Séminaire Lotharingien de Combinatoire **52** (2005), Article B52d

Architectonique des formules préférées d'Alain Lascoux

Piotr Pragacz*
Institute of Mathematics of Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
P.Pragacz@impan.gov.pl

Abstract

This article, written in December 2004, is an expanded version of the author's lecture opening the LascouxFest at the Séminaire Lotharingien de Combinatoire in Domaine Saint-Jacques, Ottrott, March 29-31, 2004. We discuss here some aspects of the work of Alain Lascoux (and some of his coworkers), related to symmetric functions and, more generally, Schubert polynomials. We illustrate some of the techniques he uses: determinants, transformations of alphabets, reproducing kernels, planar displays, divided differences, and vertex operators. The aim of this article is to show to the reader working in Algebraic Combinatorics (and others!) what we can learn from Alain to make our computations more efficient and more exciting.

1 Introduction

I[1] met Alain in Spring 1978 during his first visit to Poland. This was just after his famous Thesis [24] had been passed. He taught us the combinatorics of syzygies of determinantal ideals, using the combinatorics of the Bott theorem on cohomology of homogeneous bundles on flag varieties. His 1977 Thesis introduced to Commutative Algebra and Combinatorics *Schur functors*, a functorial version of Schur functions, allowing one to express

*Supported by KBN grant 2P03A 024 23.

[1] I believe that Marcel-Paul Schützenberger would be much more competent (than me) to tell you about the work of Alain. He was the closest coworker of Alain for many years. Also, Bernard Leclerc and Jean-Yves Thibon would surely accomplish this job better than me – their collaboration with Alain, though shorter in time, was/is more intensive. I will not cover here all aspects of Alain's work. For example, I will not touch the Kostka-Foulkes, Kazdhan-Lusztig, and Macdonald polynomials, noncommutative methods, and the singularities of Schubert varieties. I will concentrate on my 25 year collaboration with Alain and on topics that seem to be in the "intersection" of the interests of the audience of SLC and those of myself.

the syzygies of determinantal ideals (previously only symmetric and exterior powers were used, as well as *hook* Schur functors – the latter by D. Buchsbaum and D. Eisenbud [5]).

Apart from the Bott theorem (for GL_n)[2], he used a functorial version of *the Cauchy formula* for symmetric and exterior powers of the tensor product of two vector bundles. I believe that the Cauchy formula is for Alain a prototype of a "good" algebraic formula. Therefore, in the next section we discuss several incarnations of it. In particular, the Cauchy formula gives us a certain "reproducing kernel". In various places of this article, I shall show other reproducing kernels – one of the leitmotivs of Alain's algebraic way of thinking.

Already in early Alain's papers (cf. [22], [23]), some determinants show up. Determinants and more generally, minors, play an important role in Alain's work. Often he studies them via various deformations, and especially by modifying the alphabets that are arguments of Schur functions. We discuss determinantal techniques and some determinantal expressions in Section 3.

In the beginning of the 80's, Alain discovered with Marcel-Paul Schützenberger that the classical Schur functions are a very particular case of *(simple) Schubert polynomials* defined by using divided differences (of Newton). Even more natural are *(double) Schubert polynomials* which, defined originally algebro-combinatorially, have a transparent geometric interpretation: they are cohomology classes of flagged degeneracy loci. There is an important analogue of the Cauchy formula for Schubert polynomials (cf. Theorem 2 in Section 4).

Operator techniques become more and more apparent in Alain's work. Apart from divided differences, used in the theory of Schubert polynomials, and their variations (e.g. *isobaric* divided differences used in the theory of *Grothendieck polynomials*), he uses also *vertex operators*. In Section 5, I will tell you how vertex operator were helpful in the theory of $\widetilde{Q}$-polynomials.

It would be totally banal to say to the audience of SLC that planar displays are useful to see better some algebro-combinatorial properties. However, in Section 6, I wish to show you some recent examples of the use of planar displays of divided differences.

Already from Alain's Thesis, I learned about the *λ-rings of Grothendieck*. The λ-ring structure played – I believe – a large role in Alain's algebro-combinatorial thinking: it allows one to treat symmetric functions as operators on polynomials. It is truly amazing that the techniques he developed, under the general "umbrella" of λ-rings, provide a uniform approach to numerous classical polynomials (e.g. symmetric, orthogonal) and formulas (e.g. interpolation formulas or those of representation theory of general linear groups and symmetric groups). The polynomials and formulas are often

[2] Alain discovered it independently by himself (cf. [23] and [26], Addendum).

related to the names such as: E. Bézout, A. Cauchy, A. Cayley, P. Chebyshev, L. Euler, C.F. Gauss, C.G. Jacobi, J. Lagrange, E. Laguerre, A-M. Legendre, I. Newton, I. Schur, T.J. Stieltjes, J. Stirling, J.J. Sylvester, J.M. Hoene-Wroński, and others. See [30].

In fact in Section 7, I will tell you about my current work with Alain on some aspects of Euclid's algorithm. You will see again some determinants, the Lagrange interpolation, and reproducing kernels.

The last section will contain some "concluding remarks" and pictures from Nankai University in Tianjin, where Alain worked during the last years.

I will not repeat in this article the stories and jokes that I told during my talk in Ottrott, because of the rule that I try to follow: "Jokes should not be repeated."

2 The Cauchy formula

The Cauchy formula is one of Alain's favourite formulas. He told me this already during his very first lesson on symmetric functions in Paris, in November 1978. As any important formula, it has many incarnations.

I. (Cauchy kernel) Let $\mathbb{A}$, $\mathbb{B}$ be two alphabets. We have

$$\prod_{a\in A, b\in B} \frac{1}{1-ab} = \sum_I S_I(\mathbb{A}) S_I(\mathbb{B}) =: K(\mathbb{A},\mathbb{B})\,, \tag{1}$$

where the sum is over all partitions I [3].

Consider the ring $\mathfrak{Sym}$ of symmetric functions in a countable alphabet of variables. Let $(\ ,\ )$ be the scalar product on $\mathfrak{Sym}$ such that $\{S_I\}$ is an orthonormal basis. Given two such alphabets $\mathbb{A}$ and $\mathbb{B}$ and $f \in \mathfrak{Sym}$, we have

$$(f(\mathbb{A}), K(\mathbb{A},\mathbb{B})) = f(\mathbb{B}) \tag{2}$$

– a fundamental *reproducing* property.

II. (Resultant) Let $\mathrm{card}(\mathbb{A}) = m$, $\mathrm{card}(\mathbb{B}) = n$. We have

$$\prod_{a\in\mathbb{A}, b\in\mathbb{B}} (a-b) = \sum_{I\subset n^m} S_I(\mathbb{A}) S_{n^m/I}(-\mathbb{B}). \tag{3}$$

III. (Diagonal and Specht modules) Let $\mathcal{S}^I$ be the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by the partition I, $|I| = n$. Then for the diagonal embedding

$$\mathfrak{S}_n \hookrightarrow \mathfrak{S}_n \times \mathfrak{S}_n\,,$$

[3] In this section, we use French partitions in order not to offend Cauchy. The reader not familiar with Schur functions will find their definition in the next section. The notation for partitions, used in the present article, is that of [30].

we have

$$\mathrm{Ind}(\mathcal{S}^{(n)}) = \oplus_{|I|=n} \mathcal{S}^I \otimes \mathcal{S}^I\,, \tag{4}$$

where $\mathcal{S}^{(n)}$ is the one dimensional representation of $\mathfrak{S}_n$ with the trivial action.

IV. (Bitableaux) Let $X = (x_{ij})_{m\times n}$ be a matrix of indeterminates. We search for a "determinantal" additive basis of the polynomial ring $\mathbf{Z}[X]$ (instead of a familiar monomial basis). Set

$$(i_1, \ldots, i_k | j_1, \ldots, j_k) := \det(x_{i_p j_q})_{1\le p,q\le k}\,. \tag{5}$$

In the following, when speaking about *tableaux* and *standard tableaux* we mean the corresponding planar objects explained in [30], p.176. "Transposing" will mean taking reflection in the main diagonal.

A *bitableau* is a pair of *transposed tableaux*. These transposed tableaux must have the same shape, called the *shape of a bitableau*. We display planarly a bitableau as:

$$\begin{pmatrix} a_1a_2a_3 & & b_1b_2b_3 \\ c_1c_2c_3c_4c_5 & , & d_1d_2d_3d_4d_5 \\ e_1e_2e_3e_4e_5e_6e_7e_8 & & f_1f_2f_3f_4f_5f_6f_7f_8 \end{pmatrix}$$

(here the shape is 358), where $a_i, c_j, e_k \le m$ and $b_i, d_j, f_k \le n$.

Here is an example of a bitableau of shape 245:

$$\begin{pmatrix} 34 & & 78 \\ 2459 & , & 2367 \\ 23567 & & 13689 \end{pmatrix}.$$

With a bitableau, we associate the product of minors:

$$(a_1, a_2, a_3|b_1, b_2, b_3)\ (c_1, c_2, c_3, c_4, c_5|d_1, d_2, d_3, d_4, d_5) \\ \times (e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8|f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)\,.$$

Now a basic theorem says that *products of minors associated with bitableaux form an additive basis of* $\mathbf{Z}[X]$. Several mathematicians contributed to this result; the "most combinatorial" references, containing the *Rota straightening formula*, are [10] and [9].

Alain in his Thesis considers a characteristic zero variant of this result: given two vector spaces or vector bundles,

$$S^n(E \otimes F) = \oplus_{|I|=n} V_I(E) \otimes V_I(F)\,, \tag{6}$$

where $V_I(E)$ is the polynomial representation of $GL(E)$, called by Alain the *Schur functor*, corresponding to the weight I (and invented in Schur's 1901 Thesis [55]).

In fact, in Alain's Thesis the following parallel formula plays an even more important role:

$$\wedge^n(E \otimes F) = \oplus_{|I|=n} V_I(E) \otimes V_{I^\sim}(F)\,, \tag{7}$$

where $I^\sim$ is the conjugate partition of I. The importance of this formula comes from the fact that it describes the *Koszul syzygies* of the ideal generated by the entries of a generic matrix, and then via suitable derived functors (following a method introduced in G. Kempf's 1971 Thesis) allow one to describe syzygies of determinantal ideals [24], [26].

In terms of characters, the last decomposition corresponds to

$$\prod_{a\in\mathbb{A},b\in\mathbb{B}} (1+ab) = \sum_I S_I(\mathbb{A}) S_{I^\sim}(\mathbb{B})\,. \tag{8}$$

V. (Robinson-Schensted-Knuth correspondence) This is a favourite way of seeing the Cauchy formula by combinatorialists. Its simplest variant asserts a bijection between permutations in $\mathfrak{S}_n$ and pairs of standard tableaux of the same shape of weight n.

For example, the R-S-K correspondence associates with the permutation $[3,6,4,1,7,2,5]$ [4] the last pair of standard tableaux of shape 133 in the following display:

$$
\begin{array}{l}
3\,,\ \mathit{1} \longrightarrow 36\,,\ \mathit{12} \longrightarrow \begin{array}{l}6\\34\end{array},\ \begin{array}{l}\mathit{3}\\\mathit{12}\end{array} \\[2ex]
\longrightarrow \begin{array}{l}6\\3\\14\end{array},\ \begin{array}{l}\mathit{4}\\\mathit{3}\\\mathit{12}\end{array} \longrightarrow \begin{array}{l}6\\3\\147\end{array},\ \begin{array}{l}\mathit{4}\\\mathit{3}\\\mathit{125}\end{array} \\[2ex]
\longrightarrow \begin{array}{l}6\\34\\127\end{array},\ \begin{array}{l}\mathit{4}\\\mathit{36}\\\mathit{125}\end{array} \longrightarrow \begin{array}{l}6\\347\\125\end{array},\ \begin{array}{l}\mathit{4}\\\mathit{367}\\\mathit{125}\end{array}
\end{array}
$$

To be more precise, what corresponds to the Cauchy formula is a similar bijection between monomials in commutative *biletters* and pairs of tableaux of the same shape (cf. [17]).

According to Alain, the Cauchy formula gathers many (if not all) features of a "good" algebraic formula: firstly, because it gives us some useful reproducing kernel; secondly, because of its functorial character; thirdly, because it is an interesting result on determinants having a natural "bijective interpretation" with the help of planar displays. In the forthcoming sections, I will show you several variations on these themes.

[4] Here we follow the convention of writing a permutation $\sigma \in \mathfrak{S}_n$ by its sequence of values: $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$.

We also signal that a variant of a “Cauchy formula” for *loop groups* was used by G. Faltings in his derivation of the famous *Verlinde formula* (for the moduli space of vector bundles on a curve) – see Theorem 2.4 in [11].

3 Determinants

Everybody (since Jacobi) knows that Schur functions are given by determinants in *complete* or *elementary* symmetric functions. One of the very first things on symmetric functions that I learned from Alain was that the former are more important that the latter. This was also claimed by J.M. Hoene-Wroński (1778-1853), a Polish-French universal scientist admired by Alain (cf. [28]).

Castle Kórnik near Poznań in Poland, containing a large collection of the *original* manuscripts of Wroński. They probably contain interesting mathematical things still to be (re)discovered.

Working with a determinantal presentation of Schur functions, Alain always considers them as minors of a large matrix:

$$\begin{array}{cccccc} S_0 & S_1 & S_2 & S_3 & S_4 & \ldots \\ 0 & S_0 & S_1 & S_2 & S_3 & \ldots \\ 0 & 0 & S_0 & S_1 & S_2 & \ldots \\ 0 & 0 & 0 & S_0 & S_1 & \ldots \\ 0 & 0 & 0 & 0 & S_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Usually $S_i = S_i(\mathbb{A})$ where $\mathbb{A}$ is an *alphabet* or even a *virtual alphabet.* More precisely, given $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_n) \in \mathbf{Z}^n$, he sets

$$S_{J/I}(\mathbb{A}) := \det(S_{j_k - i_h + k - h}(\mathbb{A}))_{1 \le h,k \le n} \,. \tag{9}$$

If I consists of zeros, such a determinant is, up to sign, a Schur function indexed by a French partition:

$$S_{5,7,2,0,3} = \begin{vmatrix} S_5 & S_8 & S_4 & S_3 & S_7 \\ S_4 & S_7 & S_3 & S_2 & S_6 \\ S_3 & S_6 & S_2 & S_1 & S_5 \\ S_2 & S_5 & S_1 & 1 & S_4 \\ S_1 & S_4 & 1 & 0 & S_3 \end{vmatrix} = \begin{vmatrix} S_3 & S_4 & S_5 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_5 & S_6 \\ 1 & S_1 & S_2 & S_4 & S_5 \\ 0 & 1 & S_1 & S_3 & S_4 \end{vmatrix} = S_{3,3,3,4,4} \,.$$

Among minors of a given matrix, there are many relations: Laplace expansions, Plücker relations, etc. that can be used to get interesting relations for Schur functions. Alain recommended, many times, that I go through Muir's treatment of determinants in five volumes [50]. For him [50] is an example of an extremely useful "encyclopaedic" piece of work, unfortunately rather rare in Mathematics.[5]

In particular, a familiar *Binet-Cauchy formula* for the minors of a product of matrices (cf. [50], 1812, chap. IV), implies

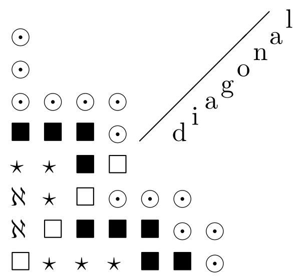
$$S_{J/I}(\mathbb{A}+\mathbb{B}) = \sum_K S_{J/K}(\mathbb{A}) S_{K/I}(\mathbb{B}) \,. \tag{10}$$

I want now to discuss some applications of the *Bazin-Sylvester identity* for minors. Suppose that an $\infty \times n$ matrix is given. For a subset $A \subset \mathbb{N}$ with $\mathrm{card}(A) = n$, we denote by $[A]$ the $n \times n$ minor of the matrix taken on rows indexed by A. Now suppose that three subsets $A, B, C \subset \mathbb{N}$ are given with $\mathrm{card}(A) = \mathrm{card}(C) = p \leq n$ and $\mathrm{card}(A) + \mathrm{card}(B) = n$. Then

$$\det([ABc \setminus a])_{a\in A, c\in C} = [AB]^{p-1}[BC] \,. \tag{11}$$

In [34] this identity was used to obtain a generalization to *skew* Schur functions of the classical *Schubert-Giambelli identity* presenting a Schur function in terms of *hook* Schur functions.

Afterwards, Alain and the author used it to derive in [35] the following determinantal formula for a Schur function in terms of *ribbon* Schur functions. Take the partition $J = 11444677$. The following picture shows the "ribbon-decomposition" of the diagram of J:



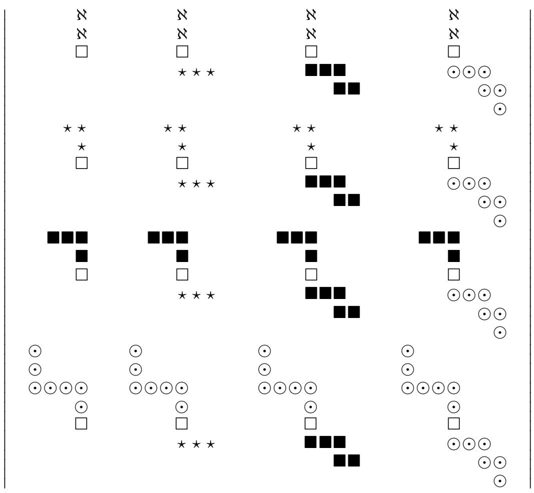
[5] Most interesting and useful determinantal relations were "systematized" by B. Leclerc in [47].

Consider now the matrix:

$$\begin{bmatrix} S_1 & S_2 & S_6 & S_7 & S_8 & S_{11} & S_{13} & S_{14} \\ S_0 & S_1 & S_5 & S_6 & S_7 & S_{10} & S_{12} & S_{13} \\ \bullet & S_0 & S_4 & S_5 & S_6 & S_9 & S_{11} & S_{12} \\ \bullet & \bullet & S_3 & S_4 & S_5 & S_8 & S_{10} & S_{11} \\ \bullet & \bullet & S_2 & S_3 & S_4 & S_7 & S_9 & S_{10} \\ \bullet & \bullet & S_1 & S_2 & S_3 & S_6 & S_8 & S_9 \\ \bullet & \bullet & S_0 & S_1 & S_2 & S_5 & S_7 & S_8 \\ \bullet & \bullet & \bullet & S_0 & S_1 & S_4 & S_6 & S_7 \\ \bullet & \bullet & \bullet & \bullet & S_0 & S_3 & S_5 & S_6 \\ \bullet & \bullet & \bullet & \bullet & \bullet & S_0 & S_2 & S_3 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & S_0 & S_1 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & S_0 \end{bmatrix},$$

where we use $\bullet$ for S_i, $i < 0$. Let $A = \{9, 12, 14, 15\}$, $B = \{\mathit{2}, \mathit{3}, \mathit{7}, \mathit{8}\}$, $C = \{\mathbf{1}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$. Then Bazin-Sylvester's formula expresses $S_{11444677}$ as the following determinant of 8×8-minors:

$$\begin{vmatrix} [\mathit{23}\mathbf{6}\mathit{78}, 12, 14, 15] & [\mathit{23}\mathbf{6}\mathit{789}, 14, 15] & [\mathit{23}\mathbf{6}\mathit{789}, 12, 15] & [\mathit{23}\mathbf{6}\mathit{789}, 12, 14] \\ [\mathit{23}\mathbf{5}\mathit{78}, 12, 14, 15] & [\mathit{23}\mathbf{5}\mathit{789}, 14, 15] & [\mathit{23}\mathbf{5}\mathit{789}, 12, 15] & [\mathit{23}\mathbf{5}\mathit{789}, 12, 14] \\ [\mathit{23}\mathbf{4}\mathit{78}, 12, 14, 15] & [\mathit{23}\mathbf{4}789, 14, 15] & [\mathit{23}\mathbf{4}\mathit{789}, 12, 15] & [\mathit{23}\mathbf{4}\mathit{789}, 12, 14] \\ [\mathbf{1}\mathit{2378}, 12, 14, 15] & [\mathbf{1}\mathit{23789}, 14, 15] & [\mathbf{1}\mathit{23789}, 12, 15] & [\mathbf{1}\mathit{23789}, 12, 14] \end{vmatrix} =$$

=  .

With this "generic example" the reader easily sees a general pattern. It is also possible to get the ribbon determinant by a suitable deformation of the Schubert-Giambelli hook determinant (cf. [35] and Section 6).

A generalization of this result to "super-Schur" functions has recently been obtained by W.Y.C. Chen, G-G. Yan, and A.L.B. Yang [6].

Ribbon Schur functions admit a lift to the noncommutative situation, and play a basic role in the theory of *noncommutative* symmetric functions (cf., e.g., [60]).

For a more systematic development of the theory of symmetric functions, the reader is referred to Macdonald's book [48]. Alain's recent book [30] exploits a more "λ-ring approach" to symmetric functions which allows one to treat them as operators on polynomials. Among the techniques used most frequently, we mention "transformations and specializations of alphabets" (see also [45], [36]).

Schur functions admit a useful generalization to (no more symmetric) *multi-Schur functions.* Given $I = (i_1, i_2, \dots, i_r) \in \mathbb{N}^r$, and alphabets $\mathbb{A}_1, \dots, \mathbb{A}_r$, $\mathbb{B}_1, \dots, \mathbb{B}_r$, the *multi-Schur function* $S_I(\mathbb{A}_1 - \mathbb{B}_1, \dots, \mathbb{A}_r - \mathbb{B}_r)$ is

$$S_I(\mathbb{A}_1 - \mathbb{B}_1, \dots, \mathbb{A}_r - \mathbb{B}_r) := \det(S_{i_k+k-h}(\mathbb{A}_k - \mathbb{B}_k))_{1 \le h,k \le r} \; . \tag{12}$$

For example, $S_{5,7,2,0,3}(\mathbb{A}-\mathbb{B}, \mathbb{C}-\mathbb{D}, \mathbb{E}-\mathbb{F}, \mathbb{G}-\mathbb{H}, \mathbb{K}-\mathbb{L})$ equals

$$\begin{vmatrix} S_5(\mathbb{A}-\mathbb{B}) & S_8(\mathbb{C}-\mathbb{D}) & S_4(\mathbb{E}-\mathbb{F}) & S_3(\mathbb{G}-\mathbb{H}) & S_7(\mathbb{K}-\mathbb{L}) \\ S_4(\mathbb{A}-\mathbb{B}) & S_7(\mathbb{C}-\mathbb{D}) & S_3(\mathbb{E}-\mathbb{F}) & S_2(\mathbb{G}-\mathbb{H}) & S_6(\mathbb{K}-\mathbb{L}) \\ S_3(\mathbb{A}-\mathbb{B}) & S_6(\mathbb{C}-\mathbb{D}) & S_2(\mathbb{E}-\mathbb{F}) & S_1(\mathbb{G}-\mathbb{H}) & S_5(\mathbb{K}-\mathbb{L}) \\ S_2(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{C}-\mathbb{D}) & S_1(\mathbb{E}-\mathbb{F}) & 1 & S_4(\mathbb{K}-\mathbb{L}) \\ S_1(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{C}-\mathbb{D}) & 1 & 0 & S_3(\mathbb{K}-\mathbb{L}) \end{vmatrix} .$$

The following simple lemma is masterfully exploited by Alain in numerous situations (cf., e.g., [45], [30]):

Lemma 1 (Jacobi-Lascoux Lemma) *Let* $\mathbb{D}_0, \mathbb{D}_1, \dots, \mathbb{D}_{n-1}$ *be alphabets such that* $\mathrm{card}(\mathbb{D}_i) \le i$. *Then*

$$S_I(\mathbb{A}_1 - \mathbb{B}_1, \dots, \mathbb{A}_r - \mathbb{B}_r) = \det(S_{j_k - i_h + k - h}(\mathbb{A}_k - \mathbb{B}_k - \mathbb{D}_{r-h}))_{1 \le h,k \le r} \, . \tag{13}$$

In other words, we can replace, in row h, $\mathbb{A}_\bullet - \mathbb{B}_\bullet$ *by* $\mathbb{A}_\bullet - \mathbb{B}_\bullet - \mathbb{D}_{r-h}$ *without changing the determinant.*

This is easily seen from the relation:

$$S_i(\mathbb{A}-\mathbb{B}-\mathbb{D}_h) = S_i(\mathbb{A}-\mathbb{B}) + S_1(-\mathbb{D}_h) S_{i-1}(\mathbb{A}-\mathbb{B}) + \cdots \\ \cdots + S_h(-\mathbb{D}_h) S_{j-h}(\mathbb{A}-\mathbb{B}) \, , \tag{14}$$

by subtracting from successive rows of the LHS-determinant, appropriate combinations of its lower rows.

For example,

$$\begin{vmatrix} S_i(\mathbb{A}) & S_{j+1}(\mathbb{B}) & S_{k+2}(\mathbb{C}) \\ S_{i-1}(\mathbb{A}) & S_j(\mathbb{B}) & S_{k+1}(\mathbb{C}) \\ S_{i-2}(\mathbb{A}) & S_{j-1}(\mathbb{B}) & S_k(\mathbb{C}) \end{vmatrix} = \begin{vmatrix} S_i(\mathbb{A}-y-z) & S_{j+1}(\mathbb{B}-y-z) & S_{k+2}(\mathbb{C}-y-z) \\ S_{i-1}(\mathbb{A}-x) & S_j(\mathbb{B}-x) & S_{k+1}(\mathbb{C}-x) \\ S_{i-2}(\mathbb{A}) & S_{j-1}(\mathbb{B}) & S_k(\mathbb{C}) \end{vmatrix} .$$

In particular, the Jacobi-Lascoux Lemma allows one to present a monomial in a determinantal way:

$$a^5b^3c^2 = S_{2,3,5}(a+b+c, a+b, a)\,,$$

which sometimes appears to be useful.

4 Schubert polynomials

Schur functions and multi-Schur functions are a very particular case of a family of *Schubert polynomials* invented by Alain and M-P. Schützenberger in 1982 ([42]). We shall follow the convention (used in [42]) that operators acts on their *left*.

Let n be a fixed positive integer. The symmetric group $\mathfrak{S}_n$ is the group with generators $s_1, \ldots, s_{n-1}$, where

$$s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n]\,, \tag{15}$$

subject to the relations:

$$s_i^2 = 1\,, \quad s_{i-1}s_is_{i-1} = s_is_{i-1}s_i\,, \quad \text{and} \quad s_is_j = s_js_i \quad \forall i,j : |i-j| > 1\,.$$

Let $\mathbb{A} = (a_1, \ldots, a_n)$ be an alphabet of variables. The group $\mathfrak{S}_n$ acts on the polynomial ring $\mathbf{Z}[\mathbb{A}]$ by permuting the variables. We define, following Newton, a (simple) divided difference $\partial_i : \mathbf{Z}[\mathbb{A}] \to \mathbf{Z}[\mathbb{A}]$, by

$$f\partial_i = \frac{f - f^{s_i}}{a_i - a_{i+1}} \tag{16}$$

for $f \in \mathbf{Z}[\mathbb{A}]$, where $f^\sigma := f(a_{\sigma_1}, \ldots, a_{\sigma_n})$ for $\sigma \in \mathfrak{S}_n$.

The ∂_i's satisfy the Moore-Coxeter relations. Thus for $\sigma \in \mathfrak{S}_n$, given its reduced decomposition $\sigma = s_{i_1}\, s_{i_2}\, \cdots s_{i_\ell}$, the operator

$$\partial_\sigma := \partial_{i_1}\, \partial_{i_2}\, \cdots \partial_{i_\ell} \tag{17}$$

is well defined (i.e. does not depend on a reduced decomposition of σ).

Let $\mathbb{B} = (b_1, \ldots, b_n)$ be another alphabet of variables. The definition of Schubert polynomials $\mathbb{X}(\mathbb{A}, \mathbb{B})$ (often called *double Schubert polynomials*[6]) of Alain and M-P. Schützenberger goes as follows. For $\omega = [n, n-1, \ldots, 1]$,

$$\mathbb{X}_\omega(\mathbb{A}, \mathbb{B}) = \prod_{i+j \le n} (a_i - b_j)\,, \tag{18}$$

[6] It seems that we need a deeper reflection on "double" versus "single" in various mathematical situations.

and for arbitrary $\sigma \in \mathfrak{S}_n$,

$$\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) = \mathbb{X}_\omega(\mathbb{A},\mathbb{B})\partial_{\omega\sigma} \,. \tag{19}$$

(In fact, it makes sense to speak about Schubert polynomials for any pair of alphabets $\mathbb{A},\mathbb{B}$ of cardinality $\geq n$, cf. [30].) Here we use divided differences w.r.t. the variables $\mathbb{A}$, but the variables from $\mathbb{B}$ appear just as systematically because of the equation:

$$X_\sigma(\mathbb{A},\mathbb{B}) = (-1)^{l(\sigma)} X_{\sigma^{-1}}(\mathbb{B},\mathbb{A}) \,. \tag{20}$$

Schubert polynomials satisfy the following fundamental *vanishing property*: all positive degree Schubert polynomials vanish under the specialization $\mathbb{A} = \mathbb{B}$.

When we specialize $\mathbb{B}$ with zeros, we get *simple Schubert polynomials* $X_\sigma(\mathbb{A})$; note that $X_\omega(\mathbb{A}) = a_1^{n-1}a_2^{n-2}\cdots a_{n-1}$. For a didactic exposition of Schubert polynomials, see [49]. *Schubert functors* having characters equal to the corresponding simple Schubert polynomials were constructed by W. Kraśkiewicz and the author in [19] (see also [46]).

There is an associated reproducing kernel in this story. Consider the ring: $\mathfrak{P}ol = \mathbf{Z}[\mathbb{A},\mathbb{B}]/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by all $f(\mathbb{A}) - f(\mathbb{B})$ where f is symmetric. On $\mathfrak{P}ol$ there is a scalar product with values in $\mathfrak{S}ym(\mathbb{A})$, defined by

$$(f,g) := fg\partial_\omega \tag{21}$$

for $f,g \in \mathfrak{P}ol$. Note that the *Jacobi symmetrizer* ∂_ω acts on $\mathbb{Z}[\mathbb{A}]$ as follows:

$$f \mapsto \sum_{\sigma\in\mathfrak{S}_n} \Big(\frac{f}{\prod_{i<j}(a_i - a_j)}\Big)^\sigma \,. \tag{22}$$

Set

$$\mathbf{K}(\mathbb{A},\mathbb{B}) := \prod_{1\leq i<j\leq n} (b_i - a_j) = \pm\mathbb{X}_\omega((a_n,\ldots,a_1),\mathbb{B}) \,. \tag{23}$$

For any polynomial f in n variables, in $\mathfrak{P}ol$ we have

$$(f(\mathbb{A}),\mathbf{K}(\mathbb{A},\mathbb{B})) = f(\mathbb{B}) \,. \tag{24}$$

There is the following nonsymmetric Cauchy-type formula for Schubert polynomials:

Theorem 2 (Cauchy-Lascoux Formula) *Let $\mathbb{A},\mathbb{B},\mathbb{C}$ be three alphabets of cardinality $\geq n$. Then for $\sigma \in \mathfrak{S}_n$,*

$$\mathbb{X}_\sigma(\mathbb{A},\mathbb{B}) = \sum_{\tau,\eta} \mathbb{X}_\tau(\mathbb{C},\mathbb{B})\mathbb{X}_\eta(\mathbb{A},\mathbb{C}) \,, \tag{25}$$

where the sum is over $\sigma = \tau\eta$ with $l(\sigma) = l(\tau) + l(\eta)$.

(Cf. [30], Theorem 10.2.6.) The most remarkable thing about the Cauchy-Lascoux formula is that we can take an arbitrary(!) alphabet $\mathbb{C}$. To be just modest, for $\mathbb{C} = (0, \ldots)$, Eq. (25) gives a quadratic expression of a Schubert polynomial in terms of simple Schubert polynomials in $\mathbb{A}$ and $\mathbb{B}$. When $\sigma = \omega$, we get an expression for the total Chern class of the full flag manifold (cf. [27] – the paper where (double) Schubert polynomials originally appeared). When $\sigma = [n+1, \ldots, n+m, 1, \ldots, m]$, we get an expression for the total Chern class of the Grassmannian (i.e., the Cauchy formula for the resultant from Section 1).

In 1978, during his first visit to Poland, Alain said to me a sentence which impressed me a lot at that time:

"The Schubert variety is a Schur function." [7]

This was probably the fastest introduction to the cohomology ring of a Grassmannian. I am pretty sure that what Alain does in Algebra or Combinatorics is often deeply motivated by Geometry (of Grassmannians, flag varieties, Schubert varieties etc.) This is also the case for Schubert polynomials. In [13], W. Fulton showed that the polynomials $\mathbb{X}_\sigma(\mathbb{A}, \mathbb{B})$ are the cohomology classes of *flagged degeneracy loci* associated with a morphism of vector bundles with *Chern roots* $\mathbb{A}$ and $\mathbb{B}$. The key point is that divided differences admit a geometric interpretation as *correspondences* in flag bundles (*loc.cit*). This is also a proper place to mention the results of I.N. Bernstein, I.M. Gelfand and S.I. Gelfand [1], and M. Demazure [7], [8] from the 70's, giving formulas for Schubert classes for flag manifolds in terms of divided differences (for any semisimple group); however in the SL_n-case their expressions were more complicated than the ones given by Schubert polynomials. In [53], one can find more details on these and related issues.

As far as I know, one of the main motivations for introducing Schubert polynomials was the authors' fascination for *Newton interpolation*. In [43] (see also [30], pp. 148–149), the following formula was given. Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two alphabets of variables of cardinalities n and ∞, respectively. Let $\mathfrak{S}_\infty$ be the infinite symmetric group acting in a natural way on $\mathbb{B}$. For any polynomial f in n variables, we have

$$f(\mathbb{A}) = \sum_\sigma f(b_1, \ldots, b_n) \partial_{\sigma^{-1}} \ \mathbb{X}_\sigma(\mathbb{A}, \mathbb{B}) \ , \tag{26}$$

where the sum is over $\sigma \in \mathfrak{S}_\infty$ satisfying $\sigma(n+1) < \sigma(n+2) < \ldots$, and the operators ∂_σ act on the variables from $\mathbb{B}$.

For $n = 1$, the sum is over the identity permutation and permutations $\sigma^{(i)} = s_i s_{i-1} \cdots s_1$, where $i \geq 1$. We have

$$\mathbb{X}_{\sigma^{(i)}}(\mathbb{A}, \mathbb{B}) = (a - b_1)(a - b_2) \cdots (a - b_i) \, .$$

[7] By extending the message of this sentence, I was happy (in 1986) to communicate to Alain that "The Lagrangian Schubert class is a Schur Q-function." (cf. [51], [52]).

Therefore (26) reads:

$$f(a) = f(b_1) + f(b_1)\partial_1 \cdot (a - b_1) + f(b_1)\partial_1\partial_2 \cdot (a - b_1)(a - b_2) \\ + f(b_1)\partial_1\partial_2\partial_3 \cdot (a - b_1)(a - b_2)(a - b_3) + \cdots ,$$

which is the Newton Interpolation Formula. Note that Newton's interpolation and that of Lagrange are used in [12], written with Amy M. Fu, to deduce some new q-identities.

Alain and M-P. Schützenberger invented also *Grothendieck polynomials* which evaluate K-theoretic classes of the structure sheaves of Schubert varieties. This was done with the help of *isobaric divided differences* and led to Combinatorial K-theory (cf., e.g., [4]). Other important polynomials defined with the help of isobaric divided differences are *Key polynomials.* Let $\mathbb{A}$ be a countable alphabet of variables. Given a permutation σ, let $c = (c_1, c_2, \ldots)$ denote its *code*, where

$$c_i = \mathrm{card}\{j > i : \sigma_j < \sigma_i\} \tag{27}$$

for $i = 1, 2 \ldots$. The group $\mathfrak{S}_\infty$ acts on codes by permuting their components.

Now, if σ is *dominant* (i.e., c is weakly decreasing), then

$$K_\sigma(\mathbb{A}) := X_\sigma(\mathbb{A}) = a^c = a_1^{c_1} a_2^{c_2} \cdots , \tag{28}$$

a monomial. Otherwise, if c and i are such that $c_i < c_{i+1}$, then

$$K_{cs_i}(\mathbb{A}) = K_c(\mathbb{A})\pi_i , \tag{29}$$

where π_i is a (simple) isobaric divided difference defined for $f \in \mathbf{Z}[\mathbb{A}]$ by

$$f\pi_i = \frac{a_i f - a_{i+1} f^{s_i}}{a_i - a_{i+1}} . \tag{30}$$

The Schubert polynomial is a *nonnegative* combination of Key polynomials. For example,

$$X_{2154367\ldots} = K_{30010\ldots} + K_{10210\ldots} + K_{2020\ldots} .$$

Since Key polynomials also have a geometric meaning (they are related to sections of line bundles on Schubert varieties (cf. [14])), this nonnegativity should have a geometric explanation.

Finally, note that Key polynomials are used in [29] to give a "Cauchy-type" quadratic expression for the product:

$$\prod_{i+j \le n+1} (1 - a_i b_j)^{-1} .$$

The Jacobi symmetrizer ∂_ω, used above to define the scalar product, is a "queen mother" of many other *symmetrizing operators.* Apart from their use in interpolation theory, numerical analysis, and cohomology theory of flag manifolds, such operators are useful in algebraic computations. Consider the operator $\pi = \pi_\omega$ (defined using isobaric divided differences and a reduced decomposition of ω). Using symmetrization, its action on the ring of series in $a_1, \ldots, a_n$ is described as follows:

$$f \mapsto \sum_{\sigma \in \mathfrak{S}_n} \Big(f \cdot \prod_{i<j} (1 - \frac{a_j}{a_i})^{-1} \Big)^\sigma . \tag{31}$$

Two basic properties of π are:

$$f \text{ symmetrical} \Rightarrow fg\pi = f(g\pi) , \tag{32}$$

$$a_1^{i_1} \cdots a_n^{i_n} \pi = S_I(\mathbb{A}) \text{ when } I \in \mathbb{N}^n . \tag{33}$$

Schur functions are *eigenfunctions* of π, i.e. for $m \le n$, $I \in \mathbb{N}^m$, $J \in \mathbb{N}^{n-m}$, we have

$$S_I(a_1 + \cdots + a_m)\ S_J(a_{m+1} + \cdots + a_n)\ \pi = S_{IJ}(\mathbb{A}) . \tag{34}$$

where IJ is the element $(i_1, \ldots, i_m, j_1, \ldots, j_{n-m})$ of $\mathbb{N}^n$. For $n \ge 2$, from Eq. (34) with $m = n-1$, $I = j0 \ldots 0$, $J = 0$, it follows that

$$\prod_{i \le n-1} (1 - a_i)^{-1} \pi = \prod_{i \le n} (1 - a_i)^{-1} . \tag{35}$$

More directly, writing

$$\prod_{i \le n-1} (1 - a_i)^{-1} = (1 - a_n) \prod_{i \le n} (1 - a_i)^{-1},$$

we can use (32) and recover Eq. (35):

$$\prod_{i \le n-1} (1 - a_i)^{-1} \pi = 1 - a_n\ \pi \cdot \prod_{i \le n} (1 - a_i)^{-1} = \prod_{i \le n} (1 - a_i)^{-1} .$$

It is not hard to check the following identities:

$$\prod_{i \le n-1} (1 - a_i a_n) \pi = 1 \ \ (\text{resp. } 1 - a_1 \cdots a_n) \ \ \text{for } n \text{ odd} \ \ (\text{resp. even}) , \tag{36}$$

$$a_n \prod_{i \le n-1} (1 - a_i a_n) \pi = 0 \ \ (\text{resp. } a_1 \cdots a_n) \ \ \text{for } n \text{ even} \ \ (\text{resp. odd}) . \tag{37}$$

We illustrate on a classical formula how to use the operator π. Let us consider

$$F(n-1) = \prod_{i \le n-1} (1 - a_i)^{-1} \prod_{i < j \le n-1} (1 - a_i a_j)^{-1} .$$

Then

$$F(n{-}1)\pi = (1{-}a_n)(1{-}a_1a_n)\cdots(1{-}a_{n-1}a_n)F(n)\pi = (1{-}a_1\cdots a_n)F(n)$$

thanks to Eqs. (32), (36), and (37). Assuming now by induction that $F(n{-}1) = \sum S_I(a_1{+}\cdots{+}a_{n-1})$, summed over all partitions $I = (i_1, \ldots, i_{n-1})$, we transform this identity with the help of π into

$$(1{-}a_1\cdots a_n)F(n) = \sum S_I(a_1 + \cdots + a_n)\,,$$

the sum over the same I's. Dividing by $(1{-}a_1\cdots a_n)$, we finally obtain the identity of Schur:

$$\prod_{i\le n}(1{-}a_i)^{-1}\prod_{i<j\le n}(1{-}a_ia_j)^{-1} = \sum S_J(a_1 + \cdots + a_n)\,, \tag{38}$$

with the sum now over all partitions $J = (j_1, \ldots, j_n)$.

Also other identities for *S-function series* can be proven in a similar way. We refer the reader to [36], where, e.g., the S-function expansion of

$$\prod_{i\le n}(1 - a_i + a_i^2)^{-1}\prod_{i<j\le n}(1 - a_ia_j)^{-1}$$

is given using the same method.

5 Divided differences versus vertex operators

In this section, I will tell you about the biggest mathematical surprise that came to me from Alain (about 1995).

Let n be a fixed positive integer. Besides the symmetric group (i.e. the Weyl group of type A) $\mathfrak{S}_n$, we shall consider two other Weyl groups.

The hyperoctahedral group (i.e. the Weyl group of type C) $\mathfrak{C}_n$, is an extension of $\mathfrak{S}_n$ by an element s_0 such that $s_0^2 = 1$, $s_0s_1s_0s_1 = s_1s_0s_1s_0$, and $s_0s_i = s_is_0$ for $i \ge 2$.

The Weyl group $\mathfrak{D}_n$ of type D is an extension of $\mathfrak{S}_n$ by an element $s_\square$ such that $s_\square^2 = 1$, $s_1s_\square = s_\square s_1$, $s_\square s_2s_\square = s_2s_\square s_2$, and $s_\square s_i = s_is_\square$ for $i > 2$.

Let $\mathbb{A} = (a_1, \ldots, a_n)$ be an alphabet of variables. The groups $\mathfrak{S}_n$, $\mathfrak{C}_n$, and $\mathfrak{D}_n$ act on $\mathbf{Z}[\mathbb{A}]$: $s_i(a_i) = a_{i+1}$, $s_0(a_1) = -a_1$, $s_\square(a_1) = -a_2$.

Apart from Newton's divided differences:

$$f\,\partial_i := \frac{f - f(\ldots, a_{i+1}, a_i, \ldots)}{a_i - a_{i+1}}\,,$$

we have two other divided differences associated to simple roots of type C and D:

$$f\,\partial_0 := \frac{f - f(-a_1, a_2, \ldots)}{2a_1}\,, \tag{39}$$

and

$$f\,\partial_\square := \frac{f - f(-a_2, -a_1, a_3, \ldots)}{a_1 + a_2}\,. \tag{40}$$

The ∂_i, ∂_0, $\partial_\square$ satisfy the Coxeter relations, together with the relations

$$\partial_\square^2 = 0 = \partial_i^2 \quad \text{for} \quad 0 \le i < n\ . \tag{41}$$

Therefore, to any element w of the groups $\mathfrak{C}_n$ and $\mathfrak{D}_n$, there corresponds a *divided difference* ∂_w. Any reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_\ell} = w$ gives rise to a factorization $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_\ell}$ of ∂_w ([1], [7]).

We now invoke single Schubert polynomials. Here it will be more handy to index them by the codes of permutations, and for a didactic reason we repeat their definition in this setting. Write ρ for the sequence $(n-1, \ldots, 1, 0)$. One defines recursively Schubert polynomials Y_α, for any sequence $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ such that $\alpha_j \le n - j$ for $j = 1, \ldots, n$, by

$$Y_\alpha\,\partial_i = Y_\beta\ , \quad \text{if}\ \ \alpha_i > \alpha_{i+1}\,, \tag{42}$$

where

$$\beta = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i - 1, \alpha_{i+2}, \ldots, \alpha_n)\,, \tag{43}$$

starting from $Y_\rho = x^\rho$.

If $\alpha \in \mathbb{N}^n$ is weakly decreasing, then Y_α is equal to the monomial x^α – a property already mentioned. If, on the contrary, $\alpha_1 \le \cdots \le \alpha_k$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$, for some $k \le n$, then Y_α coincides with the Schur polynomial $S_I(a_1 + \cdots + a_k)$, where $I = (\alpha_1, \ldots, \alpha_k)$.

Besides Schubert polynomials, we shall need the following *$\widetilde{Q}$–polynomials* of [54]. We set $\widetilde{Q}_i := \Lambda_i = \Lambda_i(\mathbb{A})$, the ith elementary symmetric polynomial in $\mathbb{A}$. Given two nonnegative integers $i \ge j$, we put

$$\widetilde{Q}_{i,j} := \widetilde{Q}_i \widetilde{Q}_j + 2 \sum_{p=1}^{j} (-1)^p \widetilde{Q}_{i+p} \widetilde{Q}_{j-p}\,. \tag{44}$$

We have $\widetilde{Q}_{i,i}(\mathbb{A}) = \Lambda_i(a_1^2 + \cdots + a_n^2)$.

Given any English partition[8] $I = (i_1 \ge \ldots \ge i_k)$, where we can assume k to be even, we set

$$\widetilde{Q}_I := \text{Pfaffian}(M)\,, \tag{45}$$

where $M = (m_{p,q})$ is the $k \times k$ skew-symmetric matrix with $m_{p,q} = \widetilde{Q}_{i_p, i_q}$ for $1 \le p < q \le k$.

For special attention of the audience of SLC: $\widetilde{Q}$-polynomials form an interesting basis of the ring of symmetric polynomials. A $\widetilde{Q}$-polynomial is not, in general, a nonnegative combination of Schur S-polynomials. It is,

[8] In this section, we use English partitions in order not to offend Ron King who is in the audience, and D.E. Littlewood whose result will be used here.

however, a nonnegative combination of monomials. It would be useful to establish a combinatorial rule for the coefficients of this expansion. Also, it is worth studying other combinatorial properties of $\widetilde{Q}$-polynomials (cf. [54], [58]).

Here is a sign that the $\widetilde{Q}_I$'s are well suited to divided differences. Consider the following operator:

$$\nabla := (\partial_0\partial_1 \cdots \partial_{n-1}) \cdots (\partial_0\partial_1)\partial_0 \,. \tag{46}$$

Denote by

$$\langle\ ,\ \rangle : \mathfrak{S}ym(\mathbb{A}) \times \mathfrak{S}ym(\mathbb{A}) \to \mathfrak{S}ym(a_1^2, \ldots, a_n^2) \tag{47}$$

the scalar product defined for $f, g \in \mathfrak{S}ym(\mathbb{A})$ by

$$\langle f, g\rangle := fg\,\nabla \,. \tag{48}$$

Let $\rho(n)$ denote the strict partition $(n, n-1, \ldots, 1)$. We have the following *orthogonality property* of $\widetilde{Q}$-polynomials: For strict partitions $I, J \subseteq \rho(n)$,

$$\langle \widetilde{Q}_I, \widetilde{Q}_{\rho(n)\smallsetminus J}\rangle = \pm\delta_{IJ}, \tag{49}$$

where $\rho(n) \smallsetminus J$ is the strict partition whose parts complement the parts of J in $\{n, n-1, \ldots, 1\}$. Given another alphabet $\mathbb{B}$ of n variables, define

$$\widetilde{Q}(\mathbb{A}, \mathbb{B}) := \sum \widetilde{Q}_I(\mathbb{A})\, \widetilde{Q}_{\rho(n)\smallsetminus I}(\mathbb{B})\ , \tag{50}$$

where the summation is over all strict partitions $I \subseteq \rho(n)$. The polynomial $\widetilde{Q}(\mathbb{A}, \mathbb{B})$ is a reproducing kernel for $\langle\ ,\ \rangle$: for a symmetric polynomial f in n variables,

$$\langle f(\mathbb{A})\,, \widetilde{Q}(\mathbb{A}, \mathbb{B})\rangle = \pm f(\mathbb{B})\ . \tag{51}$$

In the beginning of 90's, computing with SYMMETRICA [16], J. Ratajski and the author obtained evidence that the following two identities should be true for a strict partition $I \subset \rho(n)$:

Firstly,

$$\widetilde{Q}_I\, \partial_0\partial_1 \cdots \partial_{i_1-1} = \pm\widetilde{Q}_{(i_2,i_3,\ldots)} \,. \tag{52}$$

Secondly, using standard barred-permutation notation for elements of the group $\mathfrak{C}_n$, we have for $w(I) := [i_1, \ldots, i_{\ell(I)}, \overline{j_1}, \ldots, \overline{j_h}]\,,$

$$x^\rho \widetilde{Q}_{\rho(n)}\, \partial_{w(I)} = \pm\widetilde{Q}_I \,. \tag{53}$$

We tried to find conceptual proofs of these identities. More generally, let us consider the following operator for $k \le n$,

$$\nabla_k(n) := (\partial_0\partial_1 \cdots \partial_{n-1}) \cdots (\partial_0\partial_1 \cdots \partial_{n-k}) \,. \tag{54}$$

The problem is, how to compute with $\nabla_k(n)$? Here is an answer.

Theorem 3 ([37]) *Let $k \le n$ and let $\alpha = (\alpha_1 \le \cdots \le \alpha_k) \in \mathbb{N}^k$ with $\alpha_k \le n-k$. Suppose that $I \subseteq \rho(n)$ is a strict partition. Then the image of $\widetilde{Q}_I Y_\alpha$ under $\nabla_k(n)$ is 0 unless $n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k$ are parts of I. In this case, the image is $\pm\widetilde{Q}_J$, where J is the strict partition with parts $\{i_1, \ldots, i_{\ell(I)}\} \smallsetminus \{n-0-\alpha_1, \ldots, n-(k-1)-\alpha_k\}$.*

For example, for $n = 7$ and $k = 2$,

$$\begin{aligned}&\widetilde{Q}_{(5,4,3,2,1)}\, Y_{(2,5)}\, \nabla_2(7)\\ &= \widetilde{Q}_{(5,4,3,2,1)}\, Y_{(2,5)}\, (\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6)(\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5) = \widetilde{Q}_{(4,3,2)}\,;\end{aligned}$$

and for $k = 3$, we have

$$\widetilde{Q}_{(7,5,4,3,1)}\, Y_{(2,3,4)} \nabla_3(7) = \widetilde{Q}_{(7,4)}\,.$$

This theorem implies (52) and (53) (*loc.cit.*).

In fact the "key case" is the case $k = 1$. Suppose $n \ge p > 0$. Let $IpJ \subseteq \rho(n)$ be a strict partition. Then

$$x_1^{n-p} \widetilde{Q}_{IpJ}\, \partial_0\partial_1 \cdots \partial_{n-1} = \pm\widetilde{Q}_{IJ}\,. \tag{55}$$

If $H \subseteq \rho(n)$ is a strict partition with no part equal to p, then

$$x_1^{n-p} \widetilde{Q}_H\, \partial_0\partial_1 \cdots \partial_{n-1} = 0\,. \tag{56}$$

After digesting our computations, Alain conceived the idea of using *vertex operators* [9]. These are some (formally) infinite differential operators (mixing multiplication and differentiation) born in the theory of Kac-Moody algebras and inspired by Mathematical Physics. In the proof below, where operators act on symmetric functions, they serve to give "long" but useful expressions for s_0 and ∂_0.

Sketch of proof of (55) and (56) Let (,) be the standard inner product on $\mathfrak{S}ym$, the ring of symmetric functions in a countable set of variables $a_1, a_2, \ldots$. We shall use the *Foulkes' derivative* D_f for $f \in \mathfrak{S}ym$:

$$(gD_f, h) = (g, f \cdot h)\,.$$

We shall treat polynomials as operators acting by multiplication. We have (cf. [37], Lemma 5.6):

$$s_0 = 1 - 2D_{P_1}a_1 + 2D_{P_2}a_2 - 2D_{P_3}a_3 + \cdots\,. \tag{57}$$

Here, P_i is the sum of all hook Schur functions of degree i (called Schur's P-functions, well-established among the audience of SLC [32]). Thus

$$\partial_0 = D_{P_1} - D_{P_2}a_1 + D_{P_3}a_1^2 - \cdots\,. \tag{58}$$

[9]Perhaps for J-Y. Thibon [59] and some other participants of SLC, this is "bread and butter", but for us this was fantastic and absolutely genial!

Using this equality and

$$a_1^p \partial_1 \cdots \partial_{n-1} = S_{p-n+1}(a_1 + \cdots + a_n)\,, \tag{59}$$

we compute for $0 < k \leq n$,

$$a_1^{n-k} \partial_0 \partial_1 \cdots \partial_{n-1} = D_{P_k} - D_{P_{k+1}} S_1 + D_{P_{k+2}} S_2 - \cdots\,. \tag{60}$$

(after restriction to the first n variables). Now define

$$U_k^S := D_{P_k} - D_{P_{k+1}} S_1 + D_{P_{k+2}} S_2 - \cdots\,. \tag{61}$$

and

$$U_k^\Lambda := D_{P_k} - D_{P_{k+1}} \Lambda_1 + D_{P_{k+2}} \Lambda_2 - \cdots\,. \tag{62}$$

What is to be shown is:

$$\widetilde{Q}_{IkJ}\; U^S = \pm \widetilde{Q}_{IJ} \quad \text{or} \quad 0\,. \tag{63}$$

Note that U_k^S is dual to U_k^Λ for the involution: $S_I \to S_{I^\sim}$. So if we define Q_I'[10] as duals to $\widetilde{Q}_I$, then it is required to show:

$$Q'_{IkJ}\; U_k^\Lambda = \pm Q'_{IJ} \quad \text{or} \quad 0\,. \tag{64}$$

We invoke now a result of Littlewood: *The basis $\{Q'_J\}$, where J runs over strict partitions, is conjugate to $\{P_J\}$, the basis of Schur's P-functions.* Define

$$V_k^\Lambda := P_k - D_{\Lambda_1} P_{k+1} + D_{\Lambda_2} P_{k+2} - \cdots\,. \tag{65}$$

So finally we must show:

$$P_J\; V_k^\Lambda = P_{kJ} \tag{66}$$

– but this follows from:

$$P_{kJ} = \sum_{j \geq 0} (-1)^j\; P_J D_{\Lambda_j}\; P_{k+j}\,. \tag{67}$$

We refer to [37] for details. We do not know proofs of Eqs. (52) and (53) without using vertex operators.

The formulas discussed here are important for cohomology classes of Lagrangian Schubert classes and degeneracy loci [54], [37], and, consequently, for quantum cohomology of Lagrangian Grassmannians (cf., e.g., [58]).

For Weyl groups of type D, there are similar results. As $\mathfrak{D}_n$ is a subgroup of $\mathfrak{C}_n$, they can be, in fact, deduced from Theorem 3 with the help of some additional computations with divided differences of type C. It is convenient to perform them using some planar displays which is the subject of the next section.

[10]These functions were studied on another occasion by Alain, B. Leclerc, and J-Y. Thibon [31].

6 Planar displays

First we want to show another derivation of the ribbon determinant from Section 3 with the help of some deformations of diagrams. Given a skew diagram, its *right corner* is the box at the extreme right of the bottom row, and its *left corner* is the upper box of the left column. Given two skew diagrams H, K, let $\aleph$ be the right corner of H and $\blacksquare$ the left corner of K. We define $H \blacktriangleright K$ to be the skew diagram obtained by gluing the two diagrams by their corners, $\aleph$, $\blacksquare$ being on the same horizontal, and $H \blacktriangledown K$ to be the skew diagram obtained by gluing the two corners on a vertical:

$$H \blacktriangleright K = \begin{array}{cccccccccccc} \odot & \odot & & & & & & & & & & \\ \odot & \odot & \odot & \odot & & & & & & & & \\ & & \odot & \odot & \odot & \odot & & & & & & \\ & & & \odot & \odot & \aleph & \blacksquare & \star & \star & \star & \star & \star \\ & & & & & & & & \star & \star & \star & \star \\ & & & & & & & & & & \star & \star \end{array} \qquad H \blacktriangledown K = \begin{array}{ccccccccccc} \odot & \odot & & & & & & & & & \\ \odot & \odot & \odot & \odot & & & & & & & \\ & & \odot & \odot & \odot & \odot & & & & & \\ & & & \odot & \odot & \aleph & & & & & \\ & & & & & \blacksquare & \star & \star & \star & \star & \star \\ & & & & & & & \star & \star & \star & \star \\ & & & & & & & & & \star & \star \end{array}$$

Lemma 4 *If H and K are two skew diagrams, then*

$$S_H \cdot S_K = S_{H \blacktriangleright K} + S_{H \blacktriangledown K} \,. \tag{68}$$

This equality (known already to P. MacMahon for ribbon Schur functions) has a simple justification. The Schur function S_H is the sum $\sum t$ of all the tableaux of diagram H, and similarly S_K is the sum $\sum t'$ of all the tableaux of diagram K. For any pair of tableaux t, t', let y be the letter in the box $\aleph$ of t and z the letter in the box $\blacksquare$ of t'. According as $y \leq z$ or $y > z$, the product of (monomials of) tableaux is a tableau of the diagram $H \blacktriangleright K$ or $H \blacktriangledown K$ and conversely, cutting into two pieces all the pairs of tableaux of diagrams $H \blacktriangleright K$ or $H \blacktriangledown K$, one obtains all the pairs of tableaux of respective diagrams H, K (all this is a trivial consequence of the *Jeu de Taquin* which allows one to move the parts of tableaux (cf. [56]).

Jeu de Taquin is a favourite "jeu" of Alain; it is ultimately related to the notion of *Le monoïde plaxique* – an important subject developed with M-P. Schützenberger, which offers us, among other things, a noncommutative lift of Schur functions (cf. the original paper [41] and [33] which contains a more didactic exposition of these issues). The "plactic point of view" gives, according to Alain, the simplest, most natural, and most transparent proof of the famous *Littlewood-Richardson rule* (*loc.cit.*).

We now show another way of obtaining the ribbon determinant expressing a Schur function. Take, e.g., $I = 13557$. The decompositions of the diagram of I into respectively hooks and ribbons are:

$$\begin{array}{l} \odot \\ \odot\star\blacksquare \\ \odot\star\square\blacksquare\blacksquare \\ \odot\square\star\star\star \\ \square\odot\odot\odot\odot\odot\odot \end{array} \qquad \text{and} \qquad \begin{array}{l} \odot \\ \odot\odot\odot \\ \star\star\square\odot\odot \\ \blacksquare\square\star\star\odot \\ \square\blacksquare\blacksquare\star\odot\odot\odot \end{array}$$

We have

$$S_{13557} = \begin{vmatrix} S_{13} & S_{14} & S_{17} \\ S_{113} & S_{114} & S_{117} \\ S_{1113} & S_{1114} & S_{11117} \end{vmatrix}$$

$$= \begin{vmatrix} S_{13} & S_{13\blacktriangledown 1} & S_{13\blacktriangledown 1\blacktriangledown 3} \\ S_{113} & S_{113\blacktriangledown 1} & S_{13\blacktriangledown 1\blacktriangledown 3} \\ S_{11113} & S_{11113\blacktriangledown 1} & S_{11113\blacktriangledown 1\blacktriangledown 3} \end{vmatrix} = \begin{vmatrix} S_{13} & S_{13\blacktriangledown 1} & S_{13\blacktriangledown 1\blacktriangledown 3} \\ S_{1\blacktriangleright 13} & S_{1\blacktriangleright 13\blacktriangledown 1} & S_{1\blacktriangleright 3\blacktriangledown 1\blacktriangledown 3} \\ S_{11\blacktriangleright 1\blacktriangleright 13} & S_{11\blacktriangleright 1\blacktriangleright 13\blacktriangledown 1} & S_{11\blacktriangleright 1\blacktriangleright 13\blacktriangledown 1\blacktriangledown 3} \end{vmatrix},$$

which is the desirable ribbon determinant. This is a direct consequence of Lemma 4 combined with elementary operations on columns and rows of the determinants involved. We refer to [35] for more applications of such deformations of planar diagrams.

During our mathematical discussions, Alain often mentioned the Gessel-Viennot "planar" interpretation of binomial determinants in terms of non-intersecting paths in $\mathbb{N} \times \mathbb{N}$ (cf. [15]).

Now we want to discuss reduced decompositions and divided differences. Relations between reduced decompositions in the Weyl groups can be represented planarly. Let us explain it through the example of the symmetric group $\mathfrak{S}_n$. By definition, a planar display will be identified with its reading from left to right and top to bottom (*row-reading*). We shall also use *column-reading*, that is, reading successive columns downwards, from left to right.

For example, we shall write

$$\begin{array}{ccc} 2 & & \\ 1 & 2 & \end{array} \equiv \begin{array}{cc} 1 & 2 \\ & 1 \end{array}$$

for the following equality for simple transpositions:

$$s_2\, s_1\, s_2 = s_1\, s_2\, s_1 \ .$$

Suppose that a rectangle is filled row-wise from left to right, and column-wise from bottom to top with consecutive numbers from $\{1, \ldots, n-1\}$.

Then one easily checks that its row-reading and column-reading produce two words which, interpreted as words in the s_i, are congruent modulo the Coxeter relations.

Here is an example of such a congruence:

$$\begin{matrix} 3456 & & & \\ & 2345 & & \\ & & 1234 & \end{matrix} \equiv \begin{matrix} 3 & 4 & 5 & 6 \\ 2\cdot & 3\cdot & 4\cdot & 5 \\ 1 & 2 & 3 & 4 \end{matrix}$$

the congruence class being conveniently denoted by the rectangle

$$\begin{matrix} 3456 \\ 2345 \\ 1234 \end{matrix}$$

More generally, the planar arrays that we shall write, will have the property that their row-reading and column-reading are congruent modulo Coxeter relations (this is a "Jeu de Taquin" for reduced decompositions). In this notation, we have, for any integers a, b, c, d, k such that

$$1 \leq a < b, c < d \leq n, \quad a + d = b + c, \quad k < d - b,$$

the congruence

$$\begin{matrix} b+1 & \cdots & b+k & & \\ b & \cdots & \cdots & \cdots & d \\ \vdots & & & & \vdots \\ a & \cdots & \cdots & \cdots & c \end{matrix} \equiv \begin{matrix} b & \cdots & \cdots & \cdots & d \\ \vdots & & & & \vdots \\ a & \cdots & \cdots & \cdots & c \\ & & c-k & \cdots & c-1 \end{matrix} \qquad (69)$$

We shall display divided differences of type C planarly according to the same conventions as for products of s_i's. For example, the divided difference:

$$\partial_0\partial_1\partial_2\partial_3\partial_0\partial_1\partial_2\partial_0\partial_1$$

will be displayed as

$$\begin{matrix} \partial_0\partial_1\partial_2\partial_3 & & \\ & \partial_0\partial_1\partial_2 & \\ & & \partial_0\partial_1 \end{matrix}$$

As said before, the displays that we write have the property that their row-reading is congruent to their column-reading, and thus the preceding one encodes the equality

$$\partial_0\partial_1\partial_0\partial_2\partial_1\partial_0\partial_3\partial_2\partial_1 = \partial_0\partial_1\partial_2\partial_3\partial_0\partial_1\partial_2\partial_0\partial_1 \,.$$

When one wants to get an analog of Theorem 3 for type D, the following relation is, e.g., needed (cf. [38]):

$$\begin{matrix} \partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6 & & & \\ & \partial_0\partial_1\partial_2\partial_3\partial_4\partial_5 & & \\ & & \partial_0\partial_1\partial_2\partial_3\partial_4 & \\ & & & \partial_1\partial_2\partial_3 \end{matrix} = \partial_4\partial_5\partial_6 \begin{matrix} \partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6 & & \\ & \partial_0\partial_1\partial_2\partial_3\partial_4\partial_5 & \\ & & \partial_0\partial_1\partial_2\partial_3\partial_4 \end{matrix} \qquad (70)$$

Now, thanks to the relations (69) written in terms of divided differences, we have

$$
\begin{array}{r}
\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6 \\
\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5 \\
\partial_0\partial_1\partial_2\partial_3\partial_4 \\
\partial_1\partial_2\partial_3
\end{array}
$$

$$
= \begin{pmatrix} \partial_0\partial_1\partial_2 \\ \partial_0\partial_1 \\ \partial_0 \end{pmatrix}
\begin{pmatrix} \partial_3\partial_4\partial_5\partial_6 \\ \partial_2\partial_3\partial_4\partial_5 \\ \partial_1\partial_2\partial_3\partial_4 \\ \partial_1\partial_2\partial_3 \end{pmatrix}
= \begin{pmatrix} \partial_0\partial_1\partial_2 \\ \partial_0\partial_1 \\ \partial_0 \end{pmatrix}
\begin{pmatrix} \partial_4\partial_5\partial_6 \\ \partial_3\partial_4\partial_5\partial_6 \\ \partial_2\partial_3\partial_4\partial_5 \\ \partial_1\partial_2\partial_3\partial_4 \end{pmatrix}
$$

$$
= \begin{pmatrix} \partial_0\partial_1\partial_2 \\ \partial_0\partial_1 \\ \partial_0 \end{pmatrix}
(\partial_4\partial_5\partial_6)
\begin{pmatrix} \partial_3\partial_4\partial_5\partial_6 \\ \partial_2\partial_3\partial_4\partial_5 \\ \partial_1\partial_2\partial_3\partial_4 \end{pmatrix}
$$

Since $\partial_4\partial_5\partial_6$ commutes with the divided differences on its left, the last expression may be rewritten as

$$
\partial_4\partial_5\partial_6 \quad
\begin{array}{r}
\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5\partial_6 \\
\partial_0\partial_1\partial_2\partial_3\partial_4\partial_5 \\
\partial_0\partial_1\partial_2\partial_3\partial_4
\end{array}
$$

which is what we need for (70). You see that with planar displays it is much easier to compute with reduced decompositions and divided differences for Weyl groups.

Also a permutation can be displayed in a planar way using its *Rothe diagram.* This leads to Alain's favourite combinatorial presentation of Schubert polynomials with the help of *Weintrauben* of A. Kohnert [18].

7 Euclid's algorithm

Already during his first visit to Poland in 1978, Alain said me:

"Even in elementary Mathematics there are interesting things still to be discovered!"

The present section seems to be a confirmation of this sentence. I will tell you about our recent work [39], [40] on some aspects of Euclid's algorithm.

Let $\mathbb{A}$ and $\mathbb{B}$ be two alphabets of cardinalities m and n. (We can think of $\mathbb{A}$ and $\mathbb{B}$ as multi-sets of complex numbers.) Consider two monic polynomials $f(x) = S_m(x - \mathbb{A})$ and $\varphi(x) = S_n(x - \mathbb{B})$. Assume that $m \geq n$. Let us look at the iterated division of f and φ:

$$
f = *\,\varphi + c_1\mathcal{R}_1 \quad , \quad \varphi = *\,\mathcal{R}_1 + c_2\mathcal{R}_2 \quad , \quad \mathcal{R}_1 = *\,\mathcal{R}_2 + c_3\mathcal{R}_3 \quad , \quad \ldots \, . \tag{71}
$$

The successive coefficients " $*$ " are the unique polynomials such that

$$n > \deg \mathcal{R}_1 > \deg \mathcal{R}_2 > \deg \mathcal{R}_3 > \cdots \,.$$

Instead of the usual Euclidean algorithm, where $c_1 = c_2 = c_3 = \cdots = 1$ and where $\mathcal{R}_i$ are rational functions in the roots $\mathbb{A}$ and $\mathbb{B}$ of f and φ, we choose the constants c_i in such a way that the successive remainders $\mathcal{R}_i$ are polynomials in $\mathbb{A}$ and $\mathbb{B}$ with the top coefficient equal to $S_{i^{m-n+i}}(\mathbb{A}-\mathbb{B})$. The problem is to give explicit expressions for such *normalized polynomial remainders* in terms of roots.

An interesting solution to the above problem was proposed by Sylvester about 160 years ago, who found "the successive residues, divested of their allotrious factors ..." that is, who also normalized the remainders in such a way as to obtain polynomials in the roots, the last "residue" being the resultant. To state Sylvester's solution, we need the following notation. For two finite sets $\mathbb{A}$ and $\mathbb{B}$ of elements in a commutative ring, we set

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},b\in\mathbb{B}} (a-b)\,. \tag{72}$$

Let now $\mathbb{A} = (a_1,\ldots,a_m)$, $\mathbb{B} = (b_1,\ldots,b_n)$ be two alphabets of (commuting) variables. Let $0 \le p \le m$, $0 \le q \le n$ be two integers. Define, after Sylvester, the following double sum:

$$\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x) := \sum_{\mathbb{A}'\subset\mathbb{A}} \sum_{\mathbb{B}'\subset\mathbb{B}} R(x,\mathbb{A}')R(x,\mathbb{B}') \frac{R(\mathbb{A}',\mathbb{B}')R(\mathbb{A}-\mathbb{A}',\mathbb{B}-\mathbb{B}')}{R(\mathbb{A}',\mathbb{A}-\mathbb{A}')R(\mathbb{B}',\mathbb{B}-\mathbb{B}')}\,, \tag{73}$$

the sum being over all subsets $\mathbb{A}'$ of cardinality p and $\mathbb{B}'$ of cardinality q. Sometimes, we will also use the notation $\mathrm{Sylv}^{p,q}(x)$ or $\mathrm{Sylv}^{p,q}$ for this sum. The Sylvester sum $\mathrm{Sylv}^{p,q}(\mathbb{A},\mathbb{B};x)$ is a polynomial in $\mathbb{A}$, $\mathbb{B}$, and x; moreover $\deg_x \mathrm{Sylv}^{p,q} = p+q$. Observe that $\mathrm{Sylv}^{0,0} = R(\mathbb{A},\mathbb{B})$ and if $p=0$ or $q=0$ then the Sylvester sum reduces to a single sum. For example,

$$\mathrm{Sylv}^{0,q}(\mathbb{A},\mathbb{B};x) = \sum_{\mathbb{B}'\subset\mathbb{B}} R(x,\mathbb{B}') \frac{R(\mathbb{A},\mathbb{B}-\mathbb{B}')}{R(\mathbb{B}',\mathbb{B}-\mathbb{B}')}\,, \tag{74}$$

the sum being over all subsets $\mathbb{B}'$ of cardinality q. In this case, the assertion of the next theorem was already proved by C.W. Borchardt in 1860.

Note that

$$R(\mathbb{A},\mathbb{B}-\mathbb{B}') = \prod_{b\in\mathbb{B}\setminus\mathbb{B}'} f(b)\,. \tag{75}$$

For $q = n-1$, one recovers Lagrange interpolation (cf., e.g., [30]):

$$\pm\mathcal{R}_1 = \sum_{i=1}^{n} f(b_i) \prod_{j\neq i} \frac{x-b_j}{b_i-b_j}\,. \tag{76}$$

Let us come back to double Sylvester sums. For $m = 3$, $n = 2$, the Sylvester sum $\mathrm{Sylv}^{1,1}(\mathbb{A},\mathbb{B};x)$ equals:

$$\begin{aligned}
&\frac{(a_1-b_1)(a_2-b_2)(a_3-b_2)}{(a_1-a_2)(a_1-a_3)(b_1-b_2)}(x-a_1)(x-b_1)\\
+&\frac{(a_1-b_2)(a_2-b_1)(a_3-b_1)}{(a_1-a_2)(a_1-a_3)(b_2-b_1)}(x-a_1)(x-b_2)\\
+&\frac{(a_2-b_1)(a_1-b_2)(a_3-b_2)}{(a_2-a_1)(a_2-a_3)(b_1-b_2)}(x-a_2)(x-b_1)\\
+&\frac{(a_2-b_2)(a_1-b_1)(a_3-b_1)}{(a_2-a_1)(a_2-a_3)(b_2-b_1)}(x-a_2)(x-b_2)\\
+&\frac{(a_3-b_1)(a_1-b_2)(a_2-b_2)}{(a_3-a_1)(a_3-a_2)(b_1-b_2)}(x-a_3)(x-b_1)\\
+&\frac{(a_3-b_2)(a_1-b_1)(a_2-b_1)}{(a_3-a_1)(a_3-a_2)(b_2-b_1)}(x-a_3)(x-b_2)\,.
\end{aligned}$$

This is equal to $-2(x-b_1)(x-b_2)$, so you see that these sums are quite tricky. Summations of this type play nowadays an important role, for example, in representation theory, and in the description of Gysin maps (called also "integrations over a fiber") for fibrations with homogeneous spaces as fibers.

The main result connecting the polynomial remainders and Sylvester sums is the following theorem.

Theorem 5 ([39]) *If $p + q < n$, then specializing $\mathbb{A}$ and $\mathbb{B}$ to the roots of the polynomials f and φ respectively, we have*

$$\mathrm{Sylv}^{p,q} = \pm\binom{p+q}{p}\mathcal{R}_{n-(p+q)}\,. \tag{77}$$

This result was stated without complete proof in Sylvester's papers (cf., e.g., [57]). Our proof uses the following presentation of $\mathcal{R}_{n-d}$ for $d = 0, 1, \ldots, n-1$:

$$\mathcal{R}_{n-d} = \pm S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})\,.$$

(Cf. [30], [39].) This determinantal presentation will allow us to run induction (on n). Before giving a sketch of the proof of the theorem, we state some results that we shall need. Suppose that $\mathbb{B} = (b_1, \ldots, b_n)$ is an alphabet of variables. In the following, divided differences act on the variables from $\mathbb{B}$.

1) For every $k \in \mathbb{N}$, and an alphabet $\mathbb{A}$ which is independent of $\mathbb{B}$, we have

$$S_k(\mathbb{B}_i - \mathbb{A})\partial_i = S_{k-1}(\mathbb{B}_{i+1} - \mathbb{A})\,. \tag{78}$$

2) The first remainder $\mathcal{R}_1$ of the division of $f(x) = S_m(x-\mathbb{A})$ by $\varphi(x) = S_n(x-\mathbb{B})$ equals

$$\mathcal{R}_1 = \big(f(b_1)R(x,\mathbb{B}-b_1)\big)\partial_1\cdots\partial_{n-1}\,. \tag{79}$$

This is again an incarnation of the *Lagrange interpolation* in the points $a \in \mathbb{A}$. For example, for $n = 3$, $\mathcal{R}_1$ equals

$$f(b_1)\frac{(x-b_2)(x-b_3)}{(b_1-b_2)(b_1-b_3)} + f(b_2)\frac{(x-b_1)(x-b_3)}{(b_2-b_1)(b_2-b_3)} + f(b_3)\frac{(x-b_1)(x-b_2)}{(b_3-b_1)(b_3-b_2)}\,. \quad (80)$$

3) The specialization $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; b)$, $b \in \mathbb{B}$, is equal to $\pm R(b, \mathbb{A})\mathbf{c}$, where $\mathbf{c}$ is the top coefficient of $\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B} - b; x)$.

Sketch of proof of the theorem Write $d := p + q$. It is necessary to show that

$$\mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B}; x) = \pm\binom{d}{p} S_{1^d;(m-d)^{n-d}}(\mathbb{B} - x; \mathbb{B} - \mathbb{A}) \quad (81)$$

if $d < n$. I will give you a rather detailed computation because it gathers together several tricks useful while calculating with multi-Schur functions.

Since $\deg_x \mathrm{Sylv}^{p,q} < n$, $\mathrm{Sylv}^{p,q}$ coincides with its first remainder modulo $R(x, \mathbb{B})$. By Eq. (79),

$$\mathrm{Sylv}^{p,q}(x) = \big(\mathrm{Sylv}^{p,q}(b_1) R(x, \mathbb{B} - b_1)\big)\partial_1 \cdots \partial_{n-1}\,. \quad (82)$$

By property 3), we have

$$\mathrm{Sylv}^{p,q}(b_1) = \pm R(b_1, \mathbb{A}) \times \text{the top coefficient } \mathbf{c} \text{ in } \mathrm{Sylv}^{p,q}(\mathbb{A}, \mathbb{B} - b_1; x)\,. \quad (83)$$

If $d < n - 1$, then by the induction assumption,

$$\mathbf{c} = \pm\binom{d}{p} S_{\Box}(\mathbb{B} - b_1 - \mathbb{A})\,, \quad (84)$$

with $\Box = (m - d)^{n-d-1}$. This is also true if $d = n - 1$ (cf. [39]). We are left with the computation of

$$\big(R(x, \mathbb{B} - b_1) R(b_1, \mathbb{A})\big) S_{\Box}(\mathbb{B} - b_1 - \mathbb{A})\partial_1 \cdots \partial_{n-1}\,.$$

In this computation, we use three simple identities:

$$R(x, \mathbb{B} - b_1) = \pm\sum_{i=0}^{n-1} (-1)^i S_{1^{n-i-1}}(\mathbb{B} - x) b_1^i\,, \quad (85)$$

$$b_1^i R(b_1, \mathbb{A}) = S_{m+i}(b_1 - \mathbb{A})\,, \quad (86)$$

$$S_{m+i}(b_1 - \mathbb{A}) S_{\Box}(\mathbb{B} - b_1 - \mathbb{A}) = S_{\Box;m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A})\,. \quad (87)$$

Finally, using (78),

$$\mathrm{Sylv}^{p,q}(b_1) R(x, \mathbb{B} - b_1) = \pm\binom{d}{p} \sum_{i=0}^{n-1} (-1)^i S_{1^{n-1-i}}(\mathbb{B} - x) S_{\Box;m+i}(\mathbb{B} - \mathbb{A}; b_1 - \mathbb{A})$$

is sent via $\partial_1 \cdots \partial_{n-1}$ to

$$\pm\binom{d}{p}\sum_{i=0}^{n-1}(-1)^i S_{1^{n-1-i}}(\mathbb{B}-x)S_{\square;m+i-(n-1)}(\mathbb{B}-\mathbb{A})\,. \tag{88}$$

In the sum (88), the terms for $i = 0, \ldots, n-d-2$ disappear. In the remaining sum for $i = n-d-1, \ldots, n-1$:

$$\sum_{j=0}^{d}(-1)^j S_{1^{d-j}}(\mathbb{B}-x)S_{\square;m-d+j}(\mathbb{B}-\mathbb{A})\,,$$

one recognizes a Laplace expansion of $S_{1^d;(m-d)^{n-d}}(\mathbb{B}-x;\mathbb{B}-\mathbb{A})$.

This computation shows the importance of having a *proper* determinantal expression (here for a normalized polynomial remainder) that allows us to use induction.

Perform now the Euclidean algorithm with nonstandard signs:

$$f = \mathcal{Q}_0\varphi - \mathcal{R}_1,\quad \varphi = \mathcal{Q}_1\mathcal{R}_1 - \mathcal{R}_2,\quad \mathcal{R}_1 = \mathcal{Q}_2\mathcal{R}_2 - \mathcal{R}_3\ ,...$$

$$...,\mathcal{R}_{n-2} = \mathcal{Q}_{n-1}\mathcal{R}_{n-1} - \mathcal{R}_n,\quad \mathcal{R}_{n-1} = \mathcal{Q}_n\mathcal{R}_n\,.$$

We may rewrite the first equation as follows:

$$\frac{\varphi}{f} = \frac{1}{\mathcal{Q}_0 - \frac{\mathcal{R}_1}{\varphi}}\,.$$

Iterating, we get a continued fraction:

$$\frac{\varphi}{f} = \cfrac{1}{\mathcal{Q}_0 - \cfrac{1}{\mathcal{Q}_1 - \cfrac{1}{\ddots\ \cfrac{1}{\mathcal{Q}_n}}}}\,.$$

For $i = 1, \ldots, n$, we denote by $\mathcal{N}_i$ and $\mathcal{D}_i$ the numerator and denominator of the ith convergent of this continued fraction. For $i = 1, \ldots, n$, we have (cf., e.g., [40])

$$\mathcal{R}_i = \varphi\mathcal{D}_i - f\mathcal{N}_i\,. \tag{89}$$

Suppose, from now on, that $\mathbb{A}$ and $\mathbb{B}$ (the multi-sets of roots of f and φ) are alphabets of cardinalities $n+1$ and n. One finds in the book of Brioschi [3], p. 167, the following identities: for a fixed i,

$$\sum_{a\in\mathbb{A}}\mathcal{D}_i(a)\mathcal{N}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{90}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_{i-1}(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{91}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i^2(a)\mathcal{Q}_i(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,. \tag{92}$$

These three identities make use of the following functional on $\mathbb{C}[x]$:

$$\mu : g(x) \mapsto \sum_{a\in\mathbb{A}} g(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)}\ . \tag{93}$$

The functional μ is an incarnation of the Lagrange interpolation in the points $a \in \mathbb{A}$. It is characterized by the fact that it sends each x^i, $i \in \mathbb{N}$, onto the complete function $S_i(\mathbb{A}-\mathbb{B})$.

Given a linear functional, it is known how to write an orthogonal basis in terms of *moments*:

Proposition 6 *The Schur polynomials* $S_{i^i}(\mathbb{A}-\mathbb{B}-x)$ *for* $i = 0, 1, \ldots, n$, *form a unique (up to normalization) orthogonal basis with respect to the functional* μ, *of polynomials of respective degrees* $0, 1, \ldots, n$.

(More precisely, this is a rewriting, in terms of Schur functions, of the classical expressions of orthogonal polynomials in terms of *Hankel determinants* involving moments.)

By combining Eq. (89) with

$$\mathcal{R}_i \equiv \varphi\ S_{i^i}(\mathbb{A}-\mathbb{B}-x) \mod f\,, \tag{94}$$

(cf. [30], p. 49), we thus get the following basic result.

Proposition 7 *For any fixed* $0 \le i < j \le n$, *the following relations hold:*

$$\sum_{a\in\mathbb{A}} a^i\mathcal{D}_j(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,, \tag{95}$$

$$\sum_{a\in\mathbb{A}} \mathcal{D}_i(a)\mathcal{D}_j(a)\frac{\varphi(a)}{R(a,\mathbb{A}-a)} = 0\,. \tag{96}$$

Brioschi's relation (91) is just a particular case of Eq. (96) with consecutive i and j. Similarly, the two remaining relations follow from the proposition (cf. [40]).

In terms of Euclidean remainders, the proposition is rewritten as follows: for any fixed $0 \le i < j \le n$,

$$\sum_{a\in\mathbb{A}} a^i\mathcal{R}_j(a)\frac{1}{\varphi(a)R(a,\mathbb{A}-a)} = 0\,, \tag{97}$$

$$\sum_{a\in\mathbb{A}} \mathcal{R}_i(a)\mathcal{R}_j(a)\frac{1}{\varphi(a)R(a,\mathbb{A}-a)} = 0\,. \tag{98}$$

We want now to discuss Christoffel-Darboux kernels associated with these orthogonal polynomials. This will be done by investigating some bivariate polynomials called *Bézoutians*.

Suppose that f, φ is an arbitrary pair of univariate polynomials. Denoting by ∂_{xy} the Newton divided difference acting on the variables x and y, we set (after Bézout [2]):

$$\mathrm{Bez}(f,\varphi) := f(x)\varphi(y)\partial_{xy}\,, \tag{99}$$

and call this bivariate polynomial (in x and y) the *Bézoutian* (of f and φ).

In [40] the following formula (missed by classics) was proved:

Theorem 8 *With the above notation,*

$$\mathrm{Bez}(f,\varphi) = p_0\varphi(x)\varphi(y) + \sum_{i=1}^{n} p_i\mathcal{R}_i(x)\mathcal{R}_i(y)\,, \tag{100}$$

where $p_i := \mathcal{Q}_i(x)\partial_{x,y}$.

We say that a pair (f,φ) is *general* if the Euclidean quotients $\mathcal{Q}_i$ are of degree 1 for $i = 1,\ldots,n$. From now on, we assume that (f,φ) is a general pair of monic polynomials of degrees $(n{+}1, n)$ with alphabets of roots $\mathbb{A}$ and $\mathbb{B}$. The theorem gives:

$$\mathrm{Bez}(f,\varphi) = \varphi(x)\varphi(y) + \sum_{i=1}^{n} \alpha_i\mathcal{R}_i(x)\mathcal{R}_i(y)\,, \tag{101}$$

where α_i is the coefficient of x in $\mathcal{Q}_i(x)$.

Using Eq. (101), the congruence (94), and an expression for α_i from [30], we infer (cf. [40]):

Theorem 9 *With the above assumptions, we have the following congruence modulo the ideal* $(f(x), f(y))$*:*

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(x)\varphi(y)\Big(1 + \sum_{i=1}^{n}(-1)^i\frac{S_{i^i}(\mathbb{A}-\mathbb{B}-x)S_{i^i}(\mathbb{A}-\mathbb{B}-y)}{S_{(i-1)^i}(\mathbb{A}-\mathbb{B})S_{i^{i+1}}(\mathbb{A}-\mathbb{B})}\Big)\,. \tag{102}$$

We recall now an important notion from the theory of *orthogonal polynomials*. Given a family of orthogonal polynomials $P_0(x),\ldots,P_n(x)$ associated with a linear functional μ, then the *Christoffel-Darboux kernel* is:

$$K(x,y) := \sum_{i=0}^{n} P_i(x)P_i(y)/\big(P_i(x)^2\ \mu\big)\,. \tag{103}$$

To compute the Christoffel-Darboux kernel in the present situation, we record the following *normalization property*:

Lemma 10 *With the above notation,*

$$S_{i^i}(\mathbb{A}-\mathbb{B}-x)^2\ \mu = (-1)^i S_{(i-1)^i}(\mathbb{A}-\mathbb{B}) S_{i^{i+1}}(\mathbb{A}-\mathbb{B})\,. \tag{104}$$

By combining the congruence (102), Proposition 6 and Lemma 10, we infer the following congruence modulo the ideal $(f(x), f(y))$:

$$\mathrm{Bez}(\mathbb{A},\mathbb{B}) \equiv \varphi(x)\varphi(y)\, K(x,y)\,. \tag{105}$$

This congruence suggests that $\mathrm{Bez}(\mathbb{A},\mathbb{B})$ (similarly to $K(x,y)$) should have a *reproducing* property.

Theorem 11 ([40]) *For a polynomial $g(x)$,*

$$g(x)\ \mathrm{Bez}(\mathbb{A},\mathbb{B})\ \mu \equiv \varphi(y)^2\, g(y) \quad \mod f(y)\,. \tag{106}$$

Sketch of proof It follows from the *Leibnitz rule* that $\mathrm{Bez}(\mathbb{A},\mathbb{B})$ is congruent to $f(x)\partial_{xy}\cdot\varphi(x)$ modulo $f(y)$. Recall that $f(x)\partial_{xy} = S_n(x+y-\mathbb{A})$. For any $i \geq 0$, we have, modulo $f(y)$,

$$x^i\,\varphi(y)\, S_n(x+y-\mathbb{A})\ \mu \equiv \varphi(y) S_{i+n}(\mathbb{A}-\mathbb{B}+y-\mathbb{A}) = \varphi(y)^2\, y^i\,.$$

So we have arrived at another reproducing property, this time for Bézoutians. There are similar results for a (general) pair of monic polynomials of the same degree (cf. [40]).

8 Concluding remarks

I hope that I will not offend much the audience of SLC by saying that Alain is rather *algebraist* than combinatorialist. To be a bit more precise, I think that Alain is doing *Combinatorial Algebra*. His favourite mathematicians (read: algebraists) are mentioned already J.M. Hoene-Wroński and the Italian algebraic geometer G.Z. Giambelli (1879-1953). They both did very original mathematics that, to a large extent, was not accepted by the mathematical establishment of their time (cf. [28] and [20]). This is often the price you must pay for your originality:

"To reach the source you must go against the stream."

Alain is now, and has been for several years, very much involved in teaching young Chinese mathematicians at the Center of Combinatorics of Nankai University in Tianjin in the People Republic of China. The following picture of him and some of his students was taken on the occasion of his 59th Birthday in Tianjin:

Here are two other pictures from Nankai University. The first one is with S.S. Chern[11], a mathematician who translated many geometric problems to Algebra by inventing the famous *Chern characteristic classes*:

Since, by the *splitting principle* the Chern classes of a vector bundle are elementary symmetric functions in its Chern roots, many computations of Chern classes amount to calculations of symmetric functions. Given a vector bundle E and partition I, we define $S_I(E)$ to be the Schur polynomial $S_I(\mathbb{A})$ with $\mathbb{A}$ specialized to the Chern roots of E. The following problem is central here. Determine the integer coefficients c_K in the Schur polynomial expansion: $S_I(V_J(E)) = \sum c_K S_K(E)$, where V_J stands for a Schur functor. For the Chern (resp. *Segre*) classes of $S^2(E)$ and $\wedge^2(E)$ this was done in [25] (resp. [51], [21]). By [53], §7, we know that $c_K \geq 0$. Note that the expansion of the *top* Chern class of $V_J(E)$ determines those of all other Chern classes (cf. [53], Proposition 2.1).

The second picture is with Sir M. Atiyah:

[11]S.S. Chern passed away in Tianjin on December 3, 2004 at the age of 93.

Atiyah said in 1993 (on the occasion of Hirzebruch's 65th Birthday):

"It is reassuring to know that the algebraic virtuosity
of our ancestors is still present in our genes."

I wish heartily the Séminaire Lotharingien de Combinatoire, and in particular Alain, that this will be the case for Alain Lascoux's ingenious computations in Combinatorial Algebra!

Acknowledgments I thank the organizers of SLC 52 for their invitation to attend this event. My sincere thanks go to Jean-Yves Thibon whose constant encouragement allowed me to bring my talk at the LascouxFest to a paper. I am also greatly indebted to Ron King for a number of comments and suggestions that helped to ameliorate its exposition. Jan Krzysztof Kowalski from our institute in Warsaw kindly helped me with some mysteries of LaTeX. The pictures from Tianjin are courtesy of Bill Chen from the Center of Combinatorics of Nankai University.

References

[1] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and cohomology of the spaces G/P,* Russian Math. Surveys, **28:3** (1973), 1–26.

[2] M. Bézout, *Recherches sur le degré des équations résultants de l'evanouissement des inconnues, et sur les moyens qu'il convient d'employer pour trouver ces équations*, Mém. Acad. Roy. Sci. Paris (1764), 288–338.

[3] F. Brioschi, *Théorie des déterminants,* appendix to the French edition, Paris, Mallet-Bachelier 1856.

[4] A.S. Buch, *Combinatorial K-theory,* in: "Topics in cohomological studies of algebraic varieties" (P. Pragacz ed.), Trends in Mathematics, Birkhäuser 2005 – to appear.

[5] D. Buchsbaum, D. Eisenbud, *Generic free resolutions and a family of generically perfect ideals,* Adv. Math. **18** (1975), 245–301.

[6] W.Y.C. Chen, G-G. Yan, A.L.B. Yang, *The skew Schubert polynomials,* Europ. J. Comb. **25/8** in honor of Alain Lascoux (2004), 1181–1196.

[7] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion,* Invent. Math. **21** (1973), 287–301.

[8] M. Demazure, *Désingularisation des variétés de Schubert géneralisées,* Ann. Scient. École Norm. Sup. **7** (1974), 53–88.

[9] J. Désarménien, J. Kung, G-C. Rota, *Invariant theory, Young bitableaux, and combinatorics,* Adv. Math. **27** (1978), 63–92.

[10] P. Doubilet, G-C. Rota, J. Stein, *Combinatorial methods in invariant theory,* Stud. Appl. Math. **53** (1974), 185–216.

[11] G. Faltings, *A proof for the Verlinde formula,* J. Alg. Geom. **3** (1994), 347–374.

[12] A.M. Fu, A. Lascoux, *q-identities from Lagrange and Newton interpolation,* Adv. Appl. Math. **31** (2003), 527–531.

[13] W. Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas,* Duke Math. J. **65** (1992), 381–420.

[14] W. Fulton, A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle,* Duke Math. J. **76** (1994), 711–729.

[15] I. Gessel, G. Viennot, *Binomial determinants, paths, and hook length formulae,* Adv. Math. **58** (1985), 300–321.

[16] A. Kerber, A. Kohnert, A. Lascoux, *SYMMETRICA, an object oriented computer-algebra system for the symmetric group,* J. Symb. Comp. **14** (1992), 195–203.

[17] D.E. Knuth, *Permutations, matrices, and generalized Young tableaux,* Pacific J. Math. **34** (1970), 709–727.

[18] A. Kohnert, *Weintrauben, Polynome, Tableaux,* Bayreuther Math. Schriften **38** (1991), 1–97.

[19] W. Kraśkiewicz, P. Pragacz, *Schubert functors and Schubert polynomials,* Europ. J. Comb. **25/8** in honor of Alain Lascoux (2004), 1327–1344.

[20] D. Laksov, *Remarks on Giovanni Zeno Giambelli's work and life,* (A. Brigaglia et al. eds.), Rend. Circ. Mat. Palermo, serie **II** no. **36** (1994), 207–218.

[21] D. Laksov, A. Lascoux, A. Thorup, *On Giambelli's theorem on complete correlations,* Acta Math. **162** (1989), 143–199.

[22] A. Lascoux, *Puissances extérieures, déterminants et cycles de Schubert,* Bull. Soc. Math. France **102** (1974), 161–179.

[23] A. Lascoux, *Fonctions de Schur et Grassmanniennes,* C. R. Acad. Sc. Paris **281** (1975), 813–815, 851–854.

[24] A. Lascoux, *Thèse,* Université Paris 7, 1977.

[25] A. Lascoux, *Classes de Chern d'un produit tensoriel,* C. R. Acad. Sc. Paris **286** (1978), 385–387.

[26] A. Lascoux, *Syzygies des variétés déterminantales,* Adv. Math. **30** (1978), 202–237.

[27] A. Lascoux, *Classes de Chern des variétés de drapeaux,* C. R. Acad. Sc. Paris **295** (1982), 393–398.

[28] A. Lascoux, *Wroński's factorization of polynomials,* in: "Topics in Algebra" (S. Balcerzyk et al. eds.), Banach Center Publications **26(2)** (1990), 379–386.

[29] A. Lascoux, *Double crystal graphs,* in: "Studies in memory of Issai Schur" (A. Joseph et al. eds.), Progress Math. **210**, 95–114, Birkhäuser 2003.

[30] A. Lascoux, *Symmetric functions and combinatorial operators on polynomials,* CBMS/AMS Lectures Notes **99**, Providence, 2003.

[31] A. Lascoux, B. Leclerc, J-Y. Thibon, *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité,* C. R. Acad. Sc. Paris **316** (1993), 1–6.

[32] A. Lascoux, B. Leclerc, J-Y. Thibon, *Une nouvelle expression des fonctions P de Schur,* C. R. Acad. Sc. Paris **316** (1993), 221–224.

[33] A. Lascoux, B. Leclerc, J-Y. Thibon, *The Plactic Monoid,* in: "Combinatorics on Words", ed. by M. Lothaire, Cambridge University Press 2001.

[34] A. Lascoux, P. Pragacz, *Equerres et fonctions de Schur,* C. R. Acad. Sc. Paris **299** (1984), 955–958.

[35] A. Lascoux, P. Pragacz, *Ribbon Schur functions,* Europ. J. Comb. **9** (1988), 561–574.

[36] A. Lascoux, P. Pragacz, *S-functions series,* J. Phys. A: Math. Gen. **21** (1988), 4105–4114.

[37] A. Lascoux, P. Pragacz, *Operator calculus for $\widetilde{Q}$-polynomials and Schubert polynomials,* Adv. Math. **140** (1998), 1–43.

[38] A. Lascoux, P. Pragacz, *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-polynomials, and vertex operators,* Michigan Math. J. **48** in honor of William Fulton (2000), 417–441.

[39] A. Lascoux, P. Pragacz, *Double Sylvester sums for subresultants and multi-Schur functions,* J. Symb. Comp. **35** (2003), 689–710.

[40] A. Lascoux, P. Pragacz, *Bezoutians, Euclidean algorithm, and orthogonal polynomials,* preprint (2004).

[41] A. Lascoux, M-P. Schützenbeger, *Le monoïde plaxique,* in: " Non-commutative structures in algebra and geometric combinatorics", Napoli 1978, Quaderni de "La Ricerca Scientifica" **109**, C.N.R. Roma (1981), 129–156.

[42] A. Lascoux, M-P. Schützenberger, *Polynômes de Schubert,* C. R. Acad. Sc. Paris **294** (1982), 447–450.

[43] A. Lascoux, M-P. Schützenberger, *Interpolation de Newton à plusieurs variables,* Séminaire d'Algèbre Dubreil–Malliavin 1983–1984 (M-P. Malliavin ed.), Springer Lecture Notes in Math. **1146** (1985), 161–175.

[44] A. Lascoux, M-P. Schützenberger, *Keys and standard bases,* in: "Invariant Theory and Tableaux" (D. Stanton ed.), Springer 1990, 125–144.

[45] A. Lascoux, M-P. Schützenberger, *Formulaire raisonné de fonctions symétriques,* Université Paris 7, 1985.

[46] A. Lascoux, M-P. Schützenberger, *Fonctorialité des polynômes de Schubert,* in: "Invariant Theory" (R. Fossum et al. eds.), Contemporary Mathematics A.M.S. **88** (1989), 585–598.

[47] B. Leclerc, *On identities satisfied by minors of a matrix,* Adv. Math. **100** (1993), 101–132.

[48] I.G. Macdonald, *S*ymmetric functions and Hall polynomials, 2nd ed., Oxford Univ. Press 1995.

[49] I.G. Macdonald, *Notes on Schubert polynomials,* Publ. LACIM **6**, UQUAM, Montréal, 1991.

[50] T. Muir, *History of determinants*, 5 volumes, Dover reprints 1960.

[51] P. Pragacz, *Enumerative geometry of degeneracy loci,* Ann. Scient. Éc. Norm. Sup. **21** (1988), 413–454.

[52] P. Pragacz, *Algebro-geometric applications of Schur S- and Q-polynomials,* Séminaire d'Algèbre Dubreil–Malliavin 1989–1990 (M-P. Malliavin ed.), Springer Lecture Notes in Math. **1478** (1991), 130–191.

[53] P. Pragacz, *Symmetric polynomials and divided differences in formulas of intersection theory,* in: "Parameter Spaces", Banach Center Publications **36** (1996), 125–177.

[54] P. Pragacz, J. Ratajski, *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach,* Compositio Math. **107** (1997), 11–87.

[55] I. Schur, *Über eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen*, Dissertation, Berlin, 1901; see: Ges. Abhandlungen **1**, 1–72.

[56] M-P. Schützenberger, *La correspondance de Robinson,* in: "Combinatoire et Répresentation du Groupe Symétrique" (D. Foata ed.), Springer Lecture Notes in Math. **579** (1977), 59–135.

[57] J.J. Sylvester, *A theory of the syzygetic relations of two rational integral functions*, Phil. Trans. Royal Soc. London vol. CXLIII, Part III (1853), 407–548. Also: *The collected mathematical papers*, Cambridge (1904), vol. 1, paper 57, pp. 429–586.

[58] H. Tamvakis, *Gromov-Witten invariants and quantum cohomology of Grassmannians,* in: "Topics in cohomological studies of algebraic varieties" (P. Pragacz ed.), Trends in Mathematics, Birkhäuser 2005 – to appear.

[59] J-Y. Thibon, *Hopf algebras of symmetric functions and tensor products of symmetric group representations,* Internat. J. Alg. Comp. **2** (1991), 207–221.

[60] J-Y. Thibon, *Lectures on noncommutative symmetric functions,* in: " Interaction of combinatorics and representation theory", MSJ Memoirs **11** (2001), 39–94.

For Alain's explicit computations the following reference is relevant:

[61] S. Veigneau, *ACE, an algebraic combinatorics environment for the computer algebra system MAPLE,* 1998.

And other books and papers ...