(W,\xi)$ is preserved by the action of B^+ since this group consists of holomorphic symplectomorphisms preserving W:

$$\begin{array}{ccccccc} \Sigma(W,\xi) & \subset & \mathfrak{J} & \overset{\pi}{\twoheadrightarrow} & LG(V,\omega) & \overset{\tau}{\twoheadrightarrow} & X \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ B^+ & & B^+ & & Sp(V,\omega) & & \end{array}$$

We now state the main technical result of the present paper.

Theorem 5 *Suppose that the vector bundle $\mathfrak{J}$ is globally generated. Then, in $\mathfrak{J}$, the intersection of $\Sigma(W,\xi)$ with the closure of any $\pi^{-1}(Z^-_{I\lambda})$ is represented by a nonnegative cycle.*

This result is, in fact, true for any effective B^+-invariant cycle on $\mathfrak{J}$, Zariski-locally trivial fibered over X, taken instead of $\Sigma(W,\xi)$. The proof is modelled on the techniques of [12].

Before proving the theorem, we shall establish some preliminary result. For a subset $Y\subset\mathfrak{J}$ and a global section $s\in H^0(\mathfrak{J})$, we denote by $s+Y$ the fiberwise translate of Y by s. We will deform the cycle $\Sigma(W,\xi)$ using a fiberwise translate. The construction is done for each stratum $\Omega_J(F^+_\bullet,\xi)$

separately. Fix such a stratum $\Omega = \Omega_J(F_\bullet^+, \xi)$. Denote by $\mathfrak{J}_{|\Omega} = \pi^{-1}\Omega$ the restriction of the bundle to the stratum and set

$$\Xi := \Sigma(W, \xi) \cap \pi^{-1}\Omega\,. \tag{15}$$

We need the following lemma, which is a variant of the Bertini-Kleiman tranversality theorem [12].

Lemma 6 *Suppose that the vector bundle $\mathfrak{J}$ is globally generated. Let $Y \subset \mathfrak{J}_{|\Omega}$ be a subvariety. Then there exists an open, dense subset $U \subset H^0(\mathfrak{J})$ such that for any section $s \in U$, the translate $s + \Xi$ has proper intersection with Y.*

Proof. Let

$$q : H^0(\mathfrak{J}) \times \Xi \to \mathfrak{J}_{|\Omega}$$

be the fiberwise translate. We claim that this map is flat (in fact, it is a fibration). The question is local. We find an open set $X' \subset X$ such that the bundles $\alpha_{i|X'}$ and $\xi_{|X'}$ are trivial. Over X' the set Ξ is the product of X' and some variety. Therefore, it is enough to assume that X is a point. Further, note that $\Xi \to \Omega$ is a fibration since Ω is homogeneous with respect to B^+ and Ξ is a B^+-invariant subset of $\mathfrak{J}$. Thus the question reduces to a single fiber of $\mathfrak{J}$. Since $\mathfrak{J}$ is globally generated, for any $y \in \Omega$ the fiber $\mathfrak{J}_y$ is homogeneous for the action of $H^0(\mathfrak{J})$. The action map

$$q_{|H^0(\mathfrak{J})\times\Xi_y} : H^0(\mathfrak{J}) \times \Xi_y \to \mathfrak{J}_y$$

is a trivial fibration with the fiber isomorphic to $\Xi_y \times \ker(H^0(\mathfrak{J}) \to \mathfrak{J}_y)$. It follows that q is a fibration.

Applying [12, Lemma 1], the assertion of the lemma follows. $\square$

Proof of Theorem 5

For a strict partition $J \subset \rho$, we set

$$\mathfrak{J}_J := \pi^{-1}(\Omega_J(F_\bullet^+, \xi))\,. \tag{16}$$

Applying Lemma 6, we get an open dense subset $U_J \subset H^0(\mathfrak{J})$ such that for $s \in U_J$ the intersection

$$\big(s + (\Sigma(W,\xi) \cap \mathfrak{J}_J)\big) \cap \big(\pi^{-1}(\overline{Z^-_{I\lambda}}) \cap \mathfrak{J}_J\big)$$

is proper inside $\mathfrak{J}_J$. We now pick

$$s \in \bigcap_{\text{strict } J\subset\rho} U_J\,,$$

and set

$$\Sigma(W,\xi)' := s + \Sigma(W,\xi)\,. \tag{17}$$

Since the cycle $\pi^{-1}(\overline{Z^-_{I\lambda}})$ is transverse to the stratification $\{\mathfrak{J}_J\}_{\text{strict } J\subset\rho}$ of $\mathfrak{J}$, an easy dimension count shows that $\pi^{-1}(\overline{Z^-_{I\lambda}})$ intersects properly $\Sigma(W,\xi)'$ in $\mathfrak{J}$.

Theorem 5 now follows since by [6, Sect. 8.2] the intersection

$$[\Sigma(W,\xi)]\cdot[\pi^{-1}(\overline{Z^-_{I\lambda}})] = [\Sigma(W,\xi)']\cdot[\pi^{-1}(\overline{Z^-_{I\lambda}})]$$

is represented by a nonnegative cycle. $\square$

7 A family of bases in which any Legendrian Thom polynomial has positive expansion

We shall apply Theorem 5 in the situation when all α_i are equal to the same line bundle α (i.e., $W=\alpha^{\oplus n}$) and $\alpha^{-m}\otimes\xi$ is globally generated for $m\geq 3$.

Example 7 We shall consider the following three cases: the base is always $X=\mathbf{P}^n$ and

1. $$\xi_1=\mathcal{O}(-2)\,,\qquad \alpha_1=\mathcal{O}(-1)\,,$$

2. $$\xi_2=\mathcal{O}(1)\,,\qquad \alpha_2=\mathbf{1}\,,$$

3. $$\xi_3=\mathcal{O}(-3)\,,\qquad \alpha_3=\mathcal{O}(-1)\,,$$

We obtain symplectic bundles $V_i=\alpha_i^{\oplus n}\oplus(\alpha_i^*\otimes\xi_i)^{\oplus n}$ with twisted symplectic forms ω_i for $i=1,2,3$.

These cases were for the authors the key examples supporting their working conjecture that the assertion of Theorem 8 holds true.

Case 1 was the subject of [18, Remark 14], where the basis related to the distinguished cell decomposition of $LG(V_1,\omega_1)$ was investigated. In this case, for degrees $\leq n$, the cohomology $H^*(LG(V_1,\omega_1),\mathbb{Z}[\frac{1}{2}])$ is isomorphic to the ring of Legendrian characteristic classes tensored by $\mathbb{Z}[\frac{1}{2}]$.

In Case 2, the integral cohomology $H^*(LG(V_2,\omega_2),\mathbb{Z})$ is isomorphic to the ring of Legendrian characteristic classes up to degree n. The distinguished cell decomposition of $LG(V_2,\omega_2)$ gives us another basis of cohomology.

In Case 3, the cohomology of $LG(V_3,\omega_3)$ is isomorphic, up to degree n, to the ring of Legendrian characteristic classes, provided we invert the number 3 this time.

The positivity property in Case 1 was known to us (see [18, Remark 14]), whereas in Cases 2 and 3, it was Kazarian who suggested the positivity. His conjecture was supported by computation of all the Thom polynomials up to degree seven.

In general, $H^*(LG(V,\omega),\mathbb{Q})$ is isomorphic to the ring of Legendrian characteristic classes up to degree $\dim W$ if ξ is nontrivial. The case of $W = W^{(p,q)}$, $\xi = \xi^{(p,q)}$ and the corresponding $V = V^{(p,q)}$, where

$$\xi^{(p,q)} = \xi_2^{\otimes p} \otimes \xi_3^{\otimes q} \quad \text{and} \quad \alpha = \alpha^{(p,q)} = \alpha_2^{\otimes p} \otimes \alpha_3^{\otimes q} = \alpha_3^{\otimes q}$$

(p, q are integers) will be used in Section 8.

To overlap all these three cases we consider the product

$$X := \mathbf{P}^n \times \mathbf{P}^n \tag{18}$$

and set

$$W := p_1^*\mathcal{O}(-1)^{\oplus n}\,, \qquad \xi := p_1^*\mathcal{O}(-3) \otimes p_2^*\mathcal{O}(1)\,, \tag{19}$$

where $p_i : X \to \mathbf{P}^n$, $i = 1, 2$, are the projections. Restricting the bundles W and ξ to the diagonal, or to the factors, we obtain the three cases considered above. We should keep in mind that X is an approximation of the classifying space $B(U(1) \times U(1))$. We fix it because we apply algebraic geometry methods.

The space $LG(V,\omega)$ has a distinguished cell decomposition $Z^-_{I\lambda}$ where I runs over strict partitions contained in ρ, and $\lambda = (a, b)$ with a and b natural numbers smaller than or equal to n. The classes of closures of the cells of this decomposition give a basis of the homology of $LG(V,\omega)$. The dual basis of cohomology (in the sense of linear algebra) is denoted by

$$e_{I,a,b} = [\overline{Z^-_{I,a,b}}]^*\,. \tag{20}$$

By reasons of geometry, it is clear that the basis $\{e_{I,a,b}\}$ consists of the classes represented by the cycles $\overline{Z^+_{I,a,b}}$.

Let v_1 and v_2 be the first Chern classes of $p_1^*(\mathcal{O}(1))$ and $p_2^*(\mathcal{O}(1))$. By the definition of $Z^+_{I,a,b}$, in $H^*(LG(V,\omega),\mathbb{Z})$, we have

$$e_{I,a,b} = e_{I,0,0}\; v_1^a v_2^b\,. \tag{21}$$

Moreover, we have

$$e_{I,0,0} = [\overline{\Omega_I(F^+_\bullet,\xi)}]\,. \tag{22}$$

With X, W and ξ as in (18) and (19), we have

Theorem 8 *Let Σ be a Legendre singularity class. Then $[\Sigma(W,\xi)]$ has nonnegative coefficients in the basis $\{e_{I,a,b}\}$.*

Proof. Let

$$\iota : LG(V,\omega) \to \mathfrak{J}^k(W,\xi)$$

be the zero section, and

$$\iota^* : H^*(\mathfrak{J}^k(W,\xi),\mathbb{Z}) \to H^*(LG(V,\omega),\mathbb{Z})$$

be the induced map on cohomology. We write

$$\iota^*[\Sigma(W,\xi)] =: \sum_{I,a,b} \gamma_{I,a,b}[\overline{Z^+_{I,a,b}}]\,, \tag{23}$$

where $\gamma_{I,a,b}$ are integers. We claim that the coefficients $\gamma_{I,a,b}$ are nonnegative. These coefficients are equal to

$$\iota^*[\Sigma(W,\xi)]\cdot[\overline{Z^-_{I,a,b}}] \qquad \text{(intersection in } LG(V,\omega)\text{)}.$$

By the functoriality of the intersection product, the numbers $\gamma_{I,a,b}$ are equal to

$$[\Sigma(W,\xi)]\cdot[\pi^{-1}(\overline{Z^-_{I,a,b}})] \qquad \text{(intersection in } \mathfrak{J}\text{)}.$$

The vector bundle $\mathfrak{J}$ on $LG(V,\omega)$ is equal to

$$\tau^*\left(\bigoplus_{j=3}^{k+1}\mathrm{Sym}^j(W^*)\otimes\xi\right) = \tau^*\left(\bigoplus_{j=3}^{k+1}\mathrm{Sym}^j(\mathbf{1}^n)\otimes p_1^*\mathcal{O}(j-3)\otimes p_2^*\mathcal{O}(1)\right).$$

We see that $\mathfrak{J}$ is globally generated. By Theorem 5, the desired intersections in $\mathfrak{J}$ are nonnegative. $\square$

The following computation will be needed later.

Example 9 By [20, Theorem 9.3] (see also [13, Cor. 5]) if $\xi = \mathbf{1}$, $I = \{h\}$, then

$$[\overline{\Omega_h(F^+_\bullet,\mathbf{1})}] = c_h(R^* - F^+_{n+1-h})\,.$$

Hence for a general ξ, by passing to its square root, we have

$$[\overline{\Omega_h(F^+_\bullet,\xi)}] = c_h(R^*\otimes\xi^{\frac{1}{2}} - F^+_{n+1-h}\otimes\xi^{-\frac{1}{2}}) = c_h((R^*\otimes\xi - F^+_{n+1-h})\otimes\xi^{-\frac{1}{2}})\,.$$

Note that for any virtual bundle E of dimension $h-1$ and for any line bundle ζ, we have $c_h(E\otimes\zeta) = c_h(E)$. Hence

$$[\overline{\Omega_h(F^+_\bullet,\xi)}] = c_h(R^*\otimes\xi - F^+_{n+1-h})\,.$$

In our situation, ($W = \alpha^{\oplus n}$), the above formula can be written in the form

$$\begin{aligned}[\overline{\Omega_h(F^+_\bullet,\xi)}] &= c_h(R^*\otimes\xi - W + \alpha^{\oplus h-1})\\ &= c_h(W^*\otimes\xi - R + \alpha^{\oplus h-1})\\ &= c_h(A + \alpha^{\oplus h-1})\,.\end{aligned}$$

8 The parameter p/q and the basis for $p = q = 1$

Fix a Legendre singularity class Σ. By Theorem 8, we know that the Thom polynomial of Σ, evaluated at the Chern classes of

$$A = W^* \otimes \xi - R$$

and $c_1(\xi) = v_2 - 3v_1$, is a nonnegative $\mathbb{Z}$-linear combination of the following form:

$$\mathcal{T}^\Sigma = \sum_{I,a,b} \gamma_{I,a,b}\; e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} [\overline{\Omega_I(F_\bullet^+, \xi)}] v_1^a v_2^b \,.$$

We want to find an additive basis of the ring of Legendrian characteristic classes with the property that any Legendrian Thom polynomial is a nonnegative combination of basis elements. To this end, we take a geometric model of the classifying space: $LG(V^{(p,q)}, \omega^{(p,q)})$ (see the previous section), and the Z^+-basis which is dual to the Z^--basis. More precisely, dividing the cohomology ring $H^*(LG(V,\omega), \mathbb{Q})$ by the relation

$$q \cdot v_1 = p \cdot v_2 \,, \tag{24}$$

that is, specializing the parameters to $v_1 = p \cdot t$, $v_2 = q \cdot t$, we obtain the ring $H^*(LG(V^{(p,q)}, \omega^{(p,q)}), \mathbb{Q})$, isomorphic to the ring of Legendrian characteristic classes in degrees $\leq n$ (provided that $c_1(\xi) = v_2 - 3v_1$ is not specialized to 0).

From Theorem 8, we obtain:

Theorem 10 *If p and q are nonnegative, $q - 3p \neq 0$, then the Thom polynomial is a nonnegative combination of the $[\overline{\Omega_I(F_\bullet^+, \xi)}]\, t^i$'s.*

The family $[\overline{\Omega_I(F_\bullet^+, \xi)}]\, t^i$ is a one-parameter family of bases depending on the parameter p/q.

Though the main theme of the present paper is the existence of a one-parameter family of bases in which every Legendrian Thom polynomial has positive expansion, we shall also give some results on algebraic form of $[\overline{\Omega_I(F_\bullet^+, \xi)}]$.

First, we come back to Case 1 from Example 7. This corresponds to fixing the parameter to be 1, i.e. $p = 1$ and $q = 1$. This corresponds to setting $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles W and ξ to the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$, or we study $W_1 = \mathcal{O}(-1)^{\oplus n}$ and $\xi = \xi_1 = \mathcal{O}(-2)$. Set $\zeta := \mathcal{O}(-1)$. Then $\xi^{\frac{1}{2}} = \zeta$. From (10), we have

$$A^* \otimes \zeta + A \otimes \zeta^{-1} = 0 \,. \tag{25}$$

In general, when the bundle ξ admits a square root ζ it is convenient to give another description of the space $LG(V, \omega)$. Let us define

$$W' = W \otimes \zeta^{-1} \quad \text{and} \quad V' = V \otimes \zeta^{-1} \,.$$

Then V' is equipped with a symplectic form with constant coefficients, and

$$V' = W' \oplus W'^* .$$

In our case,

$$X = \mathbf{P}^n, \quad W = \mathcal{O}(-1)^{\oplus n}, \quad \text{and} \quad \xi = \mathcal{O}(-2) .$$

Then W' and V' become trivial bundles:

$$W' = \mathbf{1}^n , \qquad V' = \mathbf{1}^{2n} .$$

We have

$$LG(V,\omega) = LG(\mathbb{C}^{2n}) \times \mathbf{P}^n , \tag{26}$$

where $LG(\mathbb{C}^{2n}) = LG(\mathbb{C}^{2n},\omega)$ and ω is the standard nondegenerate symplectic form on $\mathbb{C}^{2n}$.

In algebraic expressions, we shall use $\widetilde{Q}$-functions of [20] and their geometric interpretation from [19]. The reader can find in [18, Sect. 3] a summary of their properties in the notation which also will be used here.

Let R' denote the tautological bundle on $LG(\mathbb{C}^{2n})$. Under the identification (26), R' pulled back from $LG(\mathbb{C}^{2n})$ to $LG(V,\omega)$ is equal to $R \otimes \zeta^{-1}$. We thus have

$$A \otimes \zeta^{-1} = W^* \otimes \zeta - R \otimes \zeta^{-1} = W'^* - R' = \mathbf{1}^n - R' = R'^* - \mathbf{1}^n . \tag{27}$$

The distinguished cell decomposition of $LG(V,\omega) = LG(\mathbb{C}^{2n}) \times \mathbf{P}^n$ is of the product form. By the cohomological properties of $\widetilde{Q}$-functions ([19]), the basis of cohomology consists of the following functions

$$\widetilde{Q}_I(R'^*) \cdot t^j = \widetilde{Q}_I(R^* \otimes \xi^{\frac{1}{2}}) \cdot c_1(\xi^{-\frac{1}{2}})^j ,$$

where I runs over strict partitions in ρ.

In this way, we obtain

Theorem 11 *The Thom polynomial for a Legendre singularity class Σ is a combination:*

$$\mathcal{T}^\Sigma = \sum_{j\geq 0} \sum_I \alpha_{I,j} \ \widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \cdot t^j , \tag{28}$$

where

$$t = \frac{1}{2} c_1(\xi^*) \in H^2(X, \mathbb{Z}[\frac{1}{2}]) ,$$

I runs over strict partitions in ρ, and $\alpha_{I,j}$ are nonnegative integers.

Remark 12 We get the result announced in [18, Remark 14], where it should read "$t = \frac{1}{2}c_1(\xi^*)$", and where we used the notation $L^* - \mathbf{1}^{\dim L}$ for the virtual bundle A.

9 Upper bounds of coefficients

In this section, we shall translate the positivity result in Theorem 8 into restrictions for the coefficients of Thom polynomials.

Let us fix the value of the parameter. It turns out that the nonnegativity of coefficients of the bases for some other values of the parameter can imply upper bounds of the coefficients in the basis for the given parameter. We plan to discuss this more systematically elsewhere. Here we consider the 3 bases in degree 2 from Cases 1, 2 and 3 in Example 7. We shall call them the first, second and third basis, respectively.

Let $s = c_1(\xi)$ and let us list the classes of degree 2. The first basis of the Legendrian characteristic classes consists of:

$$c_2(A+\alpha_1) = a_2+\frac{1}{2}s\,a_1\,, \quad -\frac{1}{2}c_1(\xi_1)c_1(A) = -\frac{1}{2}s\,a_1\,, \quad \left(-\frac{1}{2}c_1(\xi_1)\right)^2 = \frac{1}{4}s^2\,,$$

by Example 9 since $c_1(\xi_1) = -2c_1(\mathcal{O}(1))$ and $c_1(\alpha_1) = -c_1(\mathcal{O}(1))$. The second basis is

$$c_2(A) = a_2\,, \quad s\,c_1(A) = s\,a_1\,, \quad s^2$$

since here $\alpha_2 = 1$. The third basis is

$$c_2(A+\alpha_3) = a_2+\frac{1}{3}s\,a_1\,, \quad -\frac{1}{3}c_1(\xi_3)c_1(A) = -\frac{1}{3}s\,a_1\,, \quad \left(-\frac{1}{3}c_1(\xi_3)\right)^2 = \frac{1}{9}s^2$$

since $c_1(\xi_3) = -3c_1(\mathcal{O}(1))$ and $c_1(\alpha_3) = -c_1(\mathcal{O}(1))$.

The Thom polynomial of the singularity A_3 is of the form

$$3(a_2 + \frac{1}{2}s\,a_1) - \kappa\,\frac{1}{2}s\,a_1\,,$$

with $\kappa \geq 0$, by the positivity in the first basis. The positivity in the second basis gives the condition

$$\frac{3}{2} - \frac{1}{2}\kappa \geq 0\,,$$

that is, $\kappa \leq 3$. When we write the Thom polynomial in the third basis, we see that the coefficient of $-\frac{1}{3}s\,a_1$ is equal to $\kappa - 1$. It follows that $\kappa \geq 1$.

Recall that the only Legendrian Thom polynomial of degree 2 is the one of the singularity A_3, displayed in the first basis as:

$$\mathcal{T}^{A_3} = 3\widetilde{Q}_2 + t\widetilde{Q}_1\,,$$

i.e. with $\kappa = 1$.

Another upper bound for the coefficients of the expansions of Legendrian Thom polynomials can be obtained by the method of Example 17.

10 Legendrian vs. classical Thom polynomials

In this section, we shall use the basis from Case 1 in Example 7, i.e., we put $t = v_1 = v_2$.

Proposition 13 *For a nonempty stable Legendre singularity class* Σ*, the Lagrangian Thom polynomial (i.e.,* $\mathcal{T}^\Sigma$ *evaluated at* $t = 0$*) is nonzero.*

Proof. For a Legendre singularity class Σ, consider the associated singularity class of maps $f : M \to C$ from n-dimensional manifolds to curves (see [9, p. 729] and [10, p. 123]). We denote the related Thom polynomial by Tp^Σ.

According to [9, pp. 708–709], we have

$$Tp^\Sigma = \mathcal{T}^\Sigma \cdot c_n(T^*M \otimes f^*TC)\,. \tag{29}$$

We know by [21, Theorem 4] that the Thom polynomial Tp^Σ is nonzero. Moreover, it follows from the proof (*loc.cit.*) that Tp^Σ, specialized with $f^*TC = \mathbf{1}$, i.e., $t = 0$, is also nonzero. The assertion follows from the equation (29). $\square$

Consequently, we get an improvement of Theorem 8.

Corollary 14 *For a nonempty stable Legendre singularity class* Σ*, the (Legendrian) Thom polynomial* $\mathcal{T}^\Sigma$ *is nonzero.*

Remark 15 Suppose that we are in the setting of (29). We shall use the expansions of Legendrian Thom polynomials in the basis for $v_1 = v_2 = t$, studied in the previous section. Set $\xi := f^*TC$. If $A = T^*M \otimes \xi - TM$, having the Thom polynomial presented as in (28), we want to compute the $\widetilde{Q}$-functions of

$$A \otimes \xi^{-\frac{1}{2}} = (T^*M \otimes \xi - TM)^* \otimes \xi^{-\frac{1}{2}} = E^* - E\,,$$

where $E = TM \otimes \xi^{-\frac{1}{2}}$. Since for every strict partition I,

$$\widetilde{Q}_I(E^* - E) = Q_I(E^*)\,, \tag{30}$$

where Q_I denotes the classical *Schur Q-function* [22], we get the desired expression by changing any $\widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}})$ to $Q_I(E^*)$.

The following procedure mimics the passing from the LHS to the RHS in Eq. (29), where $\mathcal{T}^\Sigma$ is given as a $\mathbb{Z}[t]$-combination of the $\widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}})$'s, and Tp^Σ is to be written as a $\mathbb{Z}$-combination of the Schur functions $S_J(T^*M - \xi^*)$ (for Schur functions in virtual bundles we refer to [15], and for Schur Q-functions to [19]).

Procedure 16 We start from a $\mathbb{Z}[t]$-combination of the polynomials discussed in Remark 15: $\widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) = Q_I(T^*M \otimes \xi^{\frac{1}{2}})$.

- We write $Q_I(T^*M \otimes \xi^{\frac{1}{2}})$ as a combination of the $S_J(T^*M \otimes \xi^{\frac{1}{2}})$'s (here we use a combinatorial rule from [23] decomposing Schur Q-functions into S-functions);
- we expand any $S_J(T^*M \otimes \xi^{\frac{1}{2}})$ as a combination of the $S_K(T^*M)t^i$'s (here we use a formula from [14] for decomposition of the Schur polynomials of twisted bundles);
- we multiply the so-obtained combination by $c_n(T^*M \otimes \xi)$ (here we use the factorization formula for Schur functions from [3]); we eventually get a combination of the $S_L(T^*M - \xi^*)$'s with the coefficients being polynomials in n.

Example 17 We shall examine now how positivity of Schur function expansions of Thom polynomials for mappings $\mathbb{C}^n \to \mathbb{C}$, proved in [21], implies some upper bounds on the coefficients of a Legendrian Thom polynomial in the expansion (28).

Let us consider a degree 2 cohomology class already considered in Section 9, of the form

$$3\widetilde{Q}_2 + \kappa t \widetilde{Q}_1 \,, \tag{31}$$

where κ is an integer.

We fix $n \geq 2$. We apply Procedure 16 to (31):

$$\begin{aligned}
&(3\widetilde{Q}_2 + \kappa t\widetilde{Q}_1)(T^*M \otimes \xi^{\frac{1}{2}}) \cdot c_n(T^*M \otimes \xi) = \\
&(6(S_2 + S_{1^2}) + 2\kappa t S_1)(T^*M \otimes \xi^{\frac{1}{2}}) \cdot c_n(T^*M \otimes \xi) = \\
&\big(6(S_2 + S_{1^2}) + 2t(\kappa - n)S_1 + t^2 n(n - 2\kappa)\big)(T^*M) \cdot c_n(T^*M \otimes \xi)
\end{aligned}$$

By the factorization formula, the last expression is equal to

$$6(S_{1^{n-1}3} + S_{1^{n-2}2^2}) + (6n - \kappa)S_{1^n 2} + (\frac{3}{2}n^2 - \kappa\frac{n}{2})S_{1^{n+2}} \tag{32}$$

evaluated at $T^*M - \xi^*$.

Suppose that for each $n \geq 2$ we have in (32) a nonnegative combination of Schur functions, i.e.

$$6n - \kappa \geq 0 \quad \text{and} \quad \frac{3}{2}n^2 - \kappa\frac{n}{2} \geq 0 \,.$$

This implies that $\kappa \leq 6$.

11 Examples of Legendrian Thom polynomials

The Thom polynomials expanded in the basis $\{e_{I,a,b}\}$ (see Section 7) are (the summands in bold represent the Lagrangian Thom polynomials):

$\mathbf{A_2}$: $\widetilde{\mathbf{Q}}_{\mathbf{1}}$
$\mathbf{A_3}$: $\mathbf{3\widetilde{Q}_2} + v_2\widetilde{Q}_1$
$\mathbf{A_4}$: $\mathbf{12\widetilde{Q}_3} + \mathbf{3\widetilde{Q}_{21}} + (3v_1 + 7v_2)\widetilde{Q}_2 + (v_1v_2 + v_2^2)\widetilde{Q}_1$
$\mathbf{D_4}$: $\widetilde{\mathbf{Q}}_{\mathbf{21}}$.

The first (resp. last) expression means that the Thom polynomial of the singularity A_2 (resp. D_4) written in all bases from the family is equal to $\widetilde{\mathbf{Q}}_{\mathbf{1}}$ (resp. $\widetilde{\mathbf{Q}}_{\mathbf{21}}$).

Similarly the Thom polynomial of the singularity P_8 in all bases from the family is equal to $\widetilde{\mathbf{Q}}_{\mathbf{321}}$. Next we have

$\mathbf{A_5}$: $\mathbf{60\widetilde{Q}_4} + \mathbf{27\widetilde{Q}_{31}} + (6v_1 + 16v_2)\widetilde{Q}_{21} + (39v_1 + 47v_2)\widetilde{Q}_3 +$
$(6v_1^2 + 22v_1v_2 + 12v_2^2)\widetilde{Q}_2 + (2v_1^2v_2 + 3v_1v_2^2 + v_2^3)\widetilde{Q}_1$
$\mathbf{D_5}$: $\mathbf{6\widetilde{Q}_{31}} + 4v_2\widetilde{Q}_{21}$,

and analogously to D_5,

$\mathbf{P_9}$: $\mathbf{12\widetilde{Q}_{421}} + 12v_2\widetilde{Q}_{321}$.

Let us specialize $v_1 = v_2 = t$. The Thom polynomials for singularities of codimensions lower than or equal to six are listed in [18]. Here are the Legendrian Thom polynomials of the consecutive singularities A_8, D_8, E_8, X_9, P_9 displayed in the notation from (28). Again, the summands in bold represent the Lagrangian Thom polynomials. These formulas were communicated to us by Kazarian.

$\mathbf{A_8}$:

$$\mathbf{18840\widetilde{Q}_{61}} + \mathbf{20160\widetilde{Q}_7} + \mathbf{3123\widetilde{Q}_{421}} + \mathbf{5556\widetilde{Q}_{43}} + \mathbf{15564\widetilde{Q}_{52}} +$$
$$t(71856\widetilde{Q}_6 + 3999\widetilde{Q}_{321} + 55672\widetilde{Q}_{51} + 34780\widetilde{Q}_{42}) +$$
$$t^2(64524\widetilde{Q}_{41} + 24616\widetilde{Q}_{32} + 105496\widetilde{Q}_5) + t^3(36048\widetilde{Q}_{31} + 81544\widetilde{Q}_4) +$$
$$t^4(8876\widetilde{Q}_{21} + 34936\widetilde{Q}_3) + t^5 7848\widetilde{Q}_2 + t^6 720\widetilde{Q}_1\,;$$

$\mathbf{D_8}$:

$$\mathbf{1080\widetilde{Q}_{61}} + \mathbf{315\widetilde{Q}_{421}} + \mathbf{468\widetilde{Q}_{43}} + \mathbf{1332\widetilde{Q}_{52}} +$$
$$t(2754\widetilde{Q}_{42} + 2952\widetilde{Q}_{51} + 405\widetilde{Q}_{321}) + t^2(1802\widetilde{Q}_{32} + 3162\widetilde{Q}_{41}) +$$
$$t^3 1618\widetilde{Q}_{31} + t^4 344\widetilde{Q}_{21}\,;$$

$\mathbf{E_8}$:

$$\mathbf{93\widetilde{Q}_{421}} + \mathbf{108\widetilde{Q}_{43}} + \mathbf{204\widetilde{Q}_{52}} + \mathbf{72\widetilde{Q}_{61}} + t(99\widetilde{Q}_{321} + 216\widetilde{Q}_{51} + 414\widetilde{Q}_{42}) +$$
$$t^2(246\widetilde{Q}_{41} + 246\widetilde{Q}_{32}) + t^3 126\widetilde{Q}_{31} + t^4 24\widetilde{Q}_{21}\,;$$

$\mathbf{X_9}$:
$\mathbf{18\widetilde{Q}_{52}} + \mathbf{27\widetilde{Q}_{43}} + t(42\widetilde{Q}_{42} + 6\widetilde{Q}_{51}) + t^2(21\widetilde{Q}_{32} + 11\widetilde{Q}_{41}) + t^3 6\widetilde{Q}_{31} + t^4\widetilde{Q}_{21}$;

$\mathbf{P_9}$:
$\mathbf{12\widetilde{Q}_{421}} + t12\widetilde{Q}_{321}$.

The lower codimensional classes are displayed in [18], where the expression for $\mathbf{A_7}$ should read:

$\mathbf{A_7}$:
$\mathbf{135\widetilde{Q}_{321}} + \mathbf{1275\widetilde{Q}_{42}} + \mathbf{2004\widetilde{Q}_{51}} + \mathbf{2520\widetilde{Q}_{6}} +$
$t(7092\widetilde{Q}_5 + 4439\widetilde{Q}_{41} + 1713\widetilde{Q}_{32}) + t^2(3545\widetilde{Q}_{31} + 7868\widetilde{Q}_4) +$
$t^3(1106\widetilde{Q}_{21} + 4292\widetilde{Q}_3) + t^4 1148\widetilde{Q}_2 + t^5 120\widetilde{Q}_1$.

12 Appendix: A positivity result for the Lagrange Grassmann bundle

In this appendix, using the setting of Section 5, we shall give a certain new positivity result.

We assume here that X is homogeneous. For any automorphism of X which is covered by a map of ξ and α_i's, we obtain an automorphism of $LG(V,\omega) \to X$ transforming the fibers to fibers.

Inspired by the paper [1] of Anderson, we consider some "large group" action. Assume that the line bundles:

$$\alpha_i^* \otimes \alpha_j \ \text{ for } i < j \quad \text{and} \quad \alpha_i^* \otimes \alpha_j^* \otimes \xi \ \text{ for all } i, j\,,$$

are globally generated. Consider the group ΓB^- of global sections of the bundle $B^- \to X$.

Lemma 18 *For each point $x \in X$, the restriction map from ΓB^- to the fiber B_x^- is surjective.*

Proof. The group B_x^- is generated by two subgroups:

- B_W^-: the automorphisms of W inducing automorphisms of $W^* \otimes \xi$ which preserve the flag $F_\bullet^-$,
- N^-: the maps $W \to W^* \otimes \xi$ which belong to

$$\mathrm{Sym}^2(W^*) \otimes \xi \subset W^* \otimes W^* \otimes \xi = \mathrm{Hom}(W, W^* \otimes \xi)\,.$$

We identify the elements of B_W^- with matrix $\{b_{ij}\}$, whose entries over $x \in X$ belong to the fiber

$$\mathrm{Hom}(\alpha_i, \alpha_j)_x = (\alpha_i^* \otimes \alpha_j)_x$$

for $i \leq j$, or are zero for $i > j$. The group bundle B_W^- is generated by the global sections. Similarly, the group bundle N^- is a quotient bundle of

$$\mathrm{Hom}(W, W^* \otimes \xi) \cong W^* \otimes W^* \otimes \xi$$

isomorphic to $\mathrm{Sym}^2(W^*) \otimes \xi$; therefore it is globally generated. This proves the lemma. $\square$

For a strict partition $J \subset \rho$, let us denote by Ω_J^- the space of the stratum $\Omega_J(F_\bullet^-, \xi) \to X$.

The lemma implies the following result:

Corollary 19 *The group ΓB^- acts on $LG(V, \omega)$, preserving fibers, and in each fiber its orbits coincide with the strata of the stratification $\{\Omega_J^-\}$.*

Assume now, more precisely, that X is homogeneous with respect to a linear group G and the transformation group acts on the line bundles ξ and α_i. For instance, X is a product of projective spaces, and each line bundle involved is a tensor product of the $\mathcal{O}(j)$'s.

We define H to be the subgroup of $\mathrm{Aut}(LG(V, \omega))$ generated by ΓB^- and G (it is the semidirect product of these groups). The variety H is irreducible.

From the above, we obtain the following lemma:

Lemma 20 *The group H acts transitively on each stratum Ω_J^-: G transports any fiber to any other fiber, and ΓB^- acts transitively inside the fibers.*

We now state the following positivity result:

Theorem 21 *The intersection of any nonnegative cycle on $LG(V, \omega)$ with any $\overline{Z_{I\lambda}^+}$ is represented by a nonnegative cycle.*

Proof. Let Y be a nonnegative cycle on $LG(V, \omega)$. We shall find a translate $h \cdot Y$ by an element $h \in H$ which is transverse to any $Z_{I\lambda}^+$.

By Lemma 20, we can use the Bertini-Kleiman transversality theorem [12] for H acting on Ω_J^-. By this theorem, there exists an open, dense subset $U_{JI\lambda} \subset H$ with the following property: if $h \in U_{JI\lambda}$, then $h \cdot (Y \cap \Omega_J^-)$ is transverse to $Z_{I\lambda}^+ \cap \Omega_J^-$. Set

$$U_J := \bigcap_{I,\lambda} U_{JI\lambda}. \tag{33}$$

We get an open, dense subset $U_J \subset H$ with the following property: if $h \in U_J$, then $h \cdot (Y \cap \Omega_J^-)$ is transverse to any $Z_{I\lambda}^+ \cap \Omega_J^-$ (transversality in Ω_J^-). Since

Ω_J^- is transverse to all strata $Z^+_{I\lambda}$ of $LG(V,\omega)$, this transversality holds also in the whole ambient space. Set

$$U := \bigcap_{\text{strict } J\subset\rho} U_J. \tag{34}$$

Pick $h \in U$. Then $Y' = h \cdot Y$ is transverse to all the $Z^+_{I\lambda}$'s.

The theorem now follows since by [6, Sect. 8.2] the intersection

$$Y' \cdot [\overline{Z^+_{I\lambda}}]$$

is represented by a nonnegative cycle. □

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