ALGEBRAIC K-THEORY OF SCHEMES

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1. Introduction

The present notes contain the foundations of algebraic K-theory together with a series of explicit computations of the K-groups of fields and some classical varieties. Our goal is to provide an introduction to a more advanced reading, as well as to convince the reader that such a study may be useful and interesting. The exposition is by no means complete nor self-contained. We hope nevertheless, that the covered part of the theory is sufficient for effective computations in algebraic geometry.

The organization of these notes follows the historical development of algebraic K-theory in the 2nd half of the XXth century. Section 2 contains a brief outline of the theory of the Grothendieck groups $K_0(X)$, $K_0'(X)$. In Section 3 we develop the higher K-theory (of Milnor and Quillen) of fields and compute the K-groups of finite fields. The next two sections are based on Quillen's fundamental paper [23]: the definition of K-groups as homotopy groups is given and their properties are discussed. Some instructive examples not contained in [23] are also included. In Section 6 we compute the higher algebraic K-groups of projective bundles, Brauer-Severi varieties (after [23]) and quadrics (after Swan [27]). In the last section we apply these techniques to compute the Chow ring of a split quadric (here we follow the author's approach [28]) and to prove "Hilbert 90" for $K_2\left(F(\sqrt{a})\right)$ following Merkurjev's proof.

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2. Grothendieck groups

There are numerous connections between properties of a scheme/variety X and properties of the categories of sheaves of $\mathcal{O}_X$-modules on X. We are interested in two categories:

- the category $\mathcal{M}(X)$ of coherent $\mathcal{O}_X$-modules on X,
- the category $\mathcal{P}(X)$ of vector bundles (locally free $\mathcal{O}_X$-modules of finite rank) on X.

If $X = \mathrm{Spec}(R)$ is an affine scheme, the first one is equivalent to the category of finitely generated R-modules, while the second one is equivalent to its subcategory of projective R-modules.

In the 50's of the XXth century Alexander Grothendieck studied additive functions on these categories: for an abelian group A, a function $f : \mathcal{C} \to G$ is called *additive*, iff for every exact sequence

$$0 \to A \to B \to C \to 0$$

in C the equality

$$f(B) = f(A) + f(C)$$

holds.

In 1957, in the manuscript [7] along with an algebraic proof of (a generalized) Riemann-Roch-Hirzebruch Theorem, Grothendieck introduced a universal target groups for additive functions. Given an additive category $\mathcal{C}$ embedded in an abelian category $\mathcal{A}$, the *Grothendieck group* $K_0(\mathcal{C})$ of $\mathcal{C}$ is a factor of the free abelian group generated by isomorphic classes $[A]$ of objects of $\mathcal{C}$ modulo subgroup generated by all expressions

$$[B] - [A] - [C]$$

for every sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{C}$, which is exact in $\mathcal{A}$. Thus $K_0(\mathcal{C})$ has generators: one generator $[A]$ for each $A \in \mathrm{Ob}(\mathcal{C})$ and every element of $K_0(\mathcal{C})$ may be expressed in the form $[A] - [B]$ (in many ways). Moreover, split sequences are exact inependently of the embedding in an abelian category, so the addition rule may be written as

$$[A] + [B] = [A \oplus B].$$

In this way, to a scheme X there are associated two K-groups:

$$\begin{aligned} G_0(X) \text{ or } K_0'(X) &= K_0(\mathcal{M}(X)), \\ K_0(X) &= K_0(\mathcal{P}(X)) \end{aligned}$$

(this is modern notation; Grothendieck's was $K_{.}(X)$ and $K^{\cdot}(X)$ - see eg. [2]. Moreover, notation $K_0(X)$ and $K^0(X)$ is in use, e.g. [5], or even $K(X)$ and $K_1(X)$ - [9].) These groups have generators $[A]$ for $A \in \mathrm{Ob}(\mathcal{M}(X))$ (resp. $A \in \mathrm{Ob}(\mathcal{P}(X))$) which are subject to relations

$$[B] = [A] + [C]$$

associated with sequences $0 \to A \to B \to C \to 0$ which are exact in $\mathcal{M}(X)$.

The function

$$A \longmapsto [A]$$

from $\mathrm{Ob}(\mathcal{M}(X))$ (resp. $A \in \mathrm{Ob}(\mathcal{P}(X))$) to $K_0'(X)$ (resp. $K_0(X)$) is universal additive function, in the sense that every additive function factors through it.

In the affine case, $X = \mathrm{Spec}(R)$, it is customary to simplify the notation

$$\begin{aligned} K_0'(\mathrm{Spec}(R)) &= K_0'(R), \\ K_0(\mathrm{Spec}(R)) &= K_0(R). \end{aligned}$$

It is known that it is impossible to classify vector bundles up to isomorphism on all varieties; in contrast, in many cases one can compute K-groups.

Example 2.1. *If $R = F$ is a field ($X = \mathrm{Spec}(F)$ is a point), then every finitely generated F-module is free, and these modules are classified by dimension. Thus*

$$K_0'(F) = K_0(F) \cong \mathbb{Z}$$

and the isomorphism is induced by the additive function $A \longmapsto \dim A$.

Example 2.2. *If $R = D$ is a skew-field (a division ring), then finitely generated left D-modules are free. If A is such a module, then every basis of A has the same number of elements. So two finitely generated D-modules are isomorphic iff they have the same rank (or dimension). It follows that*

$$K_0'(D) = K_0(D) \cong \mathbb{Z}$$

and the isomorphism is induced by the additive function $A \longmapsto \dim A$.

Exercise 2.1. *Prove that for a (not necessarily commutative) ring R and finitely generated projective left R-modules A, B*

$$[A] = [B] \text{ in } K_0(R) \text{ iff there exists}$$
$$\text{a f.g. projective } C \text{ such that } A \oplus C \cong B \oplus C.$$

Exercise 2.2. *Prove that, in general, for a category $\mathcal{C}$ embedded in an abelian category $\mathcal{A}$,*

$$[A] = [B] \text{ in } K_0(\mathcal{C}) \text{ iff there exist objects } C', C, C'' \text{ and exact sequences}$$
$$0 \to C' \to C \to A \oplus C'' \to 0 \text{ and } 0 \to C' \to C \to B \oplus C'' \to 0.$$

2.1. **Definitions, multiplicative structure, contravariant properties, Cartan map.** If $\mathcal{F}$ is a vector bundle, then it is easy to check locally that

$$\mathcal{F} \otimes - : \mathcal{M}(X) \to \mathcal{M}(X)$$

is an exact functor (takes exact sequences to exact sequences). In fact, one has:

Proposition 2.1. *The formula*

$$[A] \cdot [B] = [A \otimes B]$$

defines a multiplication rules

$$\begin{aligned} K_0(X) \otimes_{\mathbb{Z}} K_0(X) &\longrightarrow K_0(X), \\ K_0(X) \otimes_{\mathbb{Z}} K_0'(X) &\longrightarrow K_0'(X). \end{aligned}$$

With this multiplication rule, $K_0(X)$ is a ring with unit element $1 = [\mathcal{O}_X]$, and $K_0'(X)$ is a $K_0(X)$-module. ∎

Example 2.3. *For a field F, the map $\dim : K_0(X) \to \mathbb{Z}$ is a ring isomorphism.*

In the "categorical" context to study dependence of K-groups on functors one must restrict to additive functors which preserve exactness of sequences (exact functors). To do this we will introduce the notion of exact category below. In our set-up, where exactness is defined by embedding into an abelian category, one must use exact functors defined on ambient abelian categories. It is obvious, that such a functor $f : \mathcal{A}_1 \to \mathcal{A}_2$ which takes $\mathcal{C}_1$ to $\mathcal{C}_2$ defines a homomorphism

$$f : K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_2).$$

Example 2.4. *The inclusion functor $\mathcal{P}(X) \to \mathcal{M}(X)$ preserves exactness. Thus there is a homomorphism*

$$K_0(X) \longrightarrow K_0'(X)$$

which acts naturally on generators ($[A] \mapsto [A]$.) This is a homomorpism of $K_0(X)$-modules.

This homomorphism is called the *Cartan homomorphism*.

Example 2.5. *If $f : X \to Y$ is a morphism of schemes, then the pull-back functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact and takes $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Thus K_0 and K_0' are functors from schemes to abelian groups.*

Example 2.6. *(Reduction by resolution) We keep the notation following Example 2.3. Assume that:*
1) for every object A in $\mathcal{C}_2$ there is a finite resolution of A by objects of the form $f(B)$, $B \in \mathcal{C}_1$,
2) if for $A, C \in \mathrm{Ob}(\mathcal{C}_2)$ there is a positive number n and an exact sequence

$$0 \to C \to f(B_{n-1}) \to \cdots \to f(B_0) \to A \to 0,$$

then there exists $B_n \in \mathrm{Ob}(\mathcal{C}_1)$ such that $C \cong f(B_n)$.
Then the homomorphism $f : K_0(\mathcal{C}_1) \to K_0(\mathcal{C}_2)$ is surjective. In fact, if there is an exact sequence

$$0 \to f(B_n) \to \cdots \to f(B_0) \to A \to 0,$$

then to show that

$$[A] = \sum_i (-1)^i f[B_i]$$

one needs only a standard inductive argument.

Example 2.7. *Consider $X = \mathrm{Spec}(\mathbb{Z})$. Locally free abelian groups are simply free abelian groups, so the additive function* rank *defines a ring isomorphism $K_0(\mathbb{Z}) \cong \mathbb{Z}$. Every finitely generated abelian group A has a resolution*

$$0 \to B_1 \to B_0 \to A \to 0$$

with free abelian groups B_0, B_1. Thus the inclusion functor $\mathcal{P}(\mathbb{Z}) \to \mathcal{M}(\mathbb{Z})$ induces a surjective homomorphism $K_0(\mathbb{Z}) \to K_0'(\mathbb{Z})$ which again splits by means of rank function. Thus $K_0'(\mathbb{Z}) = K_0(\mathbb{Z}) \cong \mathbb{Z}$.

Another example: if $f : X \to Y$ is a proper morphism of locally noetherian schemes and every object in $\mathcal{M}(X)$ has a finite resolution by coherent sheaves with no higher derived images, then there is a well defined group homomorphism

$$f_* : K_0'(X) \to K_0'(Y)$$

such that $f_*[\mathcal{F}] = [f_*\mathcal{F}]$ provided $\mathcal{F}$ has no higher derived images.

Proposition 2.2. *If X is a separated noetherian regular scheme, then the Cartan map $K_0(X) \longrightarrow K_0'(X)$ is an isomorphism.* ∎

2.2. **Fundamental Theorem (homotopy property), localization, cycle map, filtrations.** In the affine case $X = \mathrm{Spec}(R)$, where R is a commutative ring, every element of $K_0(R)$ may be written in following special form:

$$[A] - [B] = [A \oplus C] - [B \oplus C] = [R^n] - [B \oplus C] = n - [B \oplus C]$$

where C is such that $A \oplus C \cong R^n$. Moreover, an equality $[A] = [B]$ is equivalent to existence of a C such that $A \oplus C \cong B \oplus C$ (Exercise 2.1), and C may be chosen to be a free module (we say that A and B are *stably isomorphic*). The class $[R^1]$ of rank 1 free module is the unit of the ring $K_0(R)$, so $K_0(R) = \mathbb{Z}$ means that R^1 is a generating module and every finitely generated projective R -module is stably free. It is not very difficult to prove that every finitely generated projective module over a polynomial ring $F[x_1, x_2, \ldots, x_n]$ is stably free, so $K_0(F[x_1, x_2, \ldots, x_n]) = \mathbb{Z}$. Serre in 1955 posed the question if every finitely generated projective module is in this case actually a free module. This fact was proven independently by Quillen and Suslin in 1973.

A weaker statement - the equality $K_0\left(F[x_1, x_2, \ldots, x_n]\right) = \mathbb{Z}$ - has two generalizations.

Theorem 2.3 (Fundamental Theorem). *If R is a regular ring, then the inverse image functors $M \longmapsto R[t] \otimes_R M$, $M \longmapsto R[t, t^{-1}] \otimes_R M$ induce ismorphisms*

$$K_0(R[t]) = K_0(R) = K_0(R[t, t^{-1}]). \quad \blacksquare$$

Theorem 2.4 (Homotopy property). *If $f : X \to Y$ is a morphism of smooth varieties such that all fibers $f^{-1}(y)$ are affine spaces, then inverse image functor f^* induces a ring isomorphism $K_0(Y) = K_0(X)$.* $\blacksquare$

The fact that a (locally free) sheaf on an open subset $U \subset X$ extends to a (locally free) sheaf on whole X can be stated as follows:

Theorem 2.5 (Localization). *Let U be an open subset of a scheme X and let Z be the reduced closed complement of U in X with the reduced scheme structure. Denote by $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ the inclusions. The following sequence is exact:*

$$K_0'(Z) \xrightarrow{i_*} K_0'(X) \xrightarrow{j^*} K_0'(U) \longrightarrow 0 \quad \blacksquare$$

One of the reasons to introduce higher algebraic K-theory was to extend this sequence to a long exact localization sequence. We achieve this in Proposition 5.4 below.

Every closed subscheme Z of X defines an element $[\mathcal{O}_Z] \in K_0'(X)$. If $\mathcal{Z}^r(X)$ is the group of cycles of codimension r in X, and

$$\mathcal{Z}^\bullet(X) = \bigoplus_{r=0}^{\dim X} \mathcal{Z}^r(X)$$

then there is a group homomorphism

$$\begin{array}{ccc} \mathcal{Z}^\bullet(X) & \to & K_0'(X) \\ [Z] & \longmapsto & [\mathcal{O}_Z], \end{array}$$

called the *cycle map*. In the affine case it is easy to see that the image of the cycle map generates $K_0'(X)$ - every finitely generated R-module has a filtration with factors isomorphic to $R/\mathfrak{p}$ for suitable $\mathfrak{p} \in \operatorname{Spec} R$. This fact remains valid for smooth varieties over a field. Thus the homomorphism $\mathcal{Z}^\bullet(X) \to K_0'(X)$ is surjective ([2, Exposé X, Corollaire 1.1.4].)

There is the topological filtration

$$F^p K_0'(X) = \left\{ \alpha \in K_0'(X) : \begin{array}{c} \alpha \text{ is in the kernel of restriction} \\ K_0'(X) \to K_0'(\mathcal{O}_{X,x}) \text{ to the generic} \\ \text{point } x \text{ of every subvariety of codimension } < p \end{array} \right\},$$

and consecutive factors of this filtration are connected with the cycle groups $\mathcal{Z}^r(X)$.

Multiplication in $K_0(X)$ extends to multiplication in $K_0'(X)$:

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum (-1)^i \left[\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})\right].$$

For this multiplication, if cycles Z_1, Z_2 intersect properly, then

$$[\mathcal{O}_{Z_1}] \cdot [\mathcal{O}_{Z_2}] = [\mathcal{O}_{Z_1 \cap Z_2}].$$

2.3. **Projective Bundle Theorem.** Computation of the Grothendieck group of a projective bundle ([2, Exposé VI]) was crucial for the theory and its applications. We present here a modern proof.

Theorem 2.6. *If $\mathcal{E}$ is a vector bundle of rank n on a quasicompact scheme Y,*

$$p : X = \mathbb{P}_Y(\mathcal{E}) \to Y$$

is the projective bundle associated to $\mathcal{E}$, $\mathcal{O}_X(-1)$ is the tautological line bundle, then, via $p^* : K_0(Y) \to K_0(X)$, the $K_0(Y)$-module $K_0(X)$ is a free with the basis $[\mathcal{O}_X], [\mathcal{O}_X(-1)], \ldots, [\mathcal{O}_X(1-n)]$.

Proof. (cf. [20, Theorem 2.1] or [21, Theorem 2.1]) Define a sheaf $\mathcal{J}$ on X by the exact sequence:

$$0 \to \mathcal{J} \to p^*(\mathcal{E}) \to \mathcal{O}_X(1) \to 0.$$

The sheaf $\mathcal{J}$ is locally free of finite rank, since $p^*(\mathcal{E})$ and $\mathcal{O}_X(1)$ are such. Moreover,

$$\begin{aligned} p_*\,\mathcal{O}_X(1) &= \mathcal{E}^{\check{}} = \mathcal{H}\mathbf{om}_{\mathcal{O}_\mathbf{Y}}(\mathcal{E}, \mathcal{O}_\mathbf{Y}), \\ p_*\mathcal{J}^{\check{}} &= \mathcal{E}, \end{aligned}$$

where $\mathcal{J}^{\check{}} = \mathcal{H}\mathbf{om}_{\mathcal{O}_\mathbf{X}}(\mathcal{J}, \mathcal{O}_\mathbf{X})$. Therefore

$$H^0\left(X \times_Y X, \mathcal{O}_X(1) \boxtimes \mathcal{J}^{\check{}}\right) = H^0\left(Y, \mathcal{E}^{\check{}} \otimes \mathcal{E}\right) = \mathrm{End}_Y(\mathcal{E}),$$

where

$$\mathcal{F} \boxtimes \mathcal{G} = p_1^*(\mathcal{F}) \otimes_{\mathcal{O}_{X \times_Y X}} p_2^*(\mathcal{G})$$

is a sheaf on $X \times_Y X$, $\mathcal{F}, \mathcal{G}$ are sheaves on X, and p_1, p_2 are the projections $X \times_Y X \to X$. Consider the global section s of $\mathcal{O}_X(1) \boxtimes \mathcal{J}^{\check{}}$ which corresponds to the identity endomorphism of $\mathcal{E}$. Since the restriction to the diagonal $\Delta : X \to X \times_Y X$ is locally induced by the evaluation map $\mathcal{E}^{\check{}} \otimes \mathcal{E} \to \mathcal{O}_\mathbf{X}$, the section s vanishes exactly along the diagonal, which yields the Koszul resolution

$$\begin{aligned} 0 \quad &\to \quad \mathcal{O}_X(1-n) \boxtimes \bigwedge^{n-1} \mathcal{J} \to \cdots \\ &\to \quad \mathcal{O}_X(-2) \boxtimes \bigwedge^2 \mathcal{J} \to \mathcal{O}_X(-1) \boxtimes \mathcal{J} \to \mathcal{O}_X \boxtimes \mathcal{O}_X \to \mathcal{O}_{\Delta(X)} \to 0. \end{aligned}$$

By the projection formula, for every bundle $\mathcal{F}$ on X

$$\begin{aligned} [\mathcal{F}] &= p_{1*}\left((p_2^*[\mathcal{F}]) \cdot [\mathcal{O}_{\Delta(X)}]\right) = p_{1*}\left((p_2^*[\mathcal{F}]) \cdot \sum_{i=0}^{n-1} (-1)^i \left[\mathcal{O}_X(-i) \boxtimes \bigwedge^i \mathcal{J}\right]\right) \\ &= \sum_{i=0}^{n-1} (-1)^i p^*\left(p_*\left(\left[\bigwedge^i \mathcal{J}\right] \cdot [\mathcal{F}]\right)\right) [\mathcal{O}_X(-i)] \\ &= \sum_{i=0}^{n-1} (-1)^i p_*\left(\left[\bigwedge^i \mathcal{J}\right] \cdot [\mathcal{F}]\right) \cdot [\mathcal{O}_X(-i)], \end{aligned}$$

which implies that the map

$$\begin{aligned} \varphi \quad &: \quad \bigoplus\nolimits_{i=0}^{n-1} K_0(Y) \to K_0(X) \\ \varphi(a_0, \ldots, a_{n-1}) \quad &= \quad \sum_{i=0}^{n-1} a_i\,[\mathcal{O}_X(-i)] \end{aligned}$$

is a surjective homomorphism of $K_0(Y)$-modules.

On the other hand, it is known that

$$\begin{aligned} R^m p_* \mathcal{O}_X(i) &= 0 \text{ for every } m > 0, i \geq 0 \\ p_* \mathcal{O}_X(i) &= \mathrm{Sym}^i \mathcal{E}\check{\ } \text{ for every } i \geq 0 \\ R^m p_* \mathcal{O}_X(i) &= 0 \text{ for every } m \geq 0, 1-n \leq i \leq -1. \end{aligned}$$

Thus for the map

$$\begin{aligned} \psi &: K_0(X) \to \bigoplus\nolimits_{i=0}^{n-1} K_0(Y) \\ \psi([\mathcal{F}]) &= (p_*[\mathcal{F}], p_*[\mathcal{F}(-1)], \ldots, p_*[\mathcal{F}(1-n)]), \end{aligned}$$

the composition $\psi \circ \varphi : K_0(Y)^n \to K_0(Y)^n$ is given by an upper triangular matrix with units on the diagonal, which implies that φ is injective. ∎

Remark 2.1. *We shall prove a more general version of the Projective Bundle Theorem in Theorem 6.9 below.*

3. K-theory of fields

3.1. $K_1(F)$ **and** $K_2(F)$ **(Matsumoto Theorem).** With some effort, proper notions of $K_1(R)$, $K_2(R)$ were found even without assumption that R is commutative. Milnor's book [16] is an excellent, easy and self-contained exposition of this part of developement of the theory.

The group $\mathrm{GL}_n(R)$ may be identified with a subgroup of $\mathrm{GL}_{n+1}(R)$ of block diagonal matrices with 1 in the right lower corner:

$$A \longmapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right).$$

Definition 3.1.

$$\begin{aligned} \mathrm{GL}(R) &= \varinjlim \mathrm{GL}_n(R) \\ \mathrm{E}(R) &= \varinjlim \mathrm{E}_n(R) \end{aligned}$$

where $\mathrm{E}_n(R)$ *is a subgroup of* $\mathrm{GL}_n(R)$ *generated by all elementary matrices.*

Recall that:

- a group G is a *perfect group* iff G has trivial center and G is equal to its commutator subgroup,
- a group is perfect iff it has the universal central extension,
- the group $E(R)$ is perfect,

(see [16, §5])

Definition 3.2. *The Steinberg group* $\mathrm{St}(R)$ *is a universal central extension of the group* $\mathrm{E}(R)$*.*

Definition 3.3.

$$\begin{aligned} &K_1(R) = H_1(\mathrm{GL}(R), \mathbb{Z}) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)] = \mathrm{GL}(R)/\mathrm{E}(R) \\ &K_2(R) = H_2(\mathrm{E}(R), \mathbb{Z}) = \ker(\mathrm{St}(R) \twoheadrightarrow \mathrm{E}(R)) \\ &K_3(R) = H_3(\mathrm{St}(R), \mathbb{Z}) \end{aligned}$$

If R is a commutative ring, there is a homomorphism

$$R^* \xrightarrow{\sim} \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}(R) \twoheadrightarrow K_1(R).$$

If R is a commutative ring, then

$$\det : \mathrm{GL}(R) \longrightarrow R^*$$

factors through $K_1(R)$ and splits the homomorphism $R^* \to K_1(R)$.

The group $K_2(R)$ may be characterized as a kernel of universal central extension of perfect group $\mathrm{E}(R)$, which is a subgroup of $\mathrm{GL}(R)$ generated by (images of) elementary matrices and coincides with the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$.

In the case of a field, it is easy to see that $K_1(F)$ is a multiplicative group of F:

$$\det : K_1(F) \xrightarrow{\sim} F^*.$$

In general, the determinant map defines a splitting epimorphism

$$\det : K_1(R) \twoheadrightarrow R^*$$

and the kernel is usually denoted as $SK_1(R)$.

We will use an additive notation for the group $K_1(F)$. An element corresponding to $a \in F^*$ is denoted by $\{a\}$ (it is not a set!), so

$$\begin{array}{rcl} \{-\} & : & F^* \xrightarrow{\sim} K_1(F) \\ \{ab\} & = & \{a\} + \{b\} \\ \{1\} & = & 0. \end{array}$$

Every element of $K_1(R)$ arises from an automorphism of a free R-module. In particular, if E/F is an finite field extension, then there are canonical transfer maps

- $N_{E/F} : K_0(E) \to K_0(F)$ which as a group homomorphism coincides with multiplication by the degree $[E : F]$,
- $N_{E/F} : K_1(E) \to K_1(F)$ which coincides with usual norm, since the usual norm of $x \in E^{\cdot}$ is determinant of multiplication $x \cdot - : E \to E$.

Description of $K_2(F)$ is more complicated. A *Steinberg symbol* with values in an abelian group A is a map $f : F^{*\cdot} \times F^* \to A$ which:

- is bimultiplicative, i.e.

$$f(ab, c) = f(a, c) + f(b, c), \qquad f(a, bc) = f(a, b) + f(a, c)$$

- vanishes on pairs (a, b) such that $a + b = 1$,

$$f(a, 1 - a) = 0.$$

Matsumoto and (independently) Moore proved that $K_2(F)$ is the group of values of universal Steinberg symbol:

Theorem 3.1 (Matsumoto)**.**

$$K_2(F) \cong F^* \otimes F^*/S$$

where S is a subgroup of $F^* \otimes F^*$ generated by all $a \otimes b$ such that $a + b = 1$.

For a proof see [16]. We use the notation

$$\{a, b\} = a \otimes b \bmod S.$$

Thus the identities

$$\{ab, c\} = \{a, c\} + \{b, c\}, \qquad \{a, bc\} = \{a, b\} + \{a, c\}, \tag{3.1}$$

$$\{1, a\} = \{a, 1\} = 0, \tag{3.2}$$

$$\{a, 1 - a\} = 0 \text{ for } a \neq 1 \tag{3.3}$$

hold in $K_2(F)$. The map

$$\{-, -\} : F^* \times F^* \longrightarrow K_2(F)$$

is called *the universal Steinberg symbol.*

There is a transfer map $N_{E/F} : K_2(E) \to K_2(F)$, but it is not so easy to write formula for it (see Lemma 7.15 and Proposition 7.16 below for the case $(E : F) = 2$.)

It is worth to point out that there are natural multiplications

$$\begin{array}{rcl} K_0(R) \otimes K_i(R) & \to & K_i(R), \\ K_1(R) \otimes K_1(R) & \to & K_2(R); \end{array}$$

in the case of a field the latter is

$$\{a\}\{b\} = \{a, b\}$$

and is "graded commutative" (see Exercise 4 on page 10). For a finite field extension, the projection formula

$$N_{E/F}\left(\{a\} \cdot \{x\}\right) = \{a\} \cdot \{N_{E/F}(x)\}$$

for $a \in F^*$, $x \in E^*$, holds.

3.2. **Milnor K-theory of fields.** The Matsumoto Theorem 3.1 was a motivation for J. Milnor in 1970 ([15]) to define, for a field F, a graded ring

$$K_*^M(F) = K_0(F) \oplus K_1(F) \oplus K_2(F) \oplus K_3^M(F) \cdots \oplus K_n^M(F) \oplus \cdots$$

which has interesting connections with the theory of quadratic forms and Galois cohomology, as an approximation of K-theory.

Definition 3.4. *Let $T^\bullet(F)$ be the tensor algebra of a $\mathbb{Z}$-module F^**

$$T^\bullet(F) = \bigoplus\nolimits_{n=0}^{\infty} (F^*)^{\otimes n}$$

and let I be the two-sided ideal of $T^\bullet(F)$ generated by all expressions $a \otimes b$ such that $a + b = 1$. Then

$$K_\bullet^M(F) \stackrel{\mathrm{df}}{=} T^\bullet(F)/I.$$

The ideal I is homogeneous, so $K_\bullet^M(F)$ is a graded ring. Milnor's K-group $K_n^M(F)$ is the n-th homogeneous component of $K_\bullet^M(F)$, and $K_n^M(F) = K_n(F)$ for $n = 0, 1, 2$. We usually write

$$\{a_1, a_2, \ldots, a_n\} = a_1 \otimes a_2 \otimes \cdots \otimes a_n \bmod I.$$

Milnor was able to prove the following theorem:

Theorem 3.2. *If $F(t)$ is a field of rational functions in t, then there are split exact sequences*

$$0 \to K_n^M(F) \to K_n^M(F(t)) \to \bigoplus \pi K_{n-1}^M(F[t]/\pi) \to 0$$

for $n = 1, 2, \ldots$, where π runs over closed points of the affine line $\mathrm{Spec}F[t]$.

Proof. [15, Theorem 2.3] ∎

3.3. **Quillen K-theory of finite fields.** Known values of K-functor for a ring R led D. Quillen to use homotopy theory methods to compute appropriate homology groups.

In 1969 Quillen introduced a "+-construction": given a pointed arcwise connected CW-complex (X, x), and perfect normal subgroup $H \subset \pi_1(X, x)$, there is a map of CW-complexes $f : (X, x) \to (X^+, x^+)$ such that

- f induces an isomorphism $\pi_1(X, x)/H \xrightarrow{\sim} \pi_1(X^+, x^+)$;
- f induces an isomorphism $H_\bullet(X, f^* L) \xrightarrow{\sim} H_\bullet(X^+, L)$ for every local system of coefficients L on X^+.

There exists the *classifying space* $B\mathrm{GL}(R)$ of the group $\mathrm{GL}(R)$, such that

$$H_\bullet(\mathrm{GL}(R)) = H_\bullet(B\mathrm{GL}(R)) \text{ and } H^\bullet(\mathrm{GL}(R)) = H^\bullet(B\mathrm{GL}(R)).$$

Applying the +-construction to this classifying space yields the following definition ([22]):

Definition 3.5. *For $i > 0$,*

$$K_i(R) = \pi_i(B\mathrm{GL}(R)^+).$$

The space $B\mathrm{GL}(R)^+$ is connected, so $\pi_0(B\mathrm{GL}(R)^+) = 0$. Consider the disjoint union $K_0(R) \times B\mathrm{GL}(R)^+$ of copies of connected space $B\mathrm{GL}(R)^+$, one for each element of $K_0(R)$. This space has the same higher homotopy groups as $B\mathrm{GL}(R)^+$ (higher homotopy group do depend on the connected component of the base point only). Moreover, $\pi_0(K_0(R) \times B\mathrm{GL}(R)^+) = K_0(R)$ by the construction, so

$$K_i(R) = \pi_i(K_0(R) \times B\mathrm{GL}(R)^+)$$

for all $i \geq 0$. The product $K_0(R) \times B\mathrm{GL}(R)^+$ is the first example of so called *K-theory space* - a space which homotopy groups are desired K-groups. Note that such a space is defined only up to homotopy equivalence, since $B\mathrm{GL}$ is. The proper term "a homotopy type of spaces" is much longer than "space", so we use not-so-correct notion "K-theory space".

With this definition Quillen was able to compute K-groups of a finite field $\mathbb{F}_q$ with q elements:

Theorem 3.3. *For $i > 0$*

$$K_{2i}(\mathbb{F}_q) = 0,$$
$$K_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1).$$

If $\mathbb{F}_q \subset \mathbb{F}_r$, then $K_n(\mathbb{F}_q) \subset K_n(\mathbb{F}_r)$ for all n.

In particular, this shows that Milnor K-theory, altough useful, gives no proper values of $K_n(F)$ for $n > 1$.

Exercise 3.1.

(1) *Simplify the expression* $\dfrac{1-x}{1-x^{-1}}$.
(2) *Use this to show that* $\{x, -x\} = 0$ *in* $K_2(F)$.
(3) *Deduce that* $\{x, x\} = \{x, -1\}$.
(4) *Deduce that* $\{x, y\} + \{y, x\} = 0$.

Exercise 3.2. *Assume that F is a finite field.*

(1) *Show that for every $a, b \in F^*$, the equation $ax^2 + by^2 = 1$ has a solution in F.*

(2) *Use this and the fact that the group F^* is cyclic to prove that $K_2(F) = 0$.*
(3) *Deduce that $K_n^M(F) = 0$ for $n > 1$.*

4. Quillen Q-construction

Homotopy theory turns out to be a proper setting for developing K-theory. For a suitable category (symmetric monoidal or exact or Waldhausen), (the homotopy type of) a topological space is attached; K-theory groups are homotopy groups of this K-theory space. We discuss the first general Quillen definition of higher algebraic K-theory of an exact category in some detail.

4.1. **Exact categories, Q-construction, classifying space of a category.**

4.1.1. *Exact categories.*

Definition 4.1. Exact category $\mathfrak{M} = (\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$ *is an additive category $\mathfrak{M}$, having a set of isomorphism classes of objects, embedded as a full subcategory in abelian category $\mathfrak{A}$, closed under extensions in $\mathfrak{A}$, with a family $\mathfrak{E}$ of exact (in $\mathfrak{A}$) sequences*

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0 \tag{4.1}$$

(called admissible exact sequences*) satisfying the following conditions:*

(1) all split exact sequences of objects of $\mathfrak{M}$ are in $\mathfrak{E}$; if 4.1 is in $\mathfrak{E}$, then α is a kernel of β in $\mathfrak{M}$ and β is the cokernel of α in $\mathfrak{M}$;
We say that a monomorphism (an epimorphism) is an *admissible monomorphism* (an *admissible epimorphism*) iff it occurs in an admissible exact sequence.
(2) a composition of admissible epimorphisms (monomorphisms) is an admissible epimorphism (monomorphism)

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[dr, dashed, tail, "\beta\circ\alpha"'] & B \arrow[d, tail, "\beta"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A & B \arrow[l, two heads, "\alpha"'] \\ & C \arrow[u, two heads, "\beta"'] \arrow[ul, dashed, two heads, "\alpha\circ\beta"] \end{tikzcd}$$

a (co)base change of an admissible epimorphism (monomorphism) is an admissible epimorphism (monomorphism)

$$\begin{tikzcd} C \arrow[r, dashed, tail, "\alpha'"] & C \sqcup_A B \\ A \arrow[u, "\varphi"] \arrow[r, tail, "\alpha"'] & B \arrow[u, "\varphi'"'] \end{tikzcd} \qquad \begin{tikzcd} A \times_B C \arrow[r, dashed, two heads, "\alpha'"] \arrow[d, "\varphi'"'] & C \arrow[d, "\varphi"] \\ A \arrow[r, two heads, "\alpha"'] & B \end{tikzcd}$$

(3) if $M \to M''$ possesses a kernel in $\mathfrak{M}$ and the composition $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism; the dual statement for monomorphisms holds true.

$$\begin{tikzcd} \mathrm{Ker}\alpha \arrow[r] & B \arrow[r, dashed, two heads, "\alpha"] & A \\ & C \arrow[u] \arrow[ur, two heads] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Coker}\alpha & B \arrow[l] \arrow[d] & A \arrow[l, dashed, tail, "\alpha"'] \arrow[dl, tail] \\ & C \end{tikzcd}$$

For example, given an admissible exact sequence (4.1) and a map $N \xrightarrow{\varphi} M''$ the sequence

$$0 \to M' \xrightarrow{\overline{\alpha}} M \times_{M''} N \xrightarrow{p_2} N \to 0$$

is an admissible exact sequence.

It is now known that condition (3) is a consequence of (1) - (2) (Keller, [10, Appendix A]).

Moreover, for an exact category $(\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$ there exists a functor $\mathfrak{M} \hookrightarrow \mathfrak{A}'$ which embeds $\mathfrak{M}$ as a full, closed under extensions subcategory into an abelian category $\mathfrak{A}'$ such that $\mathfrak{E}$ is the class of all short exact sequences in $\mathfrak{M}$ which are exact in $\mathfrak{A}'$. So there is another way to define an exact category.

We will use arrows $\rightarrowtail$ and $\twoheadrightarrow$ to denote admissible monomorphisms and admissible epimorphisms respectively.

An admissible subobject is a source of an admissible monomorphism.

There is a notion of an *exact functor* between exact categories: it is an additive functor, which takes admissible exact sequences to admissible exact sequences.

Some examples:

Example 4.1. $\mathfrak{M} = \mathfrak{A}$, $\mathfrak{E}$ *is the family of all exact sequences.*

Example 4.2. $\mathfrak{M} = \mathfrak{A}$, $\mathfrak{E}$ *is the family of all split exact sequences.*

Example 4.3. $\mathfrak{A}$ *is the category of finitely generated left R-modules, $\mathfrak{M}$ is the full subcategory of projective modules, $\mathfrak{E}$ is the family of all sequences that are exact in $\mathfrak{A}$.*

Example 4.4. *X is a scheme, $\mathfrak{A}$ is the category of coherent $\mathcal{O}_X$-modules, $\mathfrak{M}$ is the full subcategory of locally free modules, $\mathfrak{E}$ is the family of all sequences that are exact in $\mathfrak{A}$.*

4.1.2. *The Q-construcion.* For an exact category $\mathfrak{M} = (\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$ Quillen defined another category, denoted by $Q\mathfrak{M}$ (called the *Quillenization* of $\mathfrak{M}$.) Given an exact category $(\mathfrak{M}, \mathfrak{A}, \mathfrak{E})$ the category $Q\mathfrak{M}$ has the same objects as $\mathfrak{M}$, and a morphism from M to M' in $Q\mathfrak{M}$ is a class of diagrams

$$M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$$

up to isomorphism, which induces the identities on M and M'.

The composition of morphisms $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ and $M' \overset{\delta}{\twoheadleftarrow} N' \overset{\gamma}{\rightarrowtail} M''$ in the category $Q\mathfrak{M}$ is defined by the fiber product as follows. Given two composable morphisms $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$, $M' \overset{\beta'}{\twoheadleftarrow} N' \overset{\gamma}{\rightarrowtail} M''$, the diagram

$$\begin{tikzcd}
 & N \arrow[dl, two heads, "\beta"'] \arrow[dr, tail, "\alpha"] & & N' \arrow[dl, two heads, "\beta'"'] \arrow[dr, tail, "\gamma"] & \\
M & & M' & & M''
\end{tikzcd}$$

is completed to the diagram

$$\begin{tikzcd}
 & & N \times_{M'} N' \arrow[dl, twoheadrightarrow, "p_1"'] \arrow[dr, tail, "p_2"] & & \\
 & N \arrow[dl, twoheadrightarrow, "\beta"'] \arrow[dr, tail, "\alpha"] & & N' \arrow[dl, twoheadrightarrow, "\beta'"'] \arrow[dr, tail, "\gamma"] & \\
M & & M' & & M''
\end{tikzcd}$$

which yields the composition

$$M \overset{\beta \circ p_1}{\twoheadleftarrow} N \times_{M'} N' \overset{\gamma \circ p_2}{\rightarrowtail} M''.$$

It is clear that composition is well-defined and associative. Since the isomorphism classes of diagrams $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ form a set for each M, M', the $Q\mathfrak{M}$ is a well defined category.

Each admissible monomorphism $N \overset{\alpha}{\rightarrowtail} M$ gives rise to a morphism $\alpha_! : N \to M$ in $Q\mathfrak{M}$ represented by the diagram

$$\alpha_! : N \overset{1_N}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M$$

and morphisms of this type are called *injective*.

Dually, each admissible epimorphism $M \overset{\beta}{\twoheadleftarrow} N$ defines a morphism $\beta^! : M \to N$ in $Q\mathfrak{M}$ represented by the diagram

$$\beta^! : M \overset{\beta}{\twoheadleftarrow} N \overset{1_N}{\rightarrowtail} N$$

and these morphisms are called *surjective*.

Note that $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M' = \alpha_! \circ \beta^!$. There is a dual decomposition: given a map $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M'$ the fiber product

$$\begin{tikzcd}
N \arrow[r, tail, "\alpha"] \arrow[d, twoheadrightarrow, "\beta"'] & M' \arrow[d, twoheadrightarrow, "\delta"] \\
M \arrow[r, tail, "\gamma"'] & M \times_N M'
\end{tikzcd}$$

defines a decomposition $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M' = \delta^! \circ \gamma_!$. Conversely, for a diagram $M \overset{\gamma}{\rightarrowtail} N' \overset{\delta}{\twoheadleftarrow} M'$ the fiber sum $N = M \amalg_{N'} M'$

$$\begin{tikzcd}
M \amalg_{N'} M' \arrow[r, tail, "\alpha"] \arrow[d, twoheadrightarrow, "\beta"'] & M' \arrow[d, twoheadrightarrow, "\delta"] \\
M \arrow[r, tail, "\gamma"'] & N'
\end{tikzcd}$$

defines a morphism $M \overset{\beta}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M' = \alpha_! \circ \beta^!$ with the decomposition. In fact, this is an alternative way to define morphisms in $Q\mathfrak{M}$. It may be convenient to regard a morphism in the category $Q\mathfrak{M}$ as a bicartesian (cartesian and cocartesian simultaneously) square

$$\begin{tikzcd} N \arrow[r, tail, "\alpha"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "\square"] & M' \arrow[d, two heads, "\delta"] \\ M \arrow[r, tail, "\gamma"] & N' \end{tikzcd}$$

and agree that usually we omit one corner of the square for short. The $\square$ sign indicates that marked square is bicartesian.

Proposition 4.1.
a) If α and α' are composable monomorphisms in $\mathfrak{M}$, then $(\alpha \circ \alpha')_! = \alpha_! \circ \alpha'_!$ in $Q\mathfrak{M}$;
b) if β and β' are composable epimorphisms in $\mathfrak{M}$, then $(\beta' \circ \beta)^! = \beta^! \circ \beta'^!$ in $Q\mathfrak{M}$;
c) $(1_M)_! = (1_M)^! = 1_M$;
d) for a bicartesian square

$$\begin{tikzcd} N \arrow[r, tail, "\alpha"] \arrow[d, two heads, "\beta"'] \arrow[dr, phantom, "\square"] & M' \arrow[d, two heads, "\delta"] \\ M \arrow[r, tail, "\gamma"'] & N' \end{tikzcd}$$

with admissible arrows in $\mathfrak{M}$ the equality $\alpha_! \circ \beta^! = \delta^! \circ \gamma_!$ holds in $Q\mathfrak{M}$. ∎

4.1.3. *The universal property of the Q-construction.* Supose given a category $\mathfrak{C}$, for each object M in $\mathfrak{M}$ an object hM of $\mathfrak{C}$, and for each $N \overset{\alpha}{\rightarrowtail} M'$ (resp. $M \overset{\beta}{\twoheadleftarrow} N$) in $\mathfrak{M}$ a map $\alpha_! : hN \to hM'$ (resp. $\beta^! : hM \to hN$) such that the properties a), b), c), d) of Proposition 4.1 hold. Then it is clear that this data induce a unique functor $F : Q\mathfrak{M} \to \mathfrak{C}$, $F(M) = hM$ compatible with the operations $\alpha \mapsto \alpha_!$, $\beta \mapsto \beta^!$.

This universal property of the Q-construction shows that an exact functor $F : \mathfrak{M} \to \mathfrak{M}'$ between exact categories induces a functor $Q\mathfrak{M} \to Q\mathfrak{M}'$, $M \mapsto FM$, $\alpha_! \mapsto (F\alpha)_!$, $\beta^! \mapsto (F\beta)^!$. Also for the dual category $\mathfrak{M}^o$ of an exact category $\mathfrak{M}$ there is an isomorphism of categories

$$Q\mathfrak{M}^o = Q\mathfrak{M}$$

such that the injective arrows in the former correspond to the surjective arrows in the latter and conversely.

4.1.4. *Isomorphisms.* For an isomorphism $N \overset{\alpha}{\rightarrowtail} M$, two maps $\alpha_! : N \to M$ and $\alpha^{-1!} : N \to M$ are equal since there is commutative diagram

$$\begin{tikzcd} N \arrow[d, equal] & N \arrow[l, two heads, "1_N"'] \arrow[r, tail, "\alpha"] \arrow[d, "\alpha"] & M \arrow[d, equal] \\ N & M \arrow[l, two heads, "\alpha^{-1}"] \arrow[r, tail, "1_M"] & M \end{tikzcd}$$

Conversely, a map in $Q\mathfrak{M}$ which is both injective and surjective is an isomorphism and it is of the form $\alpha_! = \alpha^{-1!}$ for a unique isomorphism α in $\mathfrak{M}$.

4.1.5. *Zero maps.* Let us denote $0^M : 0 \rightarrowtail M$ and $0_M : M \twoheadrightarrow 0$ unique maps in the additive category $\mathfrak{M}$. The set $\mathrm{Mor}_{Q\mathfrak{M}}(0, M)$ is in $1-1$ correspondence with the set of admissible subobjects of M (i.e. admissible monomorphisms $N \overset{\alpha}{\rightarrowtail} M$ up to automorphism of N over M), since each such a morphism by the definition is given by diagram

$$0_\alpha^M : 0 \overset{0_N}{\twoheadleftarrow} N \overset{\alpha}{\rightarrowtail} M.$$

Thus the set $\mathrm{Mor}_{Q\mathfrak{M}}(0, M)$ of morphisms from 0 to M is partially ordered with the smallest element $0_!^M$ and the greatest element $0^!_M$. There are decompositions $0_!^M = \alpha_! \circ 0_!^N$ and $0_\alpha^M = \alpha_! \circ 0^!_N$ since each such a morphism is, by definition, given by the diagram:

$$\begin{tikzcd}
0 \times_N N = 0 \arrow[r, tail] \arrow[d, equal] \arrow[dr, phantom, "\square"] & N \arrow[r, tail, "\alpha"] \arrow[d, two heads] & M \\
0 \arrow[r, tail] \arrow[d, two heads] & N \arrow[ur, "\alpha_!"'] & \\
0 \arrow[ur, "0_!^N"'] & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
N \times_N N = N \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & N \arrow[r, tail, "\alpha"] \arrow[d, equal] & M \\
N \arrow[r, equal] \arrow[d, two heads] & N \arrow[ur, "\alpha_!"'] & \\
0 \arrow[ur, "0^!_N"'] & &
\end{tikzcd}$$

Dually, given admissible epimorphism $M \overset{\beta}{\twoheadrightarrow} M''$ with kernel $N \overset{\alpha}{\rightarrowtail} M$, there are decompositions $0^!_M = \beta^! \circ 0^!_{M''}$ and $0_\alpha^M = \beta^! \circ 0_!^{M''}$ given by commutative diagrams:

$$\begin{tikzcd}
N = \mathrm{Ker}\, \beta = 0 \times_{M''} M \arrow[r, tail, "\alpha"] \arrow[d, two heads] & M \arrow[d, two heads, "\beta"] \\
0 \arrow[r, tail] & M''
\end{tikzcd}$$

Proposition 4.2. *An admissible exact sequence 4.1 produces following commutative diagram in $Q\mathfrak{M}$:*

$$\begin{tikzcd}
& & 0 \arrow[dll, "0_!^{M''}"'] \arrow[drr, "0^!_{M'}"] \arrow[dd, "0_\alpha^M"] & & \\
M'' \arrow[dr, "\beta^!"] & & & & M' \arrow[dll, "\alpha_!"'] \\
& & M & & \\
& 0 \arrow[uul, "0^!_{M''}"] \arrow[ur, "0^!_M"'] & & 0 \arrow[ul, "0_!^M"] \arrow[uur, "0_!^{M'}"'] &
\end{tikzcd}$$

Proof. There are decompositions $0_!^M = \alpha_! \circ 0_!^{M'}$ and $0_\alpha^M = \alpha_! \circ 0^!_{M'}$. ∎

4.2. **Higher K-groups and their elementary properties.** The classyfying space $B\mathfrak{C}$ of a small category $\mathfrak{C}$ is the geometric realization of the nerve $N\mathfrak{C}$ of this category. The nerve $N\mathfrak{C}$ of a small category $\mathfrak{C}$ is a simplicial set whose

(1) p-simplices are diagrams in $\mathfrak{C}$ of the form
$$X_0 \to X_1 \to \cdots \to X_p$$
(2) the i-th face of the simplex is obtained by deleting the object X_i and composing arrows which come to and go from X_i;
(3) the i-th degeneracy of the simplex is obtained by inserting $X_i \overset{1_{X_i}}{\to} X_i$ in place of X_i.

Quillen [23, I.2., Theorem 1 p. 18] proved the following

Theorem 4.3. *The fundamental group $\pi_1(B(Q\mathfrak{M}), 0)$ is canonically isomorphic to the Grothendieck group $K_0(\mathfrak{M})$.* ∎

It is obvious from this proof that the class $[M] \in K_0(\mathfrak{M})$ corresponds under this isomorphism to the loop $\left|0^!_M\right| - \left|0^M_!\right|$ formed from paths $\left|0^!_M\right|$ corresponding to 1-simplex $0^!_M : 0 \to M$ and $\left|0^M_!\right|$ corresponding to 1-simplex $0^M_! : 0 \to M$.

This theorem was a motivation for the following definition:

Definition 4.2. *The higher algebraic K-theory groups of an exact category $\mathfrak{M}$ are the homotopy groups of the classifying space of the category $Q\mathfrak{M}$:*
$$K_i(\mathfrak{M}) = \pi_{i+1}(B(Q\mathfrak{M}), 0).$$

It is known, that this definition agrees with the earlier definition of K-theory of a ring, that used the $+$-construction. In particular K-groups of a finite field are known as well as K_0, K_1 and K_2 of arbitrary field. The last Th(eorem) on page 228 of [6] states that for a ring R the loop space $\Omega Q\mathcal{P}(\mathrm{Spec}(R))$ is homotopy equivalent to $K_0(R) \times B\mathrm{GL}(R)^+$ (cf. [29, IV§5, Theorem 5.1].)

Example 4.5. *A geometrical realization of the simplicial set, associated to the commutative diagram of an admissible exact sequence 4.1 consists of the following four 2-simplices*
$$0 \xrightarrow{0^{M'}_!} M' \xrightarrow{\alpha_!} M, \qquad 0 \xrightarrow{0^{M''}_!} M'' \xrightarrow{\beta^!} M,$$
$$0 \xrightarrow{0^!_{M'}} M' \xrightarrow{\alpha_!} M, \qquad 0 \xrightarrow{0^!_{M''}} M'' \xrightarrow{\beta^!} M,$$
visible in diagram 4.2. This simplicial complex is obtained from the diagram by glueing together the three 0 vertices (fig. 1.) It may be realized as a parachute form (fig. 2) or a sphere with a hole of a clover-leaf shape (fig. 3) - boundary of each part of the leaf is a loop corresponding to one of objects M', M, M'' with such an orientation that sum of the loop of the subobject and the loop of the factor is the loop of M.

Example 4.6. *A long exact sequence, e.g.*
$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \to 0$$
may be split into two short exact sequences by $E = \operatorname{Im}\beta = \ker\gamma$ and resulting complexes are glued together along the boundary of the hole corresponding to E into a sphere with four-lobed hole.

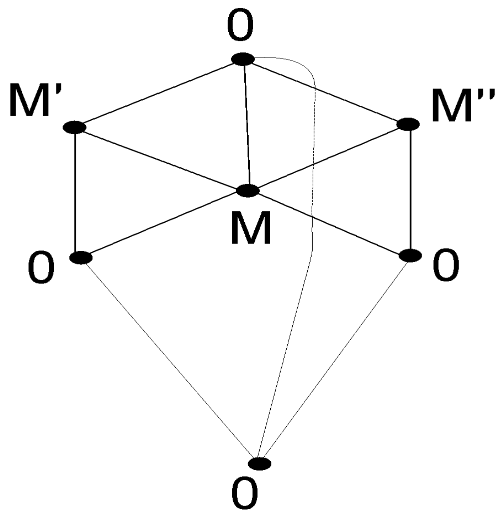

FIGURE 1. Connecting 0 vertices

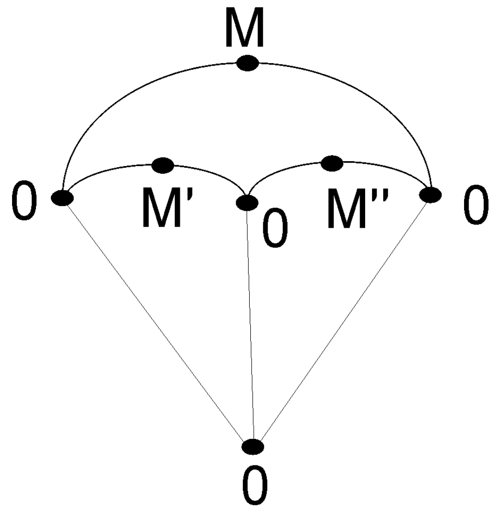

FIGURE 2. Parachute

Example 4.7. *Given two short exact sequences with the same subobject, a total object and a factor (so called double short exact sequence) there are two clover-holed spheres with common boundary of the holes. This defines an element of* $\pi_2(B(Q\mathfrak{M}), 0)$. *A. Nenashev proved (*[18], [19]*) that for arbitrary exact category* $\mathfrak{M}$, *every element of* $K_1(\mathfrak{M})$ *is given by a double short exact sequence in this way. Moreover, he gave a family of defining relations for* $K_1(\mathfrak{M})$ *in terms of double short exact sequences.*

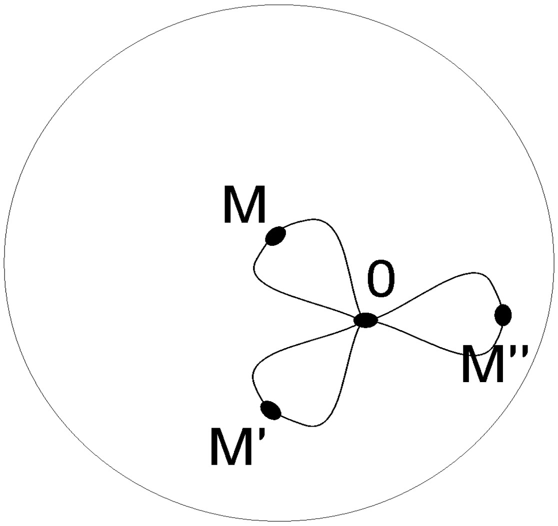

FIGURE 3. Clover-holed sphere

Any exact functor $f : \mathfrak{M} \to \mathfrak{M}'$ induces a functor $Qf : Q\mathfrak{M} \to Q\mathfrak{M}'$, and hence a homomorphism of K-groups which we denote by

$$f_* : K_\bullet(\mathfrak{M}) \to K_\bullet(\mathfrak{M}').$$

A natural transformation $Qf \to Qg$ of functors induces a homotopy between maps $B(Qf)$ and $B(Qg)$, so naturally equivalent exact functors $f, g : \mathfrak{M} \to \mathfrak{M}'$ induce the same homomorphism $f_* = g_*$.

Example 4.8. *Given two exact categories $\mathfrak{M}, \mathfrak{M}'$ one may convert the product $\mathfrak{M} \times \mathfrak{M}'$ into exact category in the obvious way. Clearly $Q(\mathfrak{M} \times \mathfrak{M}') = Q(\mathfrak{M}) \times Q(\mathfrak{M}')$ and $(\mathrm{pr}_1, \mathrm{pr}_2)$ is an isomorphism*

$$K_i(\mathfrak{M} \times \mathfrak{M}') = K_i(\mathfrak{M}) \oplus K_i(\mathfrak{M}').$$

Example 4.9. *Let*

$$f' \rightarrowtail f \twoheadrightarrow f''$$

be an exact sequence of exact functors from an exact category $\mathfrak{M}$ to an exact category $\mathfrak{N}$. Then

$$f_* = f'_* + f''_*$$

as homomorphisms $K_i(\mathfrak{M}) \to K_i(\mathfrak{N})$.

This may be generalized as follows. We say that a filtration

$$0 = f_0 \subset f_1 \subset \cdots \subset f_n = f : \mathfrak{M} \to \mathfrak{N}$$

is an *admissible filtration* iff $f_{p-1}(X) \to f_p(X)$ is an admissible monomorphism for all X and p. This implies that there exist quotient functors f_p/f_q for all $q \le p$ and if f_p/f_{p-1} are exact, then all quotients f_p/f_q are exact.

Example 4.10. *If $f : \mathfrak{M} \to \mathfrak{N}$ is an exact functor between exact categories equipped with an admissible filtration $0 = f_0 \subset f_1 \subset \cdots \subset f_n = f$ such that the consecutive* *quotients functors f_p/f_{p-1} are exact, then*

$$f_* = \sum_{p=1}^{n} (f_p/f_{p-1})_*.$$

Example 4.11. *If there is an exact sequence*

$$0 \to f_0 \to f_1 \to \cdots \to f_n \to 0$$

of exact functors $\mathfrak{M} \to \mathfrak{N}$, then

$$\sum_{p=0}^{n} (-1)^p (f_p)_* = 0.$$

Example 4.12. *Let X be a ringed space, $\mathcal{P}(X)$ - the category of vector bundles on X with the usual notion of an exact sequence, $K_i(X) = K_i(\mathcal{P}(X))$. Given E in $\mathcal{P}(X)$, there is an exact functor:*

$$E \otimes_{\mathcal{O}_X} - : \mathcal{P}(X) \to \mathcal{P}(X)$$

which induces a homomorphism $(E \otimes_{\mathcal{O}_X} -)_* : K_i(X) \to K_i(X)$. If

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of vector bundles, then

$$(E \otimes_{\mathcal{O}_X} -)_* = (E' \otimes_{\mathcal{O}_X} -)_* + (E'' \otimes_{\mathcal{O}_X} -)_*.$$

Thus there is a product

$$\begin{aligned} K_0(X) \otimes_{\mathbb{Z}} K_i(X) &\to K_i(X) \\ [E] \otimes x &\longmapsto (E \otimes_{\mathcal{O}_X} -)_*(x). \end{aligned}$$

In fact, there are products $K_i(X) \otimes_{\mathbb{Z}} K_j(X) \to K_{i+j}(X)$ which generalize the above and make $K_\bullet(X)$ into a graded-commutative ring but the construction requires more machinery.

Example 4.13. *If X is a scheme, then there are exact functors*

$$\begin{aligned} E \otimes_{\mathcal{O}_X} - &: \mathcal{P}(X) \to \mathcal{P}(X) \\ E \otimes_{\mathcal{O}_X} - &: \mathcal{M}(X) \to \mathcal{M}(X) \end{aligned}$$

which yield a $K_0(X)$-module structure on both $K_.(X)$ and $K'_.(X)$.

4.3. **Devissage and localization in abelian categories.** Technical theorems on homotopy theory of categories - Quillen's Theorem A and B ([23, §7]) - yield results on K-theory of abelian categories.

Let $\mathcal{A}$ be an abelian category having a set of isomorphism clases of objects, with all short exact sequences admissible. Let moreover $\mathcal{B}$ be a non-empty full subcategory closed under taking subobjects, quotient objects and finite products in $\mathcal{A}$, with all short exact sequences admissible. Thus $\mathcal{B}$ is an abelian category and the inclusion functor $\mathcal{B} \to \mathcal{A}$ is exact.

Then $Q\mathcal{B}$ is a full subcategory of $Q\mathcal{A}$ consisting of those objects which are also objects of $\mathcal{B}$.

Theorem 4.4 (Devissage). *Suppose that every object A of abelian category $\mathcal{A}$ has a finite filtration*

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

such that A_j/A_{j-1} is in $\mathcal{B}$ for each j. Then the inclusion $Q\mathcal{B} \to Q\mathcal{A}$ is a homotopy equivalence, so $K_i(\mathcal{B}) \cong K_i(\mathcal{A})$.

Proof. [23, §5, Theorem 4]. ∎

Example 4.14. *Let $\mathcal{A}$ be an abelian category such that:*
• *the isomorphism classes of objects of $\mathcal{A}$ form a set,*
• *every object of $\mathcal{A}$ has finite length (has a finite filtration with simple objects as consecutive factors.)*
Then

$$K_i(\mathcal{A}) \cong \coprod_{j \in J} K(D_j)$$

where $\{X_j : j \in J\}$ is a set of representatives for the isomorphism classes of simple objects in $\mathcal{A}$, and D_j is the skew field $\mathrm{End}(X_j)^{op}$.

Example 4.15. *If I is a nilpotent two-sided ideal in a noetherian ring R, then $K_i'(R/I) \cong K_i'(R)$.*

The second result - long exact homotopy sequence - is the main tool for producing long exact sequence of localization.

Theorem 4.5 (Localization). *Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$, let $\mathcal{A}/\mathcal{B}$ be the associated quotient abelian category, and let $\varepsilon : \mathcal{B} \to \mathcal{A}$, $\sigma : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ denote the canonical functors. Then there is a long exact sequence*

$$\cdots \xrightarrow{\sigma_*} K_1(\mathcal{A}/\mathcal{B}) \xrightarrow{\partial} K_0(\mathcal{B}) \xrightarrow{\varepsilon_*} K_0(\mathcal{A}) \xrightarrow{\sigma_*} K_0(\mathcal{A}/\mathcal{B}) \to 0$$

Proof. [23, §5, Theorem 5]. ∎

This is a homotopy long exact sequence of a (homotopy) fibration $B(Q\mathcal{B}) \to B(Q\mathcal{A}) \to B(Q(\mathcal{A}/\mathcal{B}))$; it is natural in $\mathcal{A}, \mathcal{B}$.

Example 4.16. *If R is a Dedekind domain (regular normal domain of dimension 1) with quotient field $R_{(0)}$, then there is a long exact sequence*

$$\cdots \to K_{i+1}\left(R_{(0)}\right) \to \coprod_{\mathfrak{m} \in \mathrm{Specm} R} K_i(R/\mathfrak{m}) \to K_i(R) \to K_i\left(R_{(0)}\right) \to \cdots$$

5. $K'_\bullet$ of noetherian schemes

The category $\mathcal{M}(X)$ of coherent $\mathcal{O}_X$-modules on a noetherian scheme X is an abelian category, so the Devissage and Localization Theorems apply. We list shortly basic results, following [23, §7].

5.1. **Fundamental Theorem.**

Theorem 5.1. *If R is a noetherian ring, then there are canonical isomorphisms*

$$K_i'(R[t]) \cong K_i'(R),$$
$$K_i'(R[t, t^{-1}]) \cong K_i'(R) \oplus K_{i-1}'(R)$$

If R is regular, there are canonical isomorphisms

$$K_i(R[t]) \cong K_i(R),$$
$$K_i(R[t,t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R).$$

Proof. [23, §6, Theorem 8]. ∎

In general, if a noetherian separated scheme X is regular, then the Cartan homomorphism $K_i(X) \to K'_i(X)$ is an isomorphism.

5.2. **Functoriality, localization, homotopy property.** If $f : X \to Y$ is a morphism of schemes, then the inverse image functor $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ is exact and induces homomorphism of K-groups which is denoted $f^* : K_i(Y) \to K_i(X)$.

If $f : X \to Y$ is a flat morphism of schemes, then the inverse image functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact and induces homomorphism of K-group which is denoted $f^* : K'_i(Y) \to K'_i(X)$.

In both cases the formula

$$(fg)^* = g^* f^*$$

holds.

Thus K_i becomes a contravariant functor from noetherian separated schemes to abelian groups and K'_i becomes a contravariant functor on the subcategory of noetherian separated schemes and flat morphisms. Some homotopy theory is needed to prove that K_i takes filtered projective limits of schemes with affine transition maps to filtered injective limits of abelian groups, and K'_i takes such a limits with flat affine transition maps to appropriate limits of abelian groups.

Let $f : X \to Y$ be a proper morphism, so that the higher derived image functors $\mathrm{R}^i f_*$ carry coherent sheaves on X to coherent sheaves. Let $\mathcal{F}(X, f)$ denote the full subcategory of $\mathcal{M}(X)$ consisting of sheaves F such that $\mathrm{R}^i f_*(F) = 0$ for $i > 0$. The functor f_* induces a homomorphism

$$f_* : K'_i(\mathcal{F}(X, f)) \to K'_i(Y).$$

In the case when f is finite, in particular, when f is a closed immersion, $\mathcal{F}(X, f) = \mathcal{M}(X)$.

In the case when X has an ample line bundle, every coherent sheaf on X can be embedded in an object of $\mathcal{F}(X, f)$, which implies that the inclusion $\mathcal{F}(X, f) \to \mathcal{M}(X)$ induces an isomorphism on K'-groups.

In both cases a homomorphism $f_* : K'_i(X) \to K'_i(Y)$ is defined.

Proposition 5.2 (Projection Formula). *If for $f : X \to Y$ both $f_* : K'_i(X) \to K'_i(Y)$ and $f^* : K'_i(Y) \to K'_i(X)$ are defined, then for $x \in K_0(X)$, $y \in K'_i(Y)$ the equlity:*

$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

in $K'_i(Y)$ holds.

Proof. [23, §7, Proposition 2.10]. ∎

Let $\iota : Z \to X$ be a closed subscheme of X, and let $\mathcal{I}$ be the coherent sheaf of ideals in $\mathcal{O}_X$ defining Z. The functor $\iota_* : \mathcal{M}(Z) \to \mathcal{M}(X)$ allows us to identify coherent sheaves on Z with coherent sheaves on X annihilated by $\mathcal{I}$.

Proposition 5.3. *If $\mathcal{I}$ is nilpotent, then $\iota_* : K'_i(Z) \to K'_i(X)$ is an isomorphism. In particular $K'_i(X_{red}) \cong K'_i(X)$.* ∎

Now we extend the exact sequence of Localization Theorem 2.5:

Proposition 5.4 (Localization). *Let $j : U \to X$ be the open complement of Z in X. The there is a long exact sequence*

$$\cdots \to K'_{i+1}(U) \xrightarrow{\partial} K'_i(Z) \xrightarrow{\iota_*} K'_i(X) \xrightarrow{j^*} K'_i(U) \to \cdots .$$

Proof. The functor $j^* : \mathcal{M}(X) \to \mathcal{M}(U)$ induces an equivalence of $\mathcal{M}(U)$ with the quotient category $\mathcal{M}(X)/\mathcal{B}$, where $\mathcal{B}$ is the Serre subcategory consisting of coherent sheaves with support in Z. Devissage implies that $\iota_* : \mathcal{M}(Z) \to \mathcal{B}$ induces isomorphism on K-groups, so the desired exact sequence results from 4.5. ∎

We get, as an immediate application, the following generalization of the Fundamental Theorem 5.1:

Proposition 5.5 (Homotopy Property). *Let $f : P \to X$ be a flat map whose fibres are affine spaces. Then $f^* : K'_i(X) \to K'_i(P)$ is an isomorphism.*

Proof. It follows by noetherian induction, starting from a closed point and passing to generic point, since the affine case is proved - see [23, §7, Proposition 4.7]. ∎

5.3. **Filtration, K-cohomology, Brown-Gersten-Quillen spectral sequence, Chow ring.** Let $\mathcal{M}_p(X)$ denote the Serre subcategory of $\mathcal{M}(X)$ consisting of those sheaves whose support is of codimension $\geq p$. It is clear that

$$K_i\left(\mathcal{M}_p(X)\right) = \varinjlim K'_i(Z)$$

where Z runs over closed subsets of codimension $\geq p$. Moreover subcategories $\mathcal{M}_p(X)$ are preserved by flat inverse image functor and by filtered projective limits with affine flat transition maps.

Theorem 5.6. *Let X_p be the set of points of codimension p in X. There is the following Brown-Gersten-Quillen spectral sequence (shortly: BGQ spectral sequence)*

$$E_1^{p,q}(X) = \coprod_{x \in X_p} K_{-p-q}\left(k(x)\right) \Longrightarrow K'_{-n}(X)$$

which is convergent when X has finite dimension. BGQ spectral sequence is contravariant for flat morphisms. If $X = \varprojlim X_i$, where $i \mapsto X_i$ is a filtered projective system with flat affine transition morphisms, then the BGQ spectral squence for X is the inductive limit of the BGQ spectral sequences for the X_i.

This sequence is concentrated in the range $p \geq 0$, $p+q \leq 0$; if $\dim X = d$ then the E_1-term looks as follows:

$$\begin{array}{ccccccc}
\coprod_{x\in X_0} K_0\left(k(x)\right) & & & & & & \\
& & & & & & \\
\coprod_{x\in X_0} K_1\left(k(x)\right) & \longrightarrow & \coprod_{x\in X_1} K_0\left(k(x)\right) & & & & \\
& & & & & & \\
\vdots & & \vdots & & & & \\
& & & & & & \\
\coprod_{x\in X_0} K_d\left(k(x)\right) & \longrightarrow & \coprod_{x\in X_1} K_{d-1}\left(k(x)\right) & \longrightarrow & \cdots & \longrightarrow & \coprod_{x\in X_d} K_0\left(k(x)\right) \\
& & & & & & \\
\coprod_{x\in X_0} K_{d+1}\left(k(x)\right) & \longrightarrow & \coprod_{x\in X_1} K_d\left(k(x)\right) & \longrightarrow & \cdots & \longrightarrow & \coprod_{x\in X_d} K_1\left(k(x)\right) \\
& & & & & & \\
\vdots & & \vdots & & & & \vdots
\end{array}$$

Proof. Consider the filtration

$$\mathcal{M}(X) = \mathcal{M}_0(X) \supset \mathcal{M}_1(X) \supset \cdots$$

of $\mathcal{M}(X)$ by Serre subcategories. There is an equivalence

$$\mathcal{M}_p(X)/\mathcal{M}_{p+1}(X) \cong \coprod_{x\in X_p} \bigcup_n \mathcal{M}\left(\mathcal{O}_{X,x} / \left(\operatorname{rad}\left(\mathcal{O}_{X,x}\right)\right)^n\right)$$

which yields an isomorphism

$$K_i\left(\mathcal{M}_p(X)/\mathcal{M}_{p+1}(X)\right) \cong \coprod_{x\in X_p} K_i\left(k(x)\right)$$

where $k(x)$ is the residue field at x.

Localization exact sequences

$$\to K_i\left(\mathcal{M}_{p+1}(X)\right) \to K_i\left(\mathcal{M}_p(X)\right) \to \coprod_{x\in X_p} K_i\left(k(x)\right) \to K_{i-1}\left(\mathcal{M}_{p+1}(X)\right) \to$$

form an exact couple:

$$\begin{tikzcd}
D \arrow[rr] & & D \arrow[dl] \\
& E \arrow[ul] &
\end{tikzcd}$$

where

$$D = \coprod_{i,p} K_i\left(\mathcal{M}_p(X)\right), \qquad E = \coprod_{i,p} \coprod_{x\in X_p} K_i\left(k(x)\right)$$

and give rise to the BGQ spectral sequence in a standard way (see eg. [12, XI.5]). ∎

The filtration, arising in $K'_n(X)$ from the BGQ spectral sequence is the one given by codimension of support: $F^{p+1}K'_n(X)$ consists of elements of $K'_n(X)$ which vanish at generic points of subschemes of codimension $\leq p$.

One of geometric applications of BGQ spectral sequence is the following:

Theorem 5.7. *Let X be regular scheme of finite dimension over a field. Then there is a canonical isomorphism*

$$E_2^{p,p}(X) = CH^p(X).$$

Proof. [23, §7, Proposition 5.14]. ∎

In fact, the group $\coprod_{x\in X_p} K_0(k(x)) = \coprod_{x\in X_p} \mathbb{Z}$ is a group of codimension p cycles on X, and one may check that image of $\coprod_{x\in X_{p-1}} K_1(k(x)) \to \coprod_{x\in X_p} K_0(k(x))$ (or $\coprod_{x\in X_{p-1}} k(x)^* \to \coprod_{x\in X_p} \mathbb{Z}$) is exactly the group of cycles linearly equivalent to 0.

For $y \in X_{p-1}$ and $x \in X_p$ the yx component of diferential $\coprod_{y\in X_{p-1}} k(y)^* \to \coprod_{x\in X_p} \mathbb{Z}$ assigns to $f \in k(y)^*$ the multiplicity of its 0 in x.

Another property of the BGQ spectral sequence is connected with the so called Gersten Conjecture.

Proposition 5.8. *The following conditions are equivalent:*
(i) For every $p \geq 0$ the inclusion $\mathcal{M}_{p+1}(X) \to \mathcal{M}_p(X)$ induces 0 on the K-groups;
(ii) For all q, $E_2^{p,q} = 0$ if $p \neq 0$ and the edge homomorphism $K'_{-q} \to E_2^{0,q}$ is an isomorphism;
(iii) For every n the sequence

$$0 \to K'_n(X) \to \coprod_{x\in X_0} K_n(k(x)) \to \coprod_{x\in X_1} K_{n-1}(k(x)) \to \cdots \tag{5.1}$$

is exact. ∎

Proposition 5.9 (Gersten). *Let $\mathcal{K}'_n$ denote the Zariski sheaf on X associated to the presheaf $U \mapsto K'_n(U)$. Assume that* Spec $(\mathcal{O}_{X,x})$ *satisfies the equivalent conditions of Proposition 5.8 for all $x \in X$. Then there is a canonical isomorphism*

$$E_2^{p,q}(X) = H^p(X, \mathcal{K}'_{-q}).$$

Proof. We use the sequences (5.1) for different open subsets of X and we sheafify them to get a sequence of sheaves

$$0 \to \mathcal{K}'_n \to \coprod_{x\in X_0} (i_x)_*(K_n(k(x))) \to \coprod_{x\in X_1} (i_x)_*(K_{n-1}(k(x))) \to \cdots$$

where $i_x : \mathrm{Spec}\,(k(x)) \to X$ denotes the canonical map. The stalk of this sequence over x is the sequence (5.1) for Spec $(\mathcal{O}_{X,x})$, which is exact by hypothesis, whence it is a flasque resolution of $\mathcal{K}'_n$. ∎

We refer to $H^p(X, \mathcal{K}'_n)$ as *K-cohomology groups of X*.

The Gersten Conjecture is the following:

Conjecture 5.10. *The conditions of Proposition 5.8 are satisfied for the spectrum of a regular local ring.*

Some partial cases of this conjecture are proved. In particular Quillen proved it even for a semilocal ring obtained from a finite type algebra over a field by localizing with respect to a finite set of regular points ([23, Theorem 5.11]). So $E_2^{p,q}(X) = H^p(X, \mathcal{K}'_{-q})$ holds for a smooth variety over a field. This is a tool for effective computation of $E_2^{p,q}(X)$, e.g.

- $H^p(-, \mathcal{K}'_{-q})$ has a homotopy property: if $f : X \to Y$ is a flat map whose fibers are affine spaces, then f^* induces an isomorphism $H^p(X, \mathcal{K}'_{-q}) \cong H^p(Y, \mathcal{K}'_{-q})$.
- if Z is a closed subset of X of pure codimension d, then
$$0 \to \cdots \to 0 \to \coprod_{x \in Z_0} K_{n-d}(k(x)) \to \coprod_{x \in Z_1} K_{n-d-1}(k(x)) \to \cdots$$
is a subcomplex of
$$\coprod_{x \in X_0} K_n\left(k(x)\right) \to \coprod_{x \in X_1} K_{n-1}\left(k(x)\right) \to$$
and the factor complex is
$$\coprod_{x \in (X \setminus Z)_0} K_n\left(k(x)\right) \to \coprod_{x \in (X \setminus Z)_1} K_{n-1}\left(k(x)\right) \to$$
which yields excision property: $H^p(X, \mathcal{K}'_n) = H^p(X \setminus Z, \mathcal{K}'_n)$ for $p = 0, 1, \ldots, d-2$ and there is an exact excision sequence:
$$\begin{aligned} 0 &\to H^{d-1}(X, \mathcal{K}'_n) \to H^{d-1}(X \setminus Z, \mathcal{K}'_n) \to H^0(Z, \mathcal{K}'_{n-d}) \to \\ &\to H^d(X, \mathcal{K}'_n) \to H^d(X \setminus Z, \mathcal{K}'_n) \to H^1(Z, \mathcal{K}'_{n-d}) \to \cdots \end{aligned}$$

Other important properties of BGQ spectral sequence are that if X is a variety over a field F, then $E_\bullet^{-,-}(X)$ is a spectral sequence of $K_\bullet(F)$-modules. In particular the constants - elements of $K_\bullet(F)$ - are in kernel of every differential. Thus if a term is constant, $E_r^{p,q}(X) = K_{-p-q}(F)$, then all differentials starting from $E_r^{p,q}(X)$ are trivial.

Example 5.1. *Affine space.*

If $X = \mathbb{A}_F^n$ is the affine space, then $E_2^{p,q}(X)$ looks as follows:

$$\begin{array}{ccccc} K_0(F(X)) & 0 & 0 & \cdots & 0 \\ \\ K_1(F) & 0 & 0 & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & & \vdots \\ \\ K_n(F) & 0 & 0 & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & & \vdots \end{array}$$

and the BGQ spectral sequence degenerates from the E_2-term on - all differentials in all E_n, $n \geq 2$, are trivial. ∎

Example 5.2. *Projective space.*

If $X = \mathbb{P}^n_F$ is the projective space, then for a hyperplane $Z = \mathbb{P}^{n-1}_F$ there is an exact excision sequence:

$$\begin{array}{rl} 0 & \to H^0(\mathbb{P}^n_F, \mathcal{K}'_{-q}) \to H^0(\mathbb{A}^n_F, \mathcal{K}'_{-q}) \to H^0(\mathbb{P}^{n-1}_F, \mathcal{K}'_{-q-1}) \to \\ & \to H^1(\mathbb{P}^n_F, \mathcal{K}'_{-q}) \to H^1(\mathbb{A}^n_F, \mathcal{K}'_{-q}) \to H^1(\mathbb{P}^{n-1}_F, \mathcal{K}'_{-q-1}) \to \cdots \end{array}$$

which in fact is:

$$\begin{array}{rl} 0 & \to H^0(\mathbb{P}^n_F, \mathcal{K}'_{-q}) \to K_{-q}(F) \to H^0(\mathbb{P}^{n-1}_F, \mathcal{K}'_{-q-1}) \to \\ & \to H^1(\mathbb{P}^n_F, \mathcal{K}'_{-q}) \to 0 \to H^1(\mathbb{P}^{n-1}_F, \mathcal{K}'_{-q-1}) \to \cdots \end{array}$$

Moreover, the homomorphism $H^0(\mathbb{A}^n_F, \mathcal{K}'_{-q}) \to H^0(\mathbb{P}^{n-1}_F, \mathcal{K}'_{-q-1})$ is induced by a differential, so it is trivial and

$$E_2^{0,q}(\mathbb{P}^n_F) = K_{-q}(F), \qquad E_2^{p,q}(\mathbb{P}^n_F) = E_2^{p-1,q-1}(\mathbb{P}^{n-1}_F) \text{ for } p > 0.$$

By induction on n we get

$$E_2^{p,q}(\mathbb{P}^n_F) = K_{-p-q}(F).$$

The BGQ spectral sequence for a projective space degenerates from E_2 onwards. In the E_2-term we have:

$$E_2(\mathbb{P}^n_F): \quad \begin{array}{c|ccccc} K_0(F) & 0 & \cdots & 0 & 0 \\ \hline K_1(F) & K_0(F) & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_n(F) & K_{n-1}(F) & \cdots & K_1(F) & K_0(F) \\ K_{n+1}(F) & K_n(F) & \cdots & K_2(F) & K_1(F) \\ \vdots & \vdots & & \vdots & \vdots \end{array}$$

It follows that there is a canonical isomorphism of $K_\bullet(F)$-modules

$$K_\bullet(\mathbb{P}^n_F) = K'_\bullet(\mathbb{P}^n_F) = K_\bullet(F)^{n+1}$$

compatible with the topological filtration: here in lower right block there is the image of $E_2(\mathbb{P}^{n-1}_F)$. Thus $F^1K_\bullet(\mathbb{P}^n_F)$ is the image of $K_\bullet(\mathbb{P}^{n-1}_F)$, where $\mathbb{P}^{n-1}_F$ is embedded in $\mathbb{P}^n_F$ as a hyperplane. The $K_0(F)$, at $p = 1, q = -1$ is the Piccard group, unit element of this group is the class $\left[\mathcal{O}_{\mathbb{P}^{n-1}_F}\right]$ of the structural sheaf of a hyperplane. ∎

Example 5.3. *A quadric.*

Let Q^d be a d-dimensional projective quadric over F defined by equation $q_d = 0$, where

$$\begin{aligned} q_{2k} &= x_0y_0 + x_1y_1 + \cdots + x_ky_k, \\ q_{2k+1} &= z^2 + x_0y_0 + x_1y_1 + \cdots + x_ky_k. \end{aligned}$$

Consider the hyperplane section $Z^d : y_k = 0$ and its open complement $U^d = Q^d \setminus Z^{d-1}$. U^d is an affine space - the spectrum of $F\left[\frac{x_0}{y_k}, \frac{y_0}{y_k}, \cdots, \frac{x_k}{y_k}\right]/(\frac{q_{2k}}{y_k})$ or

$F\left[\frac{z}{y_k}, \frac{x_0}{y_k}, \frac{y_0}{y_k}, \cdots, \frac{x_k}{y_k}\right]/(\frac{q_{2k+1}}{y_k})$, which is a polynomial ring in d variables. The excision property yields

$$H^0(Q^d, \mathcal{K}'_n) = K_n(F), \qquad H^p(Q^d, \mathcal{K}'_n) = H^{p-1}(Z^{d-1}, \mathcal{K}'_{n-1}) \text{ for } p > 0.$$

Z^{d-1} is a projective cone over Q^{d-2} with the vertex $\mathrm{pt} = (0 : 0 : \cdots : 0 : 1 : 0)$. Z^{d-1} with the vertex removed is a bundle of affine lines over Q^{d-2}, so by the homotopy property

$$H^{p-1}(Z^{d-1} \backslash \mathrm{pt}, \mathcal{K}'_{n-1}) = H^{p-1}(Q^{d-2}, \mathcal{K}'_{n-1}).$$

Starting the induction with $Q^0 = \mathrm{pt} \amalg \mathrm{pt}$ and $Q^1 \cong \mathbb{P}^1_F$ we obtain

- for odd d
$$E_2^{p,q}(Q^d) = H^p(Q^d, \mathcal{K}'_{-q}) = K_{-p-q}(F),$$
- for $d = 2k$
$$E_2^{p,q}(Q^d) = H^p(Q^d, \mathcal{K}'_{-q}) = K_{-p-q}(F) \text{ for } p \neq k,$$
$$E_2^{k,q}(Q^d) = H^k(Q^d, \mathcal{K}'_{-q}) = K_{-k-q}(F) \oplus K_{-k-q}(F) \text{ for } p = k$$
and the BGQ spectral sequence degenerates from the E_2-term on.

In particular, we see that for $i \leq d$,

$$CH^i(Q^d) = \mathbb{Z} \text{ for } 2i \neq d, \qquad CH^k(Q^{2k}) = \mathbb{Z} \oplus \mathbb{Z}. \quad \blacksquare$$

6. $K_\bullet$ of certain varieties

There are several constructions of K-theory spaces other than Q-construction, like Gillet-Grayson G-construction or Giffen K-construction (see [29, Chapter IV], i.e. `http://math.rutgers.edu./~weibel/Kbook.IV.dvi`.) If one wants to obtain results like $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$ ([11]) or $K_4(\mathbb{Z}) = 0$, then this topological machinery is required. If one wants K-theory as a tool for algebraic geometry of varieties, then it is reasonable to use above basic properties as axioms and simply do computations. So there will be no homotopy theory in the following.

6.1. **Regular sheaves and projective bundles.** We recall here the computation from §8 of [23] of K-theory ring of a projective bundle, for two reasons: firstly, this gives methods for explicit computations, secondly this provides a model for computations of K-theory of other varieties.

Let T be a scheme (not necessarily noetherian or separated), let $\mathcal{E}$ be a vector bundle of rank r over T, and let $X = \mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\mathcal{E})$ be associated projective bundle, where $\mathrm{Sym}\mathcal{E}$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_T$. Let $\mathcal{O}_X(1)$ be the canonical line bundle on X and $f : X \to T$ the structural map.

Lemma 6.1. *(i) For every quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, $R^q f_*(\mathcal{F})$ is a quasi-coherent sheaf of $\mathcal{O}_T$-modules which is zero for $q \geq r$.*
(ii) For any quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and vector bundle $\mathcal{G}$ on T, one has

$$R^q f_*(\mathcal{F}) \otimes_{\mathcal{O}_T} \mathcal{G} = R^q f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}).$$

(iii) For any quasi-coherent sheaf $\mathcal{N}$ of $\mathcal{O}_X$-modules, one has

$$R^q f_*(\mathcal{O}_X(n) \otimes_{\mathcal{O}_T} \mathcal{N}) = \begin{cases} 0 & \text{for } q \neq 0, r-1 \\ \mathrm{Sym}^n \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{N} & \text{for } q = 0 \\ \left(\mathrm{Sym}^{-r-n}\mathcal{E}\right)^{\check{}} \otimes_{\mathcal{O}_T} \mathcal{N} & \text{for } q = r-1 \end{cases} \quad \blacksquare$$

(iv) If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, and T is affine, then $\mathcal{F}$ is a quotient of $\left(\mathcal{O}_X(-1)^{\otimes n}\right)^k$ for some n, k.

The following notion is usually attributed to Mumford ([17, Lecture 14]):

Definition 6.1. *A quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X is m-regular iff*

$$R^i f_*(\mathcal{F}(m-i)) = 0$$

for all $i > 0$. "0-regular" is simply "regular". $\mathcal{F}$ is m-regular iff $\mathcal{F}(m)$ is regular.

Mumford himself attributes this notion and its properties to Castelnuovo.

The long exact sequence of higher derived images yields immediately the following Lemma.

Lemma 6.2. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules.*
(i) If $\mathcal{F}'(n)$ and $\mathcal{F}''(n)$ are m-regular, so is $\mathcal{F}(n)$.
(ii) If $\mathcal{F}(n)$ and $\mathcal{F}'(n+1)$ are m-regular, so is $\mathcal{F}''(n)$.
(iii) If $\mathcal{F}(n+1)$ and $\mathcal{F}''(n)$ are m-regular, and if $f_*(\mathcal{F}(n)) \to f_*(\mathcal{F}''(n))$ is onto, then $\mathcal{F}'(n+1)$ is m-regular. ∎

Lemma 6.3. *If $\mathcal{F}$ is regular, then $\mathcal{F}(n)$ is regular for all $n \geq 0$.*

Proof. From the canonical epimorphism $f^*\mathcal{E} \to \mathcal{O}_X(1)$ one gets an epimorphism

$$\mathcal{O}_X(-1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E}) \to \mathcal{O}_X \tag{6.1}$$

and Koszul resolutions

$$0 \to \mathcal{O}_X(-r) \otimes_{\mathcal{O}_X} f^*\left(\bigwedge^r \mathcal{E}\right) \to \cdots \to \mathcal{O}_X(-1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E}) \to \mathcal{O}_X \to 0, \tag{6.2}$$

$$0 \to \mathcal{F}(-r) \otimes_{\mathcal{O}_X} f^*\left(\bigwedge^r \mathcal{E}\right) \to \cdots \to \mathcal{F}(-1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E}) \to \mathcal{F} \to 0. \tag{6.3}$$

Assuming $\mathcal{F}$ to be regular, one sees that $\left(\mathcal{F}(-p) \otimes_{\mathcal{O}_X} f^*\left(\bigwedge^p \mathcal{E}\right)\right)(p)$ is regular. If (6.3) is split into short exact sequences

$$0 \to \mathcal{Z}_p \to \mathcal{F}(-p) \otimes_{\mathcal{O}_X} f^*\left(\bigwedge^p \mathcal{E}\right) \to \mathcal{Z}_{p-1} \to 0,$$

then by decreasing induction on p we obtain that the sheaf $\mathcal{Z}_p(p+1)$ is also regular. Thus the regularity of $\mathcal{F}$ implies that $\mathcal{Z}_0(1) = \mathcal{F}(1)$ is regular. ∎

Lemma 6.4. *If $\mathcal{F}$ is regular, then the canonical map $f^*f_*(\mathcal{F}) \to \mathcal{F}$ is surjective.*

Proof. As above there is an exact sequence

$$0 \to \mathcal{Z}_1 \to \mathcal{F}(-1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E}) \to \mathcal{F} \to 0$$

with regular $\mathcal{Z}_1(2)$. Thus $R^1 f_*(\mathcal{Z}_1(n)) = 0$ for $n \geq 1$, so the canonical map $f_*(\mathcal{F}(n-1)) \otimes_{\mathcal{O}_T} \mathcal{E} \to f_*(\mathcal{F}(n))$ is surjective for $n \geq 1$. Hence the canonical map of $\mathrm{Sym}\mathcal{E}$-modules

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_T} \mathrm{Sym}\mathcal{E} \to \coprod_{n \geq 0} f_*(\mathcal{F}(n))$$

is surjective. The Lemma follows by taking the associated sheaves. ∎

Now we shall describe a recursive procedure which gives the canonical resolution of a regular sheaf. This procedure is the key point here and in other computations below. But first suppose that a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules admits a resolution

$$0 \to f^*(\mathcal{T}_{r-1})(1-r) \to \cdots \to f^*(\mathcal{T}_0) \to \mathcal{F} \to 0$$

where $\mathcal{T}_i$ are coherent sheaves of $\mathcal{O}_T$-modules. Then $\mathcal{F}$ has to be regular. Moreover, the above exact sequence can be viewed as a resolution of the zero sheaf by acyclic objects for the δ-functor $\mathcal{F} \longmapsto R^q f_*(\mathcal{F}(n))$, where n is any fixed nonnegative integer. Applying f_* we get an exact sequence

$$0 \to \mathrm{Sym}^{n-r+1}\mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{T}_{r-1} \to \cdots \to \mathrm{Sym}^n\mathcal{E} \otimes_{\mathcal{O}_T} f_*(\mathcal{F}) \to 0$$

for $n \geq 0$. In particular, we have the exact sequences:

$$0 \to \mathcal{T}_n \to \mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{T}_{n-1} \to \cdots \to f_*(\mathcal{F}(n)) \to 0$$

for $n = 0, 1, \ldots, r-1$ which can be used to show that the sheaves $\mathcal{T}_n$ are determined by $\mathcal{F}$ up to canonical isomorphisms.

Recursive construction of a canonical resolution:

Conversely, given a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we inductively define a sequence of coherent sheaves of $\mathcal{O}_X$-modules $\mathcal{Z}_n = \mathcal{Z}_n(\mathcal{F})$ and a sequence of coherent sheaves of $\mathcal{O}_T$-modules $\mathcal{T}_n = \mathcal{T}_n(\mathcal{F})$ as follows. Starting with $\mathcal{Z}_{-1} = \mathcal{F}$, let

$$\begin{aligned} \mathcal{T}_n &= f_*(\mathcal{Z}_{n-1}(n)), \\ \mathcal{Z}_n &= \ker(f^*(\mathcal{T}_n)(-n) \to \mathcal{Z}_{n-1}). \end{aligned} \tag{6.4}$$

$\mathcal{Z}_n$ and $\mathcal{T}_n$ are additive functors of $\mathcal{F}$.

Assume now that $\mathcal{F}$ is regular. We see by induction that $\mathcal{Z}_n(n+1)$ is regular. This is clear for $n = -1$. The exact sequences

$$0 \to \mathcal{Z}_n(n) \to f^*(\mathcal{T}_n) \to \mathcal{Z}_{n-1}(n) \to 0 \tag{6.5}$$

allow one to complete the induction. In addition we have

$$f_*(\mathcal{Z}_n(n)) = 0 \text{ for } n \geq 0 \tag{6.6}$$

because the functor f_* maps $f^*(\mathcal{T}_n) \to \mathcal{Z}_{n-1}(n)$ onto an isomorphism. The functor f_* is exact on the category of regular coherent sheaves of $\mathcal{O}_X$-modules, so $\mathcal{F} \longmapsto \mathcal{T}_n(\mathcal{F})$ is an exact functor.

Example 6.1. *Projective space*

Let F be a field, $T = \mathrm{Spec}F$, $X = \mathbb{P}^m_F$. The sheaf $\mathcal{F} = \mathcal{O}_X(1)$ is regular.

$$\mathcal{T}_0 = f_*(\mathcal{F}) = H^0(X, \mathcal{O}_X(1))$$

is a vector space of dimension $m+1$. Exact sequence (6.5) for $n = 0$

$$0 \to \mathcal{Z}_0 \to \mathcal{O}_X^{m+1} \to \mathcal{O}_X(1) \to 0$$

defines $\mathcal{Z}_0$; for the next step we twist this sequence by 1:

$$0 \to \mathcal{Z}_0(1) \to \mathcal{O}_X(1)^{m+1} \to \mathcal{O}_X(2) \to 0$$

to obtain that $\dim H^0(X, \mathcal{Z}_0(1)) = (m+1)\binom{m+1}{1} - \binom{m+2}{2} = \binom{m+1}{2}$. Thus

$$\mathcal{T}_1 = f_*(\mathcal{Z}_0(1)) = H^0(X, \mathcal{Z}_0(1)) = F^{\binom{m+1}{2}}.$$

Exact sequence (6.5) for $n = 1$ is

$$0 \to \mathcal{Z}_1(1) \to \mathcal{O}_X^{\binom{m+1}{2}} \to \mathcal{Z}_0(1) \to 0.$$

For convenience we glue it with the exact sequence for $n=0$ twisted by 1:

$$0 \to \mathcal{Z}_1(1) \to \mathcal{O}_X^{\binom{m+1}{2}} \to \mathcal{O}_X(1)^{m+1} \to \mathcal{O}_X(2) \to 0.$$

Thus next twist by 1

$$0 \to \mathcal{Z}_1(2) \to \mathcal{O}_X(1)^{\binom{m+1}{2}} \to \mathcal{O}_X(2)^{m+1} \to \mathcal{O}_X(3) \to 0,$$

yields

$$\dim H^0\left(X, \mathcal{Z}_0(2)\right) = \binom{m+1}{2}(m+1) - (m+1)\binom{m+2}{2} + \binom{m+3}{3} = \binom{m+1}{3},$$

$$\mathcal{T}_2 = f_*\left(\mathcal{Z}_1(2)\right) = H^0\left(X, \mathcal{Z}_1(2)\right) = F^{\binom{m+1}{3}},$$

so the next step is

$$0 \to \mathcal{Z}_2(2) \to \mathcal{O}_X^{\binom{m+1}{3}} \to \mathcal{O}_X(1)^{\binom{m+1}{2}} \to \mathcal{O}_X(2)^{m+1} \to \mathcal{O}_X(3) \to 0.$$

The final result is $\mathcal{Z}_m = 0$ and

$$\begin{aligned} &0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\binom{m+1}{m}} \to \cdots \\ &\qquad \cdots \to \mathcal{O}_X(m-1)^{\binom{m+1}{2}} \to \mathcal{O}_X(m)^{m+1} \to \mathcal{O}_X(m+1) \to 0. \end{aligned}$$

Untwisting yields the resolution of $\mathcal{O}_X(1)$:

$$\begin{aligned} &0 \to \mathcal{O}_X(-m) \to \mathcal{O}_X(1-m)^{\binom{m+1}{m}} \to \cdots \\ &\qquad \cdots \to \mathcal{O}_X(-1)^{\binom{m+1}{2}} \to \mathcal{O}_X{}^{\binom{m+1}{1}} \to \mathcal{O}_X(1) \to 0. \end{aligned}$$

According to Proposition 2.6 above, $K_0(X)$ is a free abelian group with a basis $1 = [\mathcal{O}_X]$, $\xi = [\mathcal{O}_X(-1)]$, $\xi^2, \ldots, \xi^{m-1}, \xi^m$. Here we have an identity

$$\xi^{-1} = \binom{m+1}{1} + \binom{m+1}{2}(-\xi) + \cdots + \binom{m+1}{m}(-\xi)^{m-1} + (-\xi)^m$$

which yields the ring structure:

$$K_0(X) = Z[\xi] \,/\, \left((1-\xi)^{m+1}\right). \quad \blacksquare$$

Going back to the general case we see that the key point is that the recursive process described above terminates, i.e. that $\mathcal{Z}_{r-1} = 0$. From 6.5 we get an exact sequence

$$R^{q-1} f_*\left(\mathcal{Z}_{n+q-1}(n)\right) \to R^q f_*\left(\mathcal{Z}_{n+q}(n)\right) \to R^q f_*\left(f^*\left(\mathcal{T}_{n+q}\right)(-q)\right)$$

which allows one to prove by induction on q, starting from 6.6, that $R^q f_*\left(\mathcal{Z}_{n+q}(n)\right) = 0$ for $q, n \geq 0$. This shows that $\mathcal{Z}_{r-1}(r-1)$ is regular, since $R^q f_*$ is zero for $q \geq r$. Thus $f_*\left(\mathcal{Z}_{r-1}(r-1)\right) = 0$ and $\mathcal{Z}_{r-1}(r-1)$ by Lemma 6.4.

We have proved the following.

Proposition 6.5. *Any regular coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ has the canonical resolution of the form*

$$0 \to f^*\left(\mathcal{T}_{r-1}(\mathcal{F})\right)(1-r) \to \cdots \to f^*\left(\mathcal{T}_1(\mathcal{F})\right)(1) \to f^*\left(\mathcal{T}_1(\mathcal{F})\right) \to \mathcal{F} \to 0 \tag{6.7}$$

where the $\mathcal{T}_i(\mathcal{F})$ are coherent sheaves of $\mathcal{O}_T$-modules determined up to a unique isomorphism by $\mathcal{F}$. Moreover $\mathcal{F} \longmapsto \mathcal{T}_i(\mathcal{F})$ is an exact functor from the category of regular coherent sheaves of $\mathcal{O}_X$-modules to the category of coherent sheaves of $\mathcal{O}_T$-modules. $\blacksquare$

We state now three lemmas.

Lemma 6.6. *Assume T is quasi-compact. Then for any vector bundle $\mathcal{F}$ on X there exists an integer n_0 such that for all coherent sheaves $\mathcal{N}$ of $\mathcal{O}_T$-modules, one has*
(a) $R^q f_* (\mathcal{F}(n) \otimes_{\mathcal{O}_X} f^*(\mathcal{N})) = 0$ for $q > 0$,
(b) $f_* (\mathcal{F}(n)) \otimes_{\mathcal{O}_T} \mathcal{N} \xrightarrow{\sim} f_* (\mathcal{F}(n) \otimes_{\mathcal{O}_X} f^*(\mathcal{N}))$,
(c) $f_* (\mathcal{F}(n))$ is a vector bundle on T.

Proof. [23, §8, Lemma 1.12]. ∎

Lemma 6.7. *If $\mathcal{F}$ is a vector bundle on X such that $R^q f_* (\mathcal{F}(n)) = 0$ for $q > 0$ and $n \geq 0$, then $f_* (\mathcal{F}(n))$ is a vector bundle on T for all $n \geq 0$.*

Proof. [23, §8, Lemma 1.13]. ∎

Lemma 6.8. *If $\mathcal{F}$ is a regular vector bundle on X, then $T_i (\mathcal{F})$ is a vector bundle on T for each i.*

Proof. [23, §8, Lemma 1.14]. ∎

Now we can prove main result of this section (Theorem 2.1 of [23, §8]).

Theorem 6.9. *Let $\mathcal{E}$ be a vector bundle of rank r over a scheme T and $X = \mathbb{P}(\mathcal{E})$ the associated projective bundle. If T is quasi-compact, then one has isomorphisms*

$$(K_q(T))^r \xrightarrow{\sim} K_q(X), \qquad (a_0, a_1, \ldots, a_{r-1}) \longmapsto \sum_{i=0}^{r-1} t^i \cdot f^* a_i$$

where $t \in K_0(X)$ is the class of the canonical line bundle $\mathcal{O}_X(-1)$, product $\cdot$ is the multiplication $K_0(X) \otimes K_q(X) \to K_q(X)$ defined in Example 4.12, and $f : X \to T$ is the structural map.

Proof. Let $\mathfrak{P}_n = \mathfrak{P}_n(X)$ denote the full subcategory of $\mathcal{P}(X)$ consisting of vector bundles $\mathcal{F}$ such that $R^q f_*(\mathcal{F}(k)) = 0$ for $q \neq 0$ and $k \geq n$. Let $\mathfrak{R}_n = \mathfrak{R}_n(X)$ denote the full subcategory of $\mathcal{P}(X)$ consisting of n-regular vector bundles. Each of this subcategories is closed under extensions, so its K-groups are defined.

We prove that inclusions induce isomorphisms $K_q(\mathfrak{R}_n) \cong K_q(\mathfrak{P}_n) \cong K_q(\mathcal{P}(X)) = K_q(X)$. To see this change the exact sequence (6.2) into

$$0 \to \mathcal{O}_X \to (f^*(\mathcal{E}))\,(1) \to \cdots \to \left(f^*\left(\bigwedge^r \mathcal{E}\right)\right)(r) \to 0$$

and tensor it by $\mathcal{F}$:

$$0 \to \mathcal{F} \to f^*(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{F}(1) \to \cdots \to f^*\left(\bigwedge^r \mathcal{E}\right) \otimes_{\mathcal{O}_X} \mathcal{F}(r) \to 0.$$

For each $p > 0$, $\mathcal{F} \longmapsto \mathcal{F}(p) \otimes_{\mathcal{O}_X} f^*\left(\bigwedge^p \mathcal{E}\right)$ is an exact functor from $\mathfrak{P}_n$ to $\mathfrak{P}_{n-1}$, hence it induces a homomorphism $u_p : K_p(\mathfrak{P}_n) \to K_p(\mathfrak{P}_{n-1})$. It is clear, that $\sum\limits_{p>0} (-1)^{p-1} u_p$ is an inverse to the map induced by the inclusion $\mathfrak{P}_{n-1} \to \mathfrak{P}_n$. Thus we have $K_p(\mathfrak{P}_{n-1}) \xrightarrow{\sim} K_p(\mathfrak{P}_n)$ for all n. By Lemma 6.6 (a), $\mathcal{P}(X)$ is the union of the $\mathfrak{P}_n$'s, so $K_p(\mathfrak{P}_n) \cong K_p(\mathcal{P}(X))$ for all n. The proof that $K_p(\mathfrak{R}_n) \cong K_p(\mathcal{P}(X))$ is similar.

Put $U_n(\mathcal{N}) = (f^*(\mathcal{N}))(-n)$ for $\mathcal{N} \in \mathcal{P}(T)$. For $0 \le n < r$, U_n is an exact functor from $\mathcal{P}(T)$ to $\mathfrak{P}_0$, hence it induces a homomorphism $u_n : K_p(\mathcal{P}(T)) \to K_p(\mathfrak{P}_0)$. To prove the theorem it suffices to show that the homomorphism

$$u : K_p(\mathcal{P}(T))^r \to K_p(\mathfrak{P}_0), \qquad u(a_0, a_1, \dots, a_{r-1}) = \sum_{n=0}^{r-1} u_n(a_n),$$

is an isomorphism.

From Lemma 6.7 we know that $V_n(\mathcal{F}) = f_*(\mathcal{F}(n))$ is an exact functor from $\mathfrak{P}_0$ to $\mathcal{P}(T)$ for $n \ge 0$, hence we have a homomorphism

$$v : K_p(\mathfrak{P}_0) \to K_p(\mathcal{P}(T))^r, \qquad v(x) = (v_0(x), v_1(x), \dots, v_{r-1}(x)),$$

where v_n is a homomorphism induced by V_n. Since

$$V_n \circ U_m(\mathcal{N}) = f_*((f^*(\mathcal{N}))(n-m)) = \mathrm{Sym}^{n-m}(\mathcal{E}) \otimes_{\mathcal{O}_T} \mathcal{N},$$

it follows that the composition vu is described by a triangular matrix with 1's on the diagonal. Therefore vu is an isomorphism, so u is injective.

On the other hand, $\mathcal{T}_n$ is an exact functor from $\mathfrak{R}_0$ to $\mathcal{P}(T)$, hence we have a homomorphism

$$t : K_p(\mathfrak{R}_0) \to K_p(\mathcal{P}(T))^r, \qquad t(x) = \left(t_0(x), -t_1(x), \dots, (-1)^{r-1} t_{r-1}(x)\right),$$

where t_n is induced by $\mathcal{T}_n$. The composition ut is the map $K_p(\mathfrak{R}_0) \to K_p(\mathfrak{P}_0)$ induced by the inclusion $\mathfrak{R}_0 \to \mathfrak{P}_0$. Since ut is an isomorphism u is surjective. This concludes the proof. ∎

6.2. **Brauer-Severi varieties.** If a variety X is in some sense similar to a projective space, then K-theory of X differs slightly from the K-theory of projective space. To show similarities and differences we discuss Brauer-Severi varieties and quadric hypersurfaces.

Brauer-Severi variety is a twisted form of a projective space. Let k be a field and let $\overline{k}$ be its algebraic closure.

Definition 6.2. *A k-variety P is a* Brauer-Severi variety *over k if $P_{\overline{k}} \cong \mathbb{P}^{r-1}_{\overline{k}}$.*

(see [1].) For example a conic is a Brauer-Severi variety (Example 7.1 below.)

So a Brauer-Severi variety is similar to a projective space.

Brauer-Severi varieties over a fixed field k are classified by central simple k-algebras. Since this connection algebras-varieties is relevant, we give here some details.

Recall that a k-algebra A is *central* iff k is its center; it is *simple* iff it has no proper two-sided ideals. By Wedderburn Theorem ([24, Chapter 8, Theorem 1.5]) every central simple k-algebra is a matrix algebra over a division ring, which is central over k. In particular every central simple $\overline{k}$-algebra is a matrix algebra.

Given a central simple algebra A of rank r^2 over a field k, a Brauer-Severi variety over k may be defined directly as follows. For a field extension $k \subset K$ let P_K be the set of all left ideals L of $A \otimes_k K$ of dimension r. Let P be the k-variety which set of K-points is P_K for every K. It is clear that P possesses the structure of an algebraic variety over k. Indeed, picking a fixed basis for A over k, one embeds P as a closed subvariety of a Grassmannian $\mathrm{Gr}(r, r^2)$ of r-spaces in $A \otimes_k K$, defined by the relations stating that each L is a left ideal of A.

If A is a division ring, then P is a variety without rational points.

If $A = M_r(k)$ is a matrix algebra and $e_i = e_{ii}$ are the usual idempotents (e_{ij} is a matrix which differs from zero matrix by exactly one 1 at place i, j), then each left ideal L may be decomposed as $L = e_1 L \oplus \cdots \oplus e_r L$, when viewed as a k-module. Matrices in $e_i L$ differ form zero matrix by i-th row. Since $e_{ji} e_i L = e_j L$, $\dim_k e_i L = 1$ for each $1 \leq i \leq r$. Thus all i-rows of matrices in $e_i L$ form a line in k^r, and for $i = 1, 2, \ldots, r$ this is the same line - the common image $k^r \cdot x$ of all nonzero $x \in L$. Choose $x \neq 0$ in $e_1 L$; then x may be written as $x = \sum a_j e_{1j}$ for some $(a_1, ..., a_r) \in k^r$, where at least one $a_i \neq 0$. For another choice of x, we have $x' = \lambda x$. It follows that each left ideal L of A corresponds to a point $(a_1 : \ldots : a_r)$ of $\mathbb{P}_k^{r-1}$, which is the common image of rank 1 matrices in L. On the other hand, pick $(a_1 : \ldots : a_r)$ in $\mathbb{P}_k^{r-1}$ and let $l = \sum a_j e_{1j}$. Then we associate to it the left ideal L it generates in $M_r(k)$, which is $L = kl + ke_{21} l + ... + ke_{r1} l$. Thus the Brauer Severi-variety associated to $M_r(k)$ is just a projective $(r-1)$-space over k. In particular, a Brauer-Severi variety over an algebraically closed field is a projective space, and a Brauer-Severi variety over a field k becomes isomorphic to a projective space over $\overline{k}$.

Brauer-Severi varieties may also be defined by descent. By descent theory, if $G = \mathrm{Gal}(\overline{k}/k)$, then the pointed set of isomorphism classes of k-varieties P with the property that $P_{\overline{k}} \cong \mathbb{P}_{\overline{k}}^{r-1}$ is isomorphic to $H^1(G, PGL_r)$, because $\mathrm{Aut} P = PGL_r$. On the other hand, each k-automorphism of $M_r(k)$ is inner (Noether-Skolem Theorem, [24, Chapter 8, Theorem 4.2]), $\mathrm{Aut} M_r(k) = PGL_r(k)$. Now, let A be a central simple k-algebra and $\overline{k}$ the algebraic closure of k. Then $A \otimes_k \overline{k} = M_r(\overline{k})$. So by descent theory the pointed set of isomorphism classes of central simple k-algebras of rank r^2 is isomorphic to $H^1(G, PGL_r)$, too. We associate to a k-variety P as above the central simple algebra A of the same class in $H^1(G, PGL_r)$. If one looks at the explicit action of PGL_r on $\mathbb{P}^{r-1}$ and the variety of left ideals of rank r of M_r, then it is easy to see that both constructions yield the same result.

Example 6.2. *Conic and quaternion algebra.*

Let $r = 2$. Then there exists a quadratic extension $F = k(\sqrt{a})$ of k such that $P_F \cong \mathbb{P}_F^1$ and $A \otimes_k F \cong M_2(F)$. One may check explicitely that if P and A are defined by the same cocycle, then there exists $b \in k^*$ such that A has a basis $1, i, j, k$ such that

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

Such A is the *quaternion algebra* $\left(\frac{a,b}{k}\right)$; these are classified up to isomorphism by isomorphism classes of quadratic form (their reduced norm - see the final section below for a definition)

(6.8) $Nrd(x+yi+zj+tk) = (x+yi+zj+tk)(x-yi-zj-tk) = x^2 - ay^2 - bz^2 + abz^2.$

This is a very special quadratic form (two-fold Pfister form), its isomorphism class is uniquely defined by isomorphism class of its subform

$$Nrd(yi + zj + tk) = -(yi + zj + tk)^2 = -ay^2 - bz^2 + abz^2$$

(in the quadratic form theory it is called a *pure subform of* (6.8); a quaternion of the form $yi + zj + tk$ is called a *pure quaternion.*) We show that Brauer-Severi variety P is the projective conic defined by the equation $ay^2 + bz^2 - abz^2 = 0$. A left ideal L is of the form $A\alpha$ for $\alpha \in A$; if $N(\alpha) \neq 0$, then α is invertible, so

$N(\alpha) = 0$ for $\dim L = 2$; if $\alpha = x + yi + zj + tk$ and $x \neq 0$ and, say, $x^2 - ay^2 \neq 0$, then $x - yi$ is invertible and

$$\begin{aligned} L = Ai(x - yi)(x + yi + zj + tk) &= Ai\left(x^2 - ay^2 + (xz - byt)j + (xt - yz)k\right) \\ &= A\left(\left(x^2 - ay^2\right)i + a(xt - yz)j + (xz - byt)k + (xt - yz)k\right) \\ &= A(y'i + z'j + t'k). \end{aligned}$$

One easily checks that $A(y'i + z'j + t'k) = A(y''i + z''j + t''k)$ iff $(y' : z' : t') = (y'' : z'' : t'')$ ∎

There are relative versions of notions above. For a scheme T we introduce the following definitions:

- A sheaf $\mathcal{A}$ of $\mathcal{O}_T$-algebras of rank r^2 over T is an *Azumaya algebra* iff $\mathcal{A}$ is locally isomorphic to the sheaf $M_r(\mathcal{O}_T)$ with respect to étale topology (cf. [8, §5].)
- A T-scheme $f : X \to T$ is a *Brauer-Severi scheme* of relative dimension $r - 1$ iff it is locally isomorphic to the projective space $\mathbb{P}_T^{r-1}$ with respect to étale topology on T.

(cf. op. cit., §8.)

By descent, the connection between central simple algebras and Brauer-Severi varieties remains valid for Azumaya algebras and Brauer-Severi schemes - for details see [8, §8]. Moreover, Quillen ([23, §8.4]) has generalized the Projective Bundle Theorem 6.9 to this situation.

So let $f : X \to T$ be a Brauer-Severi scheme of relative dimension $r - 1$ corresponding to the Azumaya algebra $\mathcal{A}$ over T.

If there exists a line bundle $\mathcal{L}$ on X which restricts to $\mathcal{O}(-1)$ on each geometric fibre, then one has $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the vector bundle $f_*\mathcal{L}\check{}$ on T. In general such a line bundle $\mathcal{L}$ exists only locally for the étale topology on X. However, we shall now show that there is a canonical vector bundle of rank r on X which restricts to $\mathcal{O}(-1)^r$ on each geometric fibre.

Let the group scheme $GL(r,T)$ act on $\mathcal{O}_T^r$ in the standard way, and put $Y = \mathbb{P}_T^{r-1} = \mathbb{P}(\mathcal{O}_T^r)$. The induced action on Y factors through the projective group $PGL(r,T) = GL(r,T)/\mathbb{G}_{m,T}$. Since the multiplicative group $\mathbb{G}_{m,T}$ acts trivially on the vector bundle $\mathcal{O}_Y(-1) \otimes_{\mathcal{O}_Y} g^*(\mathcal{O}_T^r)$, where $g : Y \to T$ is the structural map, the group $PGL(r,T)$ operates on this vector bundle compatibly with its action on Y. As X is locally isomorphic to Y with respect to the étale topology on T and $PGL(r,T)$ is the group of authomorphisms of Y over T, one knows that X is the bundle over T with fibre Y associated to a torsor A under $PGL(r,T)$ locally trivial for the étale topology. Thus by a faithfully flat descent, the bundle $\mathcal{O}_Y(-1) \otimes_{\mathcal{O}_Y} g^*(\mathcal{O}_T^r)$ on Y gives rise to a vector bundle $\mathcal{J}$ on X of rank r.

It is clear that the construction of $\mathcal{J}$ is compatible with the base change, and that $\mathcal{J} = \mathcal{O}_X(-1) \otimes_{\mathcal{O}_X} f^*(\mathcal{E})$ if $X = \mathbb{P}\mathcal{E}$. In the general case, there is a cartesian square

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T' \arrow[r, "h"'] & T \end{tikzcd}$$

where h is faithfully flat (e.g. an étale surjective map over which $\mathcal{A}$ becomes trivial) such that $X' = \mathbb{P}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank r on T', and further

$$h'^*(\mathcal{J}) = \mathcal{O}_{X'}(-1) \otimes_{\mathcal{O}_{X'}} f'^*(\mathcal{E}).$$

Let $\mathcal{A}$ be the sheaf of (non-commutative) $\mathcal{O}_T$-algebras given by

$$\mathcal{A} = f_*(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J}))^{op}$$

where 'op' denotes the opposed ring structure. As h is flat, we have $h^* f_* = f'_* h'^*$. Hence

$$\begin{aligned} h^*(\mathcal{A})^{op} &= h^* f_*(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J})) = f'_* h'^*(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J})) \\ &= f'_*\left(\mathcal{E}nd_{\mathcal{O}_X}\left(\mathcal{O}_{X'}(-1) \otimes_{\mathcal{O}_{X'}} f'^*(\mathcal{E})\right)\right) = f'_* f'^*\left(\mathcal{E}nd_{\mathcal{O}_{T'}}(\mathcal{E})\right) = \mathcal{E}nd_{\mathcal{O}_{T'}}(\mathcal{E}). \end{aligned}$$

Thus $\mathcal{A}$ is an Azumaya algebra of rank r^2 over T. Moreover one has

$$f^*\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J})^{op}$$

as one verifies by pulling back both sides to X'.

Let $\mathcal{J}_n$ (resp. $\mathcal{A}_n$) be the n-fold tensor product of $\mathcal{J}$ on X (resp. $\mathcal{A}$ on T), so that $\mathcal{A}_n$ is an Azumaya algebra of rank $(r^n)^2$ such that

$$\mathcal{A}_n = f_*(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J}_n))^{op}, \qquad f^*\mathcal{A}_n = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{J}_n)^{op}.$$

Let $\mathcal{P}(T, \mathcal{A}_n)$ be the category of vector bundles on T which are left modules for $\mathcal{A}_n$. Since $\mathcal{J}_n$ is a right $f^*(\mathcal{A}_n)$-module, which locally on X is a direct summand of $f^*(\mathcal{A}_n)$, we have an exact functor

$$\mathcal{J}_n \otimes_{\mathcal{A}_n} - : \mathcal{P}(T, \mathcal{A}_n) \to \mathcal{P}(X), \qquad \mathcal{M} \longmapsto \mathcal{J}_n \otimes_{f^*(\mathcal{A}_n)} f^*(\mathcal{M})$$

and hence an induced map of K-groups.

Theorem 6.10. *If T is quasicompact, one has isomorphisms (for all i):*

$$\coprod_{n=0}^{r-1} K_i(\mathcal{A}_n) \xrightarrow{\sim} K_i(X), \qquad (x_0, x_1, \ldots, x_{r-1}) \longmapsto \sum_{n=0}^{r-1} (\mathcal{J}_n \otimes_{\mathcal{A}_n} -)_*(x_n).$$

This is actually a generalization of 6.9 because if two Azumaya algebras $\mathcal{A}$, $\mathcal{B}$ represent the same element of the Brauer group of T, then the categories $\mathcal{P}(T, \mathcal{A})$ and $\mathcal{P}(T, \mathcal{B})$ are equivalent (Morita equivalence), and hence have isomorphic K-groups. Thus $K_i(\mathcal{P}(T, \mathcal{A}_n)) = K_i(T)$ for all n if X is the projectivization of some vector bundle.

A proof of 6.10 is a modification of the proof of 6.9. One calls a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules to be a regular sheaf if its inverse image on $X' = \mathbb{P}(\mathcal{E})$ is regular. For a regular $\mathcal{F}$ one constructs a sequence

$$(6.9) \qquad 0 \to \mathcal{J}_{r-1} \otimes_{\mathcal{A}_{r-1}} \mathcal{T}_{r-1}(\mathcal{F}) \to \cdots \to \mathcal{O}_X \otimes_{\mathcal{O}_X} f^*(\mathcal{T}_0(\mathcal{F})) \to \mathcal{F} \to 0$$

recursively by

$$\begin{aligned} \mathcal{T}_n(\mathcal{F}) &= f_*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_n, \mathcal{Z}_{n-1}(\mathcal{F}))), \\ \mathcal{Z}_n(\mathcal{F}) &= \ker(\mathcal{J}_n \otimes_{\mathcal{A}_n} \mathcal{T}_n(\mathcal{F}) \to \mathcal{Z}_{n-1}(\mathcal{F})) \end{aligned}$$

starting with $\mathcal{Z}_{-1}(\mathcal{F}) = \mathcal{F}$. It is easy to see that this sequence, when lifted to X', coincides with the canonical resolution (6.7) of inverse image of $\mathcal{F}$ on X'. Since X' is faithfully flat over X, (6.9) is a resolution of $\mathcal{F}$.

We note also that there is a canonical epimorphism $\mathcal{J} \twoheadrightarrow \mathcal{O}_X$ obtained by descending (6.1), and hence a Koszul complex

$$0 \to \bigwedge^r \mathcal{J} \to \cdots \to \mathcal{J} \to \mathcal{O}_X \to 0$$

is an exact sequence of vector bundles on X, corresponding to (6.2). Therefore it is clear that all of the tools used in the proof of Projective Bundle Theorem are available in the situation under consideration.

This result was used to compute K-cohomology of Brauer-Severi variety in [14]. The field of fractions of Brauer-Severi variety is a generic splitting field for associated central simple algebra, and K-cohomology computation was used to prove that for arbitrary field F and natural number n prime to characteristic of F, there is a natural isomorphism $K_2(F)/nK_2(F) \cong H^2(F, \mu_n^{\otimes 2})$. We show below in 7.2 a particular case of the key argument leading this result, to exhibit the flavour of that kind of applications of the theory.

6.3. **Quadric hypersurfaces.** We give here a simplified account of Richard Swan paper [27]. The setup of [27] is very general: quadric is defined over a ring R - for a finitely generated projective R-module M with nonsingular quadratic form $q : M \to R$ the quadric $X(q)$ is $\mathrm{Proj}(\mathrm{Sym}(M^*))/(q)$ (here $M^* = Hom_R(M, R)$.) The main Theorem 9.1 describes K-theory of the category $\mathcal{P}(X(q), \Lambda)$ of vector bundles on $X(q)$ with left action of a generalized Azumaya algebra Λ. With such a general result it is possible to compute the K-theory of products of affine quadrics.

We content ourselves with the K-theory of $\mathcal{P}(X(q))$ of smooth quadric hypersurface $X(q)$ over a field F with characteristic different from 2.

Let us outline one of the key ingredients - the classical notion of Clifford algebra of a quadratic form.

6.3.1. *Clifford algebras.* The standard reference is [3]. [4, Chapter II §7] contains all details we need. A more recent source is [24, Chapter 8].

Let V be a vector space over a field F and let $q : V \to F$ be a quadratic form. The Clifford algebra $C(q)$ is a universal object for quadratic algebras of (V, q), i.e. F-algebras containing V as a subspace with the property

$$v^2 = q(v) \text{ for } v \in V.$$

Thus if $T(V) = \coprod_{n=0}^{\infty} V^{\otimes n}$ is the tensor algebra of V with natural grading, I is the two-sided ideal of $T(V)$ generated by all expressions $v \otimes v - q(v)$ for $v \in V$,

Definition 6.3. *The* Clifford algebra $C(q)$ *of* (V, q) *is*

$$C(q) = T(V)/I.$$

If $T_0(V) = \coprod_{n=0}^{\infty} V^{\otimes 2n}$ *is the even part of* $T(V)$*, then the even Clifford algebra* $C_0(q)$ *of* (V, q) *is*

$$C_0(q) = T_0(V)/I.$$

$C_0(q)$ is a subalgebra of $C(q)$ and $C(q)$ is a direct sum of two $C_0(q)$-modules

$$C(q) = C_0(q) \oplus C_1(q)$$

each of rank 1; this decomposition defines a $\mathbb{Z}_2$-grading in $C(q)$.

It is clear that product of vectors with more that $\dim V$ factors may be written as a shorter product, so if $v_1, v_2, \ldots v_n$ is a base of V, then set of all ordered products of distinct v_i form a base of $C(q)$; in particular $\dim C(q) = 2^{\dim V}$.

Example 6.3. *If $q = 0$ is a zero form, then $C(q)$ is the exterior algebra of V.*

It is clear that for an orthogonal direct sum of quadratic forms $(V, q) = (V_1, q_1) \oplus (V_2, q_2)$

$$\begin{aligned} C(q) &= C(q_1) \otimes_F C(q_2), \\ C_0(q) &= C_0(q_1) \otimes_F C_0(q_2) \oplus C_1(q_1) \otimes_F C_1(q_2). \end{aligned}$$

A vector $v \in V$ is called *isotropic* iff $q(v) = 0$ and is called *anisotropic* iff $q(v) \neq 0$. An anisotropic vector v is an invertible element of $C(q)$ and if v is anisotropic, then $C_1(q) = C_0(q)v = vC_0(q)$.

The identity $v^2 = q(v)$ for $v \in V$ yields in a standard way the formula

$$uw + wu = q(u + w) - q(u) - q(w) = 2B(u, w)$$

where B is the symmetric bilinear form associated to q.

Given an anisotropic vector v, any vector w may be decomposed into sum of two components: one parallel to v and one perpendicular to v. If $w = tv + u$ is such a decomposition, then

$$-vwv^{-1} = -\left(tvvv^{-1} + vuv^{-1}\right) = -tv + u$$

is the reflection with respect to the hyperplane $v^\perp$ orthogonal to v.

Example 6.4. *If $\dim V = 2$ and q has a matrix $diag(a, b)$ with respect to the basis $\{u, v\}$ of V, then $C(q)$ has the basis $\{1, u, v, uv\}$ with multiplication table $u^2 = a$, $v^2 = b$, $vu = -uv$ so the Clifford algebra is a quaternion algebra: $C(q) \cong \left(\frac{a,b}{F}\right)$.*

Example 6.5. *If $\dim V = 3$ and q has a matrix $diag(a, b, c)$ with respect to the basis $\{u, v, w\}$, then $C_0(q)$ has basis $\{1, uv, uw, -avw\}$ with multiplication table $(uv)^2 = -ab$, $(uw)^2 = -ac$, $(uw)(uv) = -(uv)(uw)$, so $C_0(q) \cong \left(\frac{-ab,-ac}{F}\right)$. The "determinant" element $\delta = uvw$ is in the centre of $C(q)$, $\delta^2 = -abc$; so $F[\delta] \cong F\left(\sqrt{-abc}\right)$. Hence*

$$C(q) = C_0(q) \oplus C_0(q)\delta \cong \left(\frac{-ab, -ac}{F\left(\sqrt{-abc}\right)}\right).$$

In particular if $-abc$ is a square in F, then $F\left(\sqrt{-abc}\right) \cong F \times F$ and $C(q) \cong \left(\frac{-ab,-ac}{F}\right) \times \left(\frac{-ab,-ac}{F}\right)$.

Exercise 6.1. *Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal basis of (V, q) (i.e. a basis of V, which is orthogonal with respect to q. Let further $\delta = v_1 \cdot v_2 \cdot \cdots \cdot v_n$.*

1. *δ commutes in $C(q)$ with every vector iff n is odd and δ anticommutes with every vector if n is even.*
2. *$\delta^2 = (-1)^{\frac{n(n-1)}{2}} \det(q)$ (this is the* discriminant *of the quadratic form q.)*
3. *If $\{w_1, w_2, \ldots, w_n\}$ is another orthogonal base, then $w_1 \cdot w_2 \cdot \cdots \cdot w_n$ differs from δ by a scalar factor.*

In general for a non-singular q and V of even dimension, $C(q)$ is a central simple F-algebra and is a tensor product of quaternion algebras; for a non-singular q and V of odd dimension, $C_0(q)$ is a central simple F-algebra and is a tensor product of quaternion algebras.

We record here three useful facts:

- for a scalar factor t, quadratic forms q and tq have isomorphic even Clifford algebras (quadratic forms q and tq are usually called *similar* forms);
- if q is an orthogonal direct sum $(V,q) = (V_1,q_1) \oplus (F, x \mapsto -x^2)$, then $C_0(q) \cong C(q_1)$;
- A pair of isotropic $e, f \in V$ such that $B(e,f) = \frac{1}{2}$ gives two orthogonal idempotents $\frac{1}{2}(1 \pm ef)$ in $C_0(q)$ (in this case $Fe \oplus Ff$ is called a *hyperbolic plane.*)

6.3.2. *Cohomology of quadric hypersurfaces.* Let $B = F[x_0, x_1, \dots, x_{d+1}]$ and let as usual $\mathbb{P}_F^{d+1}$ =Proj$B$. Let $q \in B_k$ be a nonzero (homogeneous of degree k) polynomial.

Lemma 6.11. *Let $X \subset \mathbb{P}_F^{d+1}$ be a hypersurface defined by $q \in B_k$; i.e. $X =$ProjA where $A = B/(q)$. Then $H^p(X, \mathcal{O}_X(n)) = 0$ for $p \neq 0, d$ and, fror $d \geq 1$,*
(1) $H^0(X, \mathcal{O}_X(n)) = A_n$
(2) $H^d(X, \mathcal{O}_X(n)) = A^*_{k-2-d-n}$.
If $d = 0$, we have an exact sequence

$$0 \to A_n \to H^0(X, \mathcal{O}_X(n)) \to A^*_{k-2-n} \to 0.$$

Here $A^*_n = Hom_F(A_n, F)$ is a dual space.

Proof. The exact sequence

$$0 \to B(n-k) \xrightarrow{q\cdot} B(n) \to A(n) \to 0$$

induces an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}_F^n}(n-k) \xrightarrow{q} \mathcal{O}_{\mathbb{P}_F^n}(n) \to \mathcal{O}_X(n) \to 0.$$

The Lemma follows immediately from the corresponding cohomology sequence and Lemma 6.1. ∎

It follows that $\mathcal{O}_X(n)$ is regular for $n \geq k-1$.

Lemma 6.12. *If X and d are as in 6.11 and $\mathcal{F}$ is a vector bundle on X then $H^p(X, \mathcal{F}) = 0$ for $p > d$.*

Proof. There exists an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{O}_X(-n)^r \to \mathcal{F} \to 0.$$

For $p > d+1$ we have $H^p = 0$ as a functor on quasi-coherent sheaves on $\mathbb{P}_F^{d+1}$, so the exact cohomology sequence shows that $H^{d+1}(X, \mathcal{F}) = 0$. ∎

Lemma 6.13. *If $\mathcal{F}$ is regular, then the canonical map $f^* f_*(\mathcal{F}) \to \mathcal{F}$ is surjective.*

Proof. The difference between the Lemma and Lemma 6.4 is that here we have different f^*. Nevertheless the proof remains valid. ∎

From here on we assume that q is a nonsingular quadratic form (i.e. $k = 2$.)

The first difference between the case of projective bundle and the case of quadric hypersurface is that $\mathcal{O}_X$ is no longer regular, although $\mathcal{O}_X(1)$ still is. One may recursively construct the canonical resolution of a regular sheaf as in the case of projective bundle. Nevertheless, one should remember that $\mathcal{O}_X$ here is different from $\mathcal{O}_X$ in Example 6.1. The second difference is that the recursive process (6.4) does not terminate.

6.3.3. *Generating function for the canonical resolution.* We introduce here a computational tool which is not needed to prove the Swan Theorem, but will be helpful for its applications. Let

$$\cdots \to \mathcal{O}_X(-p)^{k_p} \to \cdots \to \mathcal{O}_X(-1)^{k_1} \to {\mathcal{O}_X}^{k_0} \to \ \mathcal{F} \to 0$$

be the canonical resolution of a regular sheaf $\mathcal{F}$.

Since the functor of global sections is exact on regular sheaves, there is the following recurrence for $k_{p+1} = \dim \Gamma(X, \mathcal{Z}_p(p+1))$ in the process (6.4) of building the canonical resolution:

$$\begin{aligned}(6.10)\quad & \dim \Gamma(X,\ \mathcal{F}(p+1)) - k_0 \cdot \dim \Gamma(X, \mathcal{O}_X(p+1)) + \cdots \\ & \qquad + (-1)^{p-1} k_p \cdot \dim \Gamma(X, \mathcal{O}_X(1)) + (-1)^p k_{p+1} = 0.\end{aligned}$$

Recall that the *Poincaré series* $\Pi_{\mathcal{F}}(t)$ of a sheaf $\mathcal{F}$ is the formal power series

$$\Pi_{\mathcal{F}}(t) := \sum_{i=0}^{\infty} \dim \Gamma(X, \mathcal{F}(i)) \cdot t^i \in \mathbb{Z}[[t]].$$

The Poincaré series $\Pi_X(t)$ of a variety X is the Poincaré series of its structural sheaf:

$$\Pi_X(t) \overset{def}{=} \Pi_{\mathcal{O}_X}(t).$$

In particular if $X = \operatorname{Proj} A$ for a graded algebra A, then $\Pi_X(t)$ is the usual Poincaré series of A.

Example 6.6. *If S is the projective space, $S = \mathbb{P}^n_F$, then* $\dim \Gamma(S, \mathcal{O}_S(i)) = \binom{n+i}{i}$, *so*

$$P_n(t) := \Pi_S(t) = \sum_{i=0}^{\infty} \binom{n+i}{i} \cdot t^i = (1-t)^{-n-1}.$$

Example 6.7. *Let φ be a homogeneous polynomial of degree k in homogeneous coordinates in $\mathbb{P}^{d+1}_F = \operatorname{Proj} B,\ B = F[x_0,\ x_1,\ \ldots,\ x_{d+1}],\ A = B/(\varphi),\ X = \operatorname{Proj} A$ - a hypersurface $\varphi = 0$ in $\mathbb{P}^{d+1}_F$. Since the exact sequence*

$$0 \to B_n \xrightarrow{\phi\cdot} B_{n+k} \to A_{n+k} \to 0$$

splits for every n, the following equality holds:

$$\Pi_X(t) = P_{d+1}(t) - t^k P_{d+1}(t).$$

Thus $\Pi_S(t) = \dfrac{1-t^k}{(1-t)^{d+2}} = \dfrac{1+t+\ldots+t^{k-1}}{(1-t)^{d+1}}.$

Lemma 6.14. *For a projective quadric X of dimension d*

$$Q_d(t) := \Pi_X(t) = \frac{1+t}{(1-t)^{d+1}}. \quad ∎$$

Proposition 6.15. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$ -modules and either $\mathcal{F}'$, $\mathcal{F}''$ are regular or $\mathcal{F}$, $\mathcal{F}'(1)$ are regular, then*

$$\Pi_{\mathcal{F}}(t) = \Pi_{\mathcal{F}'}(t) + \Pi_{\mathcal{F}''}(t).$$

Proof. By Lemma 6.2, either $\mathcal{F}'$, $\mathcal{F}$, $\mathcal{F}''$ are regular or $\mathcal{F}'(1)$, $\mathcal{F}$, $\mathcal{F}''$ are regular. Hence each exact sequence of sheaves

$$0 \to \mathcal{F}'(i) \to \mathcal{F}(i) \to \mathcal{F}''(i) \to 0$$

induces an exact sequence of the corresponding spaces of global sections. ∎

The recursive method of finding a canonical resolution of a regular sheaf $\mathcal{F}$ described above, namely the identity (6.10), yields the following identities for the generating function $G_{\mathcal{F}}(t) := \sum_{i=0}^{\infty} k_i t^i$:

$$\Pi_{\mathcal{F}}(t) = G_{\mathcal{F}}(-t) \cdot \Pi_X(t) \qquad \text{and} \qquad G_{\mathcal{F}}(t) = \frac{\Pi_{\mathcal{F}}(-t)}{Q_d(-t)}.$$

Example 6.8. *The Poincaré series of the sheaf $\mathcal{O}_X(1)$ is*

$$\Pi_{\mathcal{O}_X(1)}(t) = \frac{\Pi_{\mathcal{O}_X}(t) - 1}{t},$$

so

$$G_{\mathcal{O}_X(1)}(t) = \frac{\Pi_{\mathcal{O}_X(1)}(-t)}{Q_d(-t)} = \frac{Q_d(-t) - 1}{-tQ_d(-t)} = \frac{\frac{1-t}{(1+t)^{d+1}} - 1}{-t\frac{1-t}{(1+t)^{d+1}}} = \frac{(1+t)^{d+1} - (1-t)}{t(1-t)}.$$

One may easily check that

$$\frac{(1+t)^{d+1} - (1-t)}{t(1-t)} - \frac{2^{d+1}t^{d+1}}{t\,(1-t)}$$

is a polynomial of degree $d-1$, so $k_i = 2^{d+1}$ for $i \geq d$.

6.3.4. *The canonical resolution of a regular sheaf.*

Lemma 6.16. *Assume that $H^q(X, \mathcal{F}) = 0$ for $q > d$ and all vector bundles $\mathcal{F}$. Let $\mathcal{Z}_p = \mathcal{Z}_p(\mathcal{F})$ be as in 6.4 above. If $\mathcal{G}$ is d-regular, then $\operatorname{Ext}^q(\mathcal{Z}_{d-1}, \mathcal{G}) = 0$ for all $q > 0$.*

Proof. The sequence $0 \to \mathcal{Z}_p \to f^*(\mathcal{T}_p) \otimes \mathcal{O}_X(-p) \to \mathcal{Z}_{p-1} \to 0$ gives

$$\begin{aligned} \cdots \to \operatorname{Ext}^q(f^*(\mathcal{T}_p) \otimes \mathcal{O}_X(-p), \mathcal{G}) \to \operatorname{Ext}^q(\mathcal{Z}_p, \mathcal{G}) \to \operatorname{Ext}^{q+1}(\mathcal{Z}_{p-1}, \mathcal{G}) \\ \to \operatorname{Ext}^{q+1}(f^*(\mathcal{T}_p) \otimes \mathcal{O}_X(-p), \mathcal{G}) \to \cdots . \end{aligned}$$

Now $\operatorname{Ext}^q(f^*(\mathcal{T}_p) \otimes \mathcal{O}_X(-p), \mathcal{G}) = \mathcal{T}_p^* \otimes H^q(X, \mathcal{G}(p)) = 0$ for $q > 0$, $p + q \geq d$; so for $q > 0$ we have

$$\operatorname{Ext}^q(\mathcal{Z}_{d-1}, \mathcal{G}) \cong \operatorname{Ext}^{q+1}(\mathcal{Z}_{d-2}, \mathcal{G}) \cong \cdots \cong \operatorname{Ext}^{q+d}(\mathcal{Z}_{-1}, \mathcal{G}) = 0 \quad ∎$$

Corollary 6.17. *Assume $H^q(X,\mathcal{F}) = 0$ for $q > d$ and all vector bundles $\mathcal{F}$. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of regular sheaves, then*

$$0 \to \mathcal{Z}_{d-1}(\mathcal{F}') \to \mathcal{Z}_{d-1}(\mathcal{F}) \to \mathcal{Z}_{d-1}(\mathcal{F}'') \to 0$$

is split exact.

Proof. Here $\mathcal{Z}_{d-1}(\mathcal{F}')$ is d-regular, so the appropriate Ext^1 is 0. ∎

Corollary 6.18. *$K_0(X)$ is torsion-free.*

Proof. Let $K_0(X,\oplus)$ be the K-group of exact category of vector bundles on X with only split exact sequences being admissible. The natural map $K_0(X,\oplus) \to K_0(X)$ has a right inverse given by

$$[\mathcal{F}] \mapsto \sum_{p=0}^{n} (-1)^p \left[\mathcal{O}_X(-p) \otimes f^*(\mathcal{T}_p(\mathcal{F}))\right] + (-1)^{n+1}\left[\mathcal{Z}_n(\mathcal{F})\right]$$

for n large enough and regular $\mathcal{F}$. Since $K_0(X,\oplus)$ is torsion-free by the Krull-Schmidt Theorem, the same is true for $K_0(X)$. ∎

Swan's method of computing K-theory of quadric hypersurfaces uses truncation of the canonical resolution of a (-1)-regular sheaf.

If $\mathcal{F}$ is (-1)-regular, then $\mathcal{F}(-1)$ is regular; so we have an epimorphism

$$\mathcal{O}_X \otimes \Gamma(X, \mathcal{F}(-1)) \twoheadrightarrow \mathcal{F}(-1).$$

Since $\Gamma(X,\mathcal{F}(-1))$ has finite dimension m, this is an epimorphism $\mathcal{O}_X^m \twoheadrightarrow \mathcal{F}(-1)$. Therefore there is an exact sequence of vector bundles $0 \to \mathcal{G} \to \mathcal{O}_X(1)^m \to \mathcal{F} \to 0$. If d is as in Lemma 6.16, then $\mathcal{Z}_{d-1}(\mathcal{G}) \oplus \mathcal{Z}_{d-1}(\mathcal{F}) \cong \mathcal{Z}_{d-1}(\mathcal{O}_X(1))^m$. Note that $\mathcal{G}$ is regular, so its canonical resolution is defined. Let $\mathcal{U} = \mathcal{Z}_{d-1}(\mathcal{O}_X(1))$ and let $E = \mathrm{End}_X(\mathcal{U})$ acting on $\mathcal{U}$ from the right. For any vector bundle $\mathcal{W}$, $Hom_X(\mathcal{U},\mathcal{W})$ is a left E-module.

Lemma 6.19. *If $\mathcal{W}$ is a direct summand of $\mathcal{U}^m$ for some $m < \infty$ then*

$$\mathcal{U} \otimes_E Hom_X(\mathcal{U},\mathcal{W}) \xrightarrow{\sim} \mathcal{W}.$$

Proof. This is clear for $\mathcal{W} = \mathcal{U}$ and the property is obviously preserved by direct sums and inherited by direct summands. ∎

Note that if $\mathcal{W}$ is a direct summand of $\mathcal{U}^m$, then $Hom_X(\mathcal{U},\mathcal{W})$ is a finitely generated projective E-module since it is a direct summand of $Hom_X(\mathcal{U},\mathcal{U}^m) = E^m$.

Definition 6.4. *For a projective quadric X of dimension d:*
- *the Swan bundle of X is $\mathcal{U} := \mathcal{Z}_{d-1}(\mathcal{O}_X(1))$;*
- *the functor $\mathcal{T}$ from $\mathfrak{R}_{-1}(X)$ to F-modules is $\mathcal{T}(\mathcal{F}) = Hom_X(\mathcal{U}, \mathcal{Z}_{d-1}(\mathcal{F}))$;*
- *the truncated canonical resolution of $\mathcal{F} \in \mathfrak{R}_{-1}(X)$ is*

$$0 \to \mathcal{U} \otimes_E \mathcal{T}(\mathcal{F}) \to \mathcal{O}_X(1-d) \otimes \mathcal{T}_{d-1}(\mathcal{F}) \to \cdots$$
$$\cdots \to \mathcal{O}_X \otimes \mathcal{T}_0(\mathcal{F}) \to \mathcal{F} \to 0.$$

Each $\mathcal{T}_p$ is an exact functor from $\mathfrak{R}_{-1}(X)$ to $\mathcal{P}(F)$ and $\mathcal{T}$ is also an exact functor.

We omit technical details of Swan's computations with several kinds of resolutions and state the result.

Let $\{v_0, v_1, \ldots, v_{d+1}\}$ be an orthogonal basis of the vector space V; let $\{z_0, z_1, \ldots, z_{d+1}\}$ be the dual basis of V^*. Denote by C_1 the odd part of the Clifford algebra $C(q)$. The subscripts in C_i will be taken $\mod 2$. Put

$$\varphi = \sum_{i=0}^{d+1} z_i \otimes v_i, \qquad \varphi \in \Gamma(X, \mathcal{O}_X(1) \otimes V).$$

The complex

$$\begin{aligned} \cdots \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n) \otimes C_{n+d+1} &\xrightarrow{\varphi\cdot} \mathcal{O}_X(1-n) \otimes C_{n+d} \\ &\xrightarrow{\varphi\cdot} \mathcal{O}_X(2-n) \otimes C_{n+d-1} \xrightarrow{\varphi\cdot} \cdots \end{aligned} \tag{6.11}$$

is exact and locally splits ([27, Proposition 8.2.(a)].) Moreover, its part for $n > d-1$ coincides with the $n > d-1$ part of the canonical resolution of $\mathcal{O}_X(1)$. Thus if we denote

$$\mathcal{U}_n := \operatorname{Coker}\left(\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2}\right)$$

then $\mathcal{U} = \mathcal{U}_{d-1}$ ([27, Corollary 8.6].)

Since the complex (6.11) is - up to twist - periodical with period two, we have

$$\mathcal{U}_{n+2} = \mathcal{U}_n(-2).$$

Consider the exact sequences

$$\mathcal{O}_X(-n-2) \otimes C_{n+d+3} \xrightarrow{\varphi\cdot} \mathcal{O}_X(-n-1) \otimes C_{n+d+2} \to \mathcal{U}_n \to 0$$

for two consecutive values of n; twist the first of them by 1. For any anisotropic vector $w \in V$ the isomorphism given by right multiplication by $1 \otimes w$ fits into commutative diagram:

$$\begin{tikzcd} \mathcal{O}_X(-n-2) \otimes C_{n+d+4} \arrow[r, "\varphi\cdot"] \arrow[d, "\cong"', "\cdot 1\otimes w"] & \mathcal{O}_X(-n-1) \otimes C_{n+d+3} \arrow[r] \arrow[d, "\cong"', "\cdot 1\otimes w"] & \mathcal{U}_{n+1}(1) \arrow[r] & 0 \\ \mathcal{O}_X(-n-2) \otimes C_{n+d+3} \arrow[r, "\varphi\cdot"] & \mathcal{O}_X(-n-1) \otimes C_{n+d+2} \arrow[r] & \mathcal{U}_n \arrow[r] & 0. \end{tikzcd}$$

Thus we have proven the following Lemma:

Lemma 6.20. *One has*

$$\mathcal{U}_{n+1} \cong \mathcal{U}_n(-1) \quad \textit{and} \quad \mathcal{U}_n \cong \mathcal{U}_0(-n)$$

for an arbitrary integer n. ∎

There is an exact sequence

$$0 \to \mathcal{U}_0 \xrightarrow{\varphi} \mathcal{O}_X \otimes C_0 \to \mathcal{U}_{-1} \to 0 \tag{6.12}$$

where the isomorphism $\cdot(1 \otimes w)$ was used to replace $\mathcal{O}_X \otimes C_1$ by $\mathcal{O}_X \otimes C_0$ for even d. In particular

$$\operatorname{rank}(\mathcal{U}) = \frac{1}{2} \dim C_0 = 2^d. \tag{6.13}$$

Lemma 6.21. $\operatorname{End}_X(\mathcal{U}_n) \cong C_0$ *acts on* $\mathcal{U}_n$ *from the right.*

Proof. [27, Lemma 8.7]. ∎

Now we can compute the K-theory of the quadric hypersurface X. For $n = 0, 1, \ldots, d-1$ let $U_n : \mathcal{P}(F) \to \mathcal{P}(X)$ be $U_n(M) := \mathcal{O}_X(-n) \otimes M$; moreover define $U : \mathcal{P}(C_0(q)) \to \mathcal{P}(X)$ by $U(M) := \mathcal{U} \otimes_{C_0(q)} M$. These functors induce maps $u_n : K_i(F) \to K_i(X)$ and $u : K_i(C_0(q)) \to K_i(X)$. Therefore we get

$$\overline{u} = (u_0, u_1, \ldots, u_{d-1}, u) : K_i(F)^d \oplus K_i(C_0(q)) \to K_i(X).$$

Theorem 6.22. *Let $X = X(q) \subset \mathbb{P}_F^{d+1}$ be a quadratic hypersurface of dimension d defined by a nonsingular quadratic space (V, q). Then the map $\overline{u} : K_i(F)^d \oplus K_i(C_0(q)) \to K_i(X)$ is an isomorphism.*

Proof. We already know that $K_i(\mathfrak{R}_n) \cong K_i(\mathfrak{P}_n) \cong K_i(\mathcal{P}(X))$. Moreover, we have exact functors $\mathcal{T}_n : \mathfrak{R}_{-1}(X) \to \mathcal{P}(F)$ and $\mathcal{T} : \mathfrak{R}_{-1}(X) \to \mathcal{P}(C_0(q))$, and induced maps $t_n : K_i(X) \to K_i(F)$, $t : K_i(X) \to K_i(C_0(q))$; therefore

$$\overline{t} := (t_0, t_1, \ldots, t_{d-1}, t)$$

sends $K_i(X)$ to $K_i(F)^d \oplus K_i(C_0(q))$. The truncated canonical resolution shows that $\overline{u}\overline{t}$ is the isomorphism $K_i(\mathfrak{R}_n) \cong K_i(\mathfrak{P}_n)$. This shows that $\overline{u}$ is onto.

Now define functors $W_n : \mathfrak{P}_0(X) \to \mathcal{P}(F)$ by $W_n(\mathcal{F}) = \Gamma(X, \mathcal{F}(n))$ and $W : \mathfrak{P}_0(X) \to \mathcal{P}(C_0(q))$ by $W(\mathcal{F}) = Hom_X(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{U}^{\vee} \otimes \mathcal{F})$. Since

$$\begin{aligned} W_i U_j(M) &= \Gamma\left(X, \left(\mathcal{O}_X(-j) \otimes M\right)(i)\right) = \Gamma\left(X, \mathcal{O}_X(i-j)\right) \otimes M, \\ WU(M) &= Hom_X\left(\mathcal{U}, \mathcal{U} \otimes_{C_0(q)} M\right) = Hom_X\left(\mathcal{U}, \mathcal{U}\right) \otimes_{C_0(q)} M = M, \\ \Gamma\left(X, \mathcal{U}(n)\right) &= 0 \text{ for } n = 0, 1, \ldots, d-1, \end{aligned}$$

the induced map $\overline{w} : K_i(X) \to K_i(F)^d \oplus K_i(C_0(q))$ has the property that $\overline{wu}$ is given by a triangular matrix with the identity maps on diagonal. Thus $\overline{u}$ is injective. ∎

Swan gave also an explicit formula for the map induced by an embedding of a nonsingular hyperplane section into X [27, Theorem 10.5], and hence computed the K-theory of a smooth affine quadric. Moreover, Swan computed the K-theory of a cone like $\mathrm{Proj} R[x_1, x_2, \ldots, x_n]/\left(q(x_2, \ldots, x_n)\right)$ - [27, Theorem 11.7].

7. Applications

7.1. **Chow ring of a split smooth quadric.** In theory of quadratic forms a subspace U of a space V with a quadratic form q is said to be an *isotropic subspace* iff U contains a nonzero isotropic vector; a subspace U is said to be a *totally isotropic subspace* iff $q|_U = 0$. This convention is convenient if one is interested in classification of quadratic forms, since nonsingular isotropic space contains a hyperbolic plane. Geometers prefer term "isotropic subspace" for a totally isotropic space. The dimension of maximal totally isotropic subspace is called a *Witt index* of q, or simply an *index* of q. A quadric $q = 0$ is *split* iff the index of q is maximal possible for quadratic forms of given dimension.

We shall apply the results of section 6.3 in the simplest possible case of a split quadric: X is a projective quadric hypersurface over a field F, $\mathrm{char}\, F \neq 2$, defined by the quadratic form of maximal index.

7.1.1. *Notation.* Consider a vector space V with a basis $\{v_0, v_1, \ldots, v_{d+1}\}$ over a field F, $\operatorname{char} F \neq 2$. Denote by $\{z_0, z_1, \ldots, z_{d+1}\}$ the dual basis of $V^* = Hom_F(V, F)$. Let q be the quadratic form

$$q = \sum_{i=0}^{d+1} (-1)^i z_i^2.$$

Moreover, let $e_i = \frac{1}{2}(v_{2i} - v_{2i+1})$, $f_i = \frac{1}{2}(v_{2i} + v_{2i+1})$ for all possible values of i. Thus $e_0, e_1, \ldots, e_m$ span a maximal totally isotropic subspace and $f_0, f_1, \ldots, f_m$ span a maximal totally isotropic subspace. The index of q is $m+1$, which is maximal possible value for quadratic forms of dimenssion $d+2$.

- if $d = 2m$, then $e_0, f_0, e_1, f_1, \ldots, e_m, f_m$ form a basis of V with the dual basis $\{x_0, y_0, x_1, y_1, \ldots, x_m, y_m\}$ and
$$q = \sum_{i=0}^{m} x_i y_i.$$
- if $d = 2m+1$, then $f_0, e_1, f_1, \ldots, e_m, f_m, v_{d+1}$ form a basis of V with the dual basis $\{x_0, y_0, x_1, y_1, \ldots, x_m, y_m, z_{d+1}\}$ and
$$q = \sum_{i=0}^{m} x_i y_i + z_{d+1}^2.$$

Note that $v_{2i+1} = f_i - e_i$, and $q(f_i) = q(e_i)$, so that we have:

Lemma 7.1. *The reflection with respect to the hyperplane ${v_{2i+1}}^{\perp}$ interchanges f_i with e_i and interchanges x_i with y_i.* ∎

We shall compute table of multiplication in $K_0(X)$ for a d-dimensional projective quadric X defined by the equation $q = 0$ in $\mathbb{P}_F^{d+1}$, i.e., for

$$X = \operatorname{Proj} S(V^*)/(q) \cong \operatorname{Proj} F[z_0, z_1, \ \ldots, z_{d+1}]/(q).$$

7.1.2. *Clifford algebra.* In the case of $d = 2m+1$ the even part $C_0 = C_0(q)$ of the Clifford algebra $C(q)$ is isomorphic to the matrix algebra $\mathrm{M}_{2^{m+1}}(F)$ (a standard fact),

$$C_0 \cong \mathrm{M}_{2^{m+1}}(F).$$

In particular, the Morita equivalence

$$\begin{array}{rcl} M & \longmapsto & Hom_F(F^N, F) \otimes_{C_0} M \in \mathrm{Ob}(\mathcal{P}(F)) \\ W & \longmapsto & F^N \otimes_F W \in \mathrm{Ob}(\mathcal{P}(C_0)) \end{array}$$

of categories of C_0-modules and F-modules induces isomorphisms $K_p(C_0) \cong K_p(F)$ for all p.

In the case of even $d = 2m$, the algebra C_0 has the centre $F \oplus F \cdot \delta$, where $\delta = v_0 \cdot v_1 \cdot \ldots \cdot v_{d+1}$ and $\delta^2 = 1$. Thus $\frac{1}{2}(1+\delta)$, $\frac{1}{2}(1-\delta)$ are orthogonal central idempotents of C_0, so

$$C_0 = \frac{1}{2}(1+\delta)C_0 \oplus \frac{1}{2}(1-\delta)C_0$$

where each direct summand is isomorphic to the matrix algebra $\mathrm{M}_{2^m}(F)$ (yet another standard fact).

For every anisotropic vector $w \in V$, the reflection $\alpha \mapsto -w\alpha w^{-1}$ with respect to the hyperplane $w^{\perp}$ induces an automorphism ρ_w of C_0, which interchanges δ with its opposite:

$$\rho_w(\delta) = -\delta. \tag{7.1}$$

Regarding subscripts $i \mod 2$ denote

$$P_i = (1 + (-1)^i \delta)C_0 \text{ for even } d.$$

Lemma 7.2. *For any anisotropic vector $w \in V$, $\rho_w(P_i) = P_{i+1}$.* ∎

We are now ready to compute $\mathcal{U}_n\check{}$.

Lemma 7.3. *$\mathcal{U}_n\check{} \cong \mathcal{U}_n(2n+1)$; in particular, $\mathcal{U}\check{} \cong \mathcal{U}(2d-1)$.*

Proof. We have chosen a basis $\{v_0, v_1, \ldots, v_{d+1}\}$ of V in 7.1.1 above. The set of naturally ordered products of several v_i's with even number of factors forms a basis of C_0. Define a quadratic form Q on C_0 as follows: let the distinct products of elements of the basis $\{v_0, v_1, \ldots, v_{d+1}\}$ be orthogonal to each other and let

$$Q(v_{i_1} \cdot v_{i_2} \cdot \ldots \cdot v_{i_k}) = q(v_{i_1}) \cdot q(v_{i_2}) \cdot \ldots \cdot q(v_{i_k}).$$

The form Q is nonsingular and defines - by scalar extension - a nonsingular symmetric bilinear form Δ on $\mathcal{O}_X \otimes C_0$. Since $(q(v_i))^2 = 1$, a direct computation shows that $\mathrm{Im}(\mathcal{O}_X(-1) \otimes C_1 \xrightarrow{\varphi} \mathcal{O}_X \otimes C_0) = \varphi \cdot \mathcal{U}_0 \cong \mathcal{U}_0$ is a totally isotropic subspace of $\mathcal{O}_X \otimes C_0$. Therefore

$$\mathcal{U}_0 \cong \varphi \cdot \mathcal{U}_0 = (\varphi \cdot \mathcal{U}_0)^{\perp} \cong ((\mathcal{O}_X \otimes C_0)/(\varphi \cdot \mathcal{U}_0))\check{} \cong \mathcal{U}_{-1}\check{}.$$

Thus

$$\mathcal{U}_0\check{} \cong \mathcal{U}_{-1} \cong \mathcal{U}_0(1)$$

and, in general,

$$\mathcal{U}_n\check{} \cong (\mathcal{U}_0(-n))\check{} \cong \mathcal{U}_0\check{}(n) \cong \mathcal{U}_0(n+1) \cong \mathcal{U}_n(2n+1). \quad ∎$$

Corollary 7.4. i) *$[\mathcal{U}\check{}] = [\mathcal{U}(2d-1)]$ and $[\mathcal{U}(d-1)] + [\mathcal{U}(d-1)]\hat{} = 2^{d+1}$ in $K_0(X)$;*

ii) $\mathrm{rank}\,\mathcal{U} = \frac{1}{2}\dim C_0 = 2^d$. ∎

In case of $d = 2m$ the algebra $\mathrm{End}_X(\mathcal{U}) = C_0$ splits into the direct product of subalgebras defined in 7.1.2 above: $C_0 = P_0 \times P_1$.

Definition 7.1. *In case of even d:*

$$\mathcal{U}_n' = \mathcal{U}_n \otimes_{C_0} P_0,\ \mathcal{U}_n'' = \mathcal{U}_n \otimes_{C_0} P_1,$$
$$\mathcal{U}' = \mathcal{U} \otimes_{C_0} P_0,\ \mathcal{U}'' = \mathcal{U} \otimes_{C_0} P_1.$$

Note that $\mathcal{U}_n = \mathcal{U}_n' \oplus \mathcal{U}_n''$ and $\mathcal{U} = \mathcal{U}' \oplus \mathcal{U}''$. The summands $\mathcal{U}_0'$ and $\mathcal{U}_0''$ correspond to spinor representation and we shall use here the standard argument on dualization.

In the case of an even $d = 2m$, another property of φ and the quadratic form Q introduced in the proof of Lemma 7.3, may be verified by a direct computation:

Lemma 7.5. *In case of $d = 2m$*

i) *if m is even, then $P_i = (1 \pm \delta)C_0$ are orthogonal to each other, hence self-dual;*

ii) *if m is odd, then $P_i = (1 \pm \delta)C_0$ are totally isotropic, hence dual to each other;*

iii) $\varphi(1 \pm \delta) = (1 \mp \delta)\varphi$. ∎

Corollary 7.6. *In case of* $d = 2m$,

i) $\mathcal{U}'\check{} \cong \mathcal{U}'(2d-1)$ *and* $\mathcal{U}''\hat{} \cong \mathcal{U}''(2d-1)$ *for even* m*;*
ii) $\mathcal{U}'\check{} \cong \mathcal{U}''(2d-1)$ *and* $\mathcal{U}''\hat{} \cong \mathcal{U}'(2d-1)$ *for odd* m*;*
iii) $\mathrm{End}_X(\mathcal{U}') \cong \mathrm{End}_X(\mathcal{U}'') \cong M_{2^m}(F)$*;*
iv) *the exact sequence (6.12) splits into two exact sequences*

$$0 \to \mathcal{U}_0' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_0 \to \mathcal{U}_0''(1) \to 0,$$
$$0 \to \mathcal{U}_0'' \xrightarrow{\varphi\cdot} \mathcal{O}_X \otimes P_1 \to \mathcal{U}_0'(1) \to 0. \;∎$$

The trivial observation, that an automorphism of X induced by a reflection with respect to a nonsingular hyperplane interchanges $\mathcal{U}'$ with $\mathcal{U}''$, will be important in the following.

A standard way to determine indecomposable components is tensoring by the simple left module over an appropriate endomorphism algebra.

Definition 7.2.
i) in case of $d = 2m+1$*:* $\mathcal{V} = \mathcal{U} \otimes_{C_0} F^{2^{m+1}}$*;*
ii) in case of $d = 2m$*:* $\mathcal{V}_0 = \mathcal{U}' \otimes_{\mathrm{M}_{2^m}(F)} F^{2^m}$, $\mathcal{V}_1 = \mathcal{U}'' \otimes_{\mathrm{M}_{2^m}(F)} F^{2^m}$.

For convenience we will use mod 2 subscripts in $\mathcal{V}_i$. Since $M_n(F) = (F^n)^n$ as a left $M_n(F)$-module, indecomposable components inherit properties of the Swan bundle. We have:

Proposition 7.7. a) *In case of* $d = 2m+1$*:*
i) $\mathcal{U} \cong \mathcal{V}^{2^{m+1}}$*;*
ii) $\mathcal{V}\check{} = \mathcal{V}(2d-1)$*;*
iii) $\mathrm{End}_X(\mathcal{V}) \cong F$ *and* $\mathrm{rank}\,\mathcal{V} = 2^m$*;*
iv) $[\mathcal{V}(d-1)] + [\mathcal{V}(d)] = 2^m$ *in* $K_0(X)$.

b) *In case of* $d = 2m$*:*
i) $\mathcal{U}' = \mathcal{V}_0^{2^m}$ *and* $\mathcal{U}'' = \mathcal{V}_1^{2^m}$*;*
ii) $\mathcal{V}_i\check{} = \mathcal{V}_{i+m}(2d-1)$*;*
iii) $\mathrm{End}_X(\mathcal{V}_i) \cong F$ *and* $\mathrm{rank}\,\mathcal{V}_i = 2^{m-1}$*;*
iv) $[\mathcal{V}_i(d-1)] + [\mathcal{V}_{i+1}(d)] = 2^m$ *in* $K_0(X)$. ∎

In particular $Hom_X(\mathcal{V}_i, \mathcal{V}_{i+1}) = 0$. Moreover, once again a reflection interchanges $\mathcal{V}_i$ with $\mathcal{V}_{i+1}$.

Example 7.1. *If* $d = 1$ *the conic* $X \overset{i}{\hookrightarrow} \mathbb{P}^2$ *given by* $x_0y_0 + {z_2}^2 = 0$ *is isomorphic to the projective line* $\mathbb{P}^1$*:*

$$\chi \;:\; \mathbb{P}^1 \xrightarrow{\sim} X,$$
$$\chi(t \;:\; u) = (t^2 : -u^2 : uv).$$

In this case

$$\mathcal{O}_X(-1) = i^*\left(\mathcal{O}_{\mathbb{P}^2}(-1)\right),$$

and the canonical truncated resolution of $\mathcal{O}_X(1)$ *is*

$$0 \to \mathcal{U} \to {\mathcal{O}_X}^3 \to \mathcal{O}_X(1) \to 0.$$

It follows that $\mathcal{U}$ has rank 2, and

$$\begin{aligned}\bigwedge^3 \mathcal{O}_X{}^3 &\cong \bigwedge^2 \mathcal{U} \otimes \mathcal{O}_X(1),\\ \bigwedge^2 \mathcal{U} &\cong \mathcal{O}_X(-1).\end{aligned}$$

Moreover,

$$\mathcal{V} = \chi_*\left(\mathcal{O}_{\mathbb{P}^1}(-1)\right) = \left(\chi^{-1}\right)^*\left(\mathcal{O}_{\mathbb{P}^1}(-1)\right),$$
$$\mathcal{V} \oplus \mathcal{V} = \mathcal{U},$$
$$\mathcal{V} \otimes \mathcal{V} = \bigwedge^2 \mathcal{U} = \mathcal{O}_X(-1). \blacksquare$$

Example 7.2. *In the case $d = 2$, the quadric surface $X \hookrightarrow \mathbb{P}^3$ given by $x_0y_0 + x_1y_1 = 0$ has two projections*

$$\begin{aligned} p_0 &: X \longrightarrow \mathbb{P}^1, \qquad p_0\left(x_0 : x_1 : y_0 : y_1\right) = \begin{cases} (x_0 : x_1) \\ (-y_1 : y_0) \end{cases}, \\ p_1 &: X \longrightarrow \mathbb{P}^1, \qquad p_1\left(x_0 : x_1 : y_0 : y_1\right) = \begin{cases} (x_0 : y_1) \\ (-x_1 : y_0) \end{cases} \end{aligned}$$

which define an isomorphism

$$\psi : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$

In this case one may check that

$$\mathcal{V}_0(1) = p_0{}^*\left(\mathcal{O}_{\mathbb{P}^1}(-1)\right), \qquad \mathcal{V}_1(1) = p_1{}^*\left(\mathcal{O}_{\mathbb{P}^1}(-1)\right)$$

or vice versa, and

$$\mathcal{O}_X(-1) = \mathcal{V}_0(1) \otimes \mathcal{V}_1(1). \quad \blacksquare$$

Let $H = [\mathcal{O}_X] - [\mathcal{O}_X(-1)]$ be the class of a hyperplane section in $K_0(X)$.

Corollary 7.8. *In case of $d = 2m$ the following identities hold in $K_0(X)$:*

i) $([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot H = 0$;
ii) $([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot [\mathcal{O}_X(n)] = [\mathcal{V}_0] - [\mathcal{V}_1]$;
iii) $([\mathcal{V}_0] - [\mathcal{V}_1])\check{} = (-1)^m([\mathcal{V}_0] - [\mathcal{V}_1])$.

Proof. Proposition 7.7.b) iv) yields

$$[\mathcal{V}_0(d-1)] + [\mathcal{V}_1(d)] = [\mathcal{V}_1(d-1)] + [\mathcal{V}_0(d)].$$

Tensoring by $\mathcal{O}_X(-d)$ one obtains

$$[\mathcal{V}_0] - [\mathcal{V}_1] = ([\mathcal{V}_0] - [\mathcal{V}_1]) \cdot [\mathcal{O}_X(-1)],$$

hence i) and ii). Thus iii) results from 7.7. b) ii). $\blacksquare$

Proposition 7.9. *$K_\bullet(X)$ is a free $K_\bullet(F)$-module of rank $2m + 2$; moreover*

i) *in the case of $d = 2m + 1$ the classes $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\dots$, $[\mathcal{O}_X(1-d)]$, $[\mathcal{V}]$ form a basis of $K_\bullet(X)$;*
ii) *in the case of $d = 2m$ the classes $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\dots$, $[\mathcal{O}_X(1-d)]$, $[\mathcal{V}_0]$, $[\mathcal{V}_1]$ form a basis of $K_\bullet X$.*

Proof. Apply Theorem 6.22. $\blacksquare$

In case of a (-1)-regular $\mathcal{F}$ to obtain an expression for $[\mathcal{F}] \in K_0(X)$ in terms of the basis from Proposition 7.9 one truncates the canonical resolution of $\mathcal{F}$:

$$0 \to \mathcal{Z}_{d-1} \to \mathcal{O}_X(1-d)^{k_{d-1}} \to \cdots \to \mathcal{O}_X(-1)^{k_1} \to \mathcal{O}_X{}^{k_0} \to \mathcal{F} \to 0$$

and replaces $\mathcal{Z}_{d-1}$ by $\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1}) \cong \mathcal{Z}_{d-1}$. Then in $K_0(X)$

$$[\mathcal{F}] = \sum_{i=0}^{d-1} (-1)^i k_i [\mathcal{O}_X(-i)] + [\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})].$$

Depending on the parity of d we have either

$$[\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})] = a[\mathcal{V}]$$

or

$$[\mathcal{U} \otimes_{C_0} \operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})] = a[\mathcal{V}_0] + b[\mathcal{V}_1],$$

where the integers a, b in turn depend on the decomposition of $\operatorname{Hom}_X(\mathcal{U}, \mathcal{Z}_{d-1})$ into a direct sum of simple left C_0 - modules. Conversely, if for a given $\mathcal{F}$ the equality

$$[\mathcal{F}] = \sum_{i=0}^{d-1} (-1)^i k_i [\mathcal{O}_X(-i)] + W$$

holds, where W is either $a[\mathcal{V}]$ or $a[\mathcal{V}_0] + b[\mathcal{V}_1]$, then k_0 is the Euler characteristic $\sum (-1)^i \dim \mathrm{H}^i(X, \mathcal{F})$ of $\mathcal{F}$. So if $\mathcal{F}$ is regular, then $k_0 = \dim \Gamma(X, \mathcal{F})$. Next, $\mathcal{Z}_0(1) = \operatorname{Ker}\left(\mathcal{O}_X(1)^{k_0} \to \mathcal{F}(1)\right)$ is regular, and iterating this as in the recursive process of constructing the canonical resolution, one obtains that for a regular $\mathcal{F}$ the congruence

$$[\mathcal{F}] \equiv \sum_{i=0}^{d-1} (-1)^i k_i [\mathcal{O}_X(-i)] \qquad \bmod \operatorname{Im}(K_0(C_0) \xrightarrow{u} K_0(X))$$

holds if and only if the integers the k_i satisfy (6.10). In case of $d = 2m+1$, in order to express class $[\mathcal{F}]$ of a regular sheaf $\mathcal{F}$ in terms of the basis of Proposition 7.9, it is enough to know the dimensions of $\Gamma(X, \mathcal{F}(i))$ for $i = 0, 1, 2, \ldots, d-1$ to determine the k_i's. Then the rank of $\mathcal{F}$ is sufficient to determine the coefficient a of $[\mathcal{V}]$. An analogous statement remains valid for an arbitrary sheaf $\mathcal{F}$ with the Euler characteristic of $\mathcal{F}(i)$ in place of $\dim \Gamma(X, \mathcal{F}(i))$. In case of $d = 2m$, in view of Corollary 7.8 ii) and Proposition 7.7 ii), the bundles $\mathcal{V}_0$ and $\mathcal{V}_1$ have the same Euler characteristic, rank and even the highest exterior power. Thus, without special considerations, one can express a class $[\mathcal{F}]$ in terms of the basis of Proposition 7.9 only up to a multiple of $[\mathcal{V}_0] - [\mathcal{V}_1]$.

7.1.3. *Generating function for a truncated canonical resolution.* Now we compute some generating functions for canonical resolutions.

Example 7.3. *For a linear section $H^l = (1 - [\mathcal{O}_X(-1)])^l$ of codimension l in X*

$$G_{H^l}(t) = (1+t)^l. \quad \blacksquare$$

Example 7.4. *We follow the notation of 7.1.1. Since X splits, it contains linear subvarieties $S_k = \operatorname{Proj} F[x_0, \ldots, x_k]$ given by the following equations:*
a) in case of $d = 2m$:

$$y_0 = \ldots = y_m = x_{k+1} = \ldots = x_m = 0 \text{ for } k < m \text{ and}$$
$$y_0 = \ldots = y_m = 0 \text{ for } k = m;$$

b) in case of $d = 2m+1$*:*

$$y_0 = \ldots = y_m = z_d = x_{k+1} = \ldots = x_m = 0 \text{ for } k < m \text{ and}$$
$$y_0 = \ldots = y_m = z_d = 0 \text{ for } k = m.$$

S_k is isomorphic to $\mathbb{P}^k_F$, in particular its structural sheaf $\mathcal{L}_k$ is regular. Therefore

$$G_{\mathcal{L}_k}(t) = \frac{P_k(-t)}{Q_d(-t)} = \frac{(1+t)^{-k-1}}{(1-t)/(1+t)^{d+1}} = \frac{(1+t)^{d-k}}{1-t}.$$

Lemma 7.10.

$$2G_{\mathcal{L}_k} - G_{\mathcal{L}_{k-1}} = (1+t)^{d-k}. \quad \blacksquare$$

To obtain a compact formula for the truncated canonical resolution of a sheaf $\mathcal{F}$, we truncate $G_{\mathcal{F}}$, neglecting all terms of degree $\geq d$. Truncating generating function $G_{\mathcal{F}}$ one obtains a polynomial $T_{\mathcal{F}}$. For $l < d$ the canonical resolution for H^l is itself truncated:

$$T_{H^l}(t) = (1+t)^l \qquad \text{for} \qquad l < d.$$

The sequence (c_i) of coefficients of the canonical resolution of the sheaf $\mathcal{L}_k$ stabilizes starting from the degree $d-k$ onwards:

$$G_{\mathcal{L}_k}(t) = \frac{(1+t)^{d-k}}{1-t} = (1+t)^{d-k} \cdot \sum_{i=0}^{\infty} t^i = \sum_{i=0}^{\infty} c_i t^i$$

so

$$c_{d-k} = c_{d-k+1} = \ldots = 2^{d-k}.$$

Thus

$$T_{\mathcal{L}_k}(t) = \frac{(1+t)^{d-k} - 2^{d-k} t^d}{1-t}.$$

Proposition 7.11. *If, for a fixed k, $\mathcal{L}_k$ is the structural sheaf of a linear subvariety S_k of dimension k in X, then in $K_0(X)$:*
a) in case of $d = 2m+1$

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + 2^{m-k}[\mathcal{V}];$$

b) in case of $d = 2m$ for a suitable integer a,

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (2^{m-k} - a)[\mathcal{V}_1];$$

c) $2[\mathcal{L}_k] - [\mathcal{L}_{k-1}] = H^{d-k}$.

Proof. Substituting $t = -[\mathcal{O}_X(-1)]$ into the expansion for $T_{\mathcal{L}_k}(t)$ yields, depending on the parity of d, the expressions

$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}];$$
$$[\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + b[\mathcal{V}_1].$$

for suitable integers a, b. Thus

$$0=\operatorname{rank}[\mathcal{L}_k]=\begin{cases} T_{\mathcal{L}_k}(-1)+(-1)^d a\cdot 2^m = & \\ \quad =(-1)^d(2^m a-2^{d-k-1}) & \text{for } d=2m+1\\ T_{\mathcal{L}_k}(-1)+(-1)^d(a+b)\cdot 2^{m-1} = & \\ \quad =(-1)^d(2^{m-1}(a+b)-2^{d-k-1}) & \text{for } d=2m.\end{cases}$$

To prove c) it is enough to show that $2T_{\mathcal{L}_k}(t)-T_{\mathcal{L}_{k-1}}(t)=(1+t)^{d-k}$, which follows directly from Lemma 7.10. ∎

7.1.4. *The topological filtration.* Now we shall find a basis of $K_0(X)$ which is convenient for computations. Since the quadric X is regular, $K_0'(X)=K_0(X)$ and one may transfer the topological filtration

$$\mathrm{F}^p\,K_0'(X)= \text{ subgroup generated by } \left\{[\mathcal{F}]:\ \begin{array}{c}\text{the stalk } \mathcal{F}_x=0 \text{ for all generic points}\\ x \text{ of subvarieties of codimension} < p\end{array}\right\}$$

of $K_0'(X)$ to $K_0(X)$. We know that for a split projective quadric X the Chow groups $CH^p(X)$ are isomorphic to the corresponding factors of the topological filtration:

$$CH^p(X)\cong \mathrm{F}^p\,K_0(X)/\,\mathrm{F}^{p+1}\,K_0(X).$$

We follow the notation of 7.1.1. The K-cohomology computation 5.3 yields Chow groups of the quadric X.

Proposition 7.12. *For a split projective quadric X of dimension d*
a) in case of $d=2m$,

$$CH^p(X)\cong\mathbb{Z} \text{ for } p\neq m,\ 0\le p\le 2m \text{ and } CH^m(X)\cong\mathbb{Z}\oplus\mathbb{Z};$$

b) in case of $d=2m+1$,

$$CH^p(X)\cong\mathbb{Z} \text{ for all } 0\le p\le 2m+1.$$

Explicit generators are given as follows:
Case $d=2m$:
i) for $p>m$, the class of any linear subvariety of dimension $d-p$, e.g., $S_{d-p}: y_0=\ldots=y_m=x_{d-p+1}=\ldots=x_m=0$;
ii) for $p<m$, the class H^p of a linear section of codimension p;
iii) for $p=m$, $CH^m(X)$ is generated by two classes of linear subvarieties $S_m':\ x_0=\ldots=x_m=0$ and $S_m'':\ y_0=x_1=\ldots=x_m=0$; the classes in $CH^m(X)$ remain unchanged if an even number of x_i's is replaced by corresponding y_i in these equations.
Case $d=2m+1$:
i) for $p>m$, the class of any linear subvariety of dimension $d-p$, e.g. $S_{d-p}: y_0=\ldots=y_m=z_{d+1}=x_{d-p+1}=\ldots=x_m=0$;
ii) for $p\le m$, a class H^p of a linear section of codimension p.

Note that the reflection of Lemma 7.1 interchanges one x_i with y_i; so it interchanges S_m' with S_m''. Moreover, this reflection interchanges $[\mathcal{V}_0]$ with $[\mathcal{V}_1]$.

Now we can give an explicit description of the ring structure in $K_0(X)$. To do this denote $L_p=[\mathcal{L}_p]$ the class of the structural sheaf of the linear subvariety S_p of dimension p. Moreover, in case of an even $d=2m$ denote by L_m' and L_m'' the class of the structural sheaf of S_m' and S_m'' respectively.

Theorem 7.13. *Let X be a split projective quadric of dimension d. Then*

i) *in the case of $d = 2m+1$ the classes $1,\ H,\ H^2,\ \ldots,\ H^m,\ L_m,\ \ldots,\ L_0$ form a basis of the free Abelian group $K_0(X)$;*
ii) *in the case of $d = 2m$ the classes $1,\ H,\ H^2,\ \ldots,\ H^{m-1},\ L'_m, L''_m,\ L_{m-1},\ \ldots,\ L_0$ form a basis of the free Abelian group $K_0(X)$;*
iii) *in the case of $d = 2m$ the classes of (co)dimesion m may be chosen as follows:*

$$L'_m = \sum_{i=0}^{d-1}\left(\sum_{p=0}^{i}\binom{m}{p}\right)(-1)^i[\mathcal{O}_X(-i)] + [\mathcal{V}_0],$$
$$L''_m = \sum_{i=0}^{d-1}\left(\sum_{p=0}^{i}\binom{m}{p}\right)(-1)^i[\mathcal{O}_X(-i)] + [\mathcal{V}_1]$$

and for dimensions $k < m$

$$L_k = \sum_{i=0}^{d-1}\left(\sum_{p=0}^{i}\binom{d-k}{p}\right)(-1)^i[\mathcal{O}_X(-i)] + 2^{m-k-1}([\mathcal{V}_0]+[\mathcal{V}_1]);$$

iv) *if $d = 2m$, then $H^m = L'_m + L''_m - L_{m-1}$;*
v) $H \cdot L_p = L_{p-1}$, $H \cdot L'_m = H \cdot L''_m = L_{m-1}$;
vi) $H^{d-k} = 2L_k - L_{k-1}$ *for* $k \le \dfrac{d-1}{2}$, $H^d = 2L_0$, $H^{d+1} = 0$;
vii) $L_p \cdot L_q = L_p \cdot L'_m = L_p \cdot L''_m = 0$;
viii) *if $d = 2m$ and m is even, then ${L'_m}^2 = {L''_m}^2 = L_0$, $L'_m \cdot L''_m = 0$, if $d = 2m$ and m is odd, then ${L'_m}^2 = {L''_m}^2 = 0$, $L'_m \cdot L''_m = L_0$.*

Proof. Obviously H^i belongs to i-th group $F^iK_0(X)$ of topological filtration. We first check that $H^{d+1} = 0$. If $\pi : X \to \mathbb{P}^d_F$ is a two-fold cover ramified along non-singular hyperplane section, then $\pi^* : K_0\left(\mathbb{P}^d_F\right) \to K_0(X)$ is a ring homomorphism and

$$\pi^*\left(\mathcal{O}_{\mathbb{P}^d_F}(n)\right) = \mathcal{O}_X(n)$$

so $\pi^*(H) = H$ and $H^{d+1} = 0$.

Next, $L_p \in F^{d-p}K_0(X)$, $L'_m, L''_m \in F^mK_0(X)$. To verify iii), recall that the reflection ρ_{v_1} fixes $v_0,\ v_2,\ \ldots,\ v_{d+1}$ and $\rho_{v_1}(v_1) = -v_1$ (7.1.2 above). Thus, this reflection induces an automorphism of the symmetric algebra $S(V\check{\ })$, which interchanges x_0 with y_0 and fixes other coordinates and q. Therefore it induces an automorphism of $S(V\check{\ })/(q)$, $X = \operatorname{Proj} S(V\check{\ })/(q)$, a semilinear automorphism of $\mathcal{O}_X(n)$ for all n, and an automorphism of $K_0(X)$. By Lemma 7.2 ii), the reflection ρ_{v_1} interchanges the P_i's. So the induced automorphism of $\mathcal{U}$ interchanges direct summands $\mathcal{U}' = \mathcal{U} \otimes_{C_0} P_0$ and $\mathcal{U}'' = \mathcal{U} \otimes_{C_0} P_1$ of $\mathcal{U}$ and their indecomposable components $\mathcal{V}_0, \mathcal{V}_1$. Therefore, the induced automorphism of $K_0(X)$ fixes the basic elements $[\mathcal{O}_X]$, $[\mathcal{O}_X(-1)]$, $\ldots$, $[\mathcal{O}_X(1-d)]$ and interchanges $[\mathcal{V}_0]$ with $[\mathcal{V}_1]$. This automorphism fixes $L_0,\ \ldots,\ L_{m-1}$. The explicit description given in Proposition 7.12 ii) shows that this automorphism interchanges L'_m with L''_m. Hence, by the explicit formula of Proposition 7.11 ii), for $k < m$, the integer a in the following formula

$$L_k = [\mathcal{L}_k] = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (2^{m-k} - a)[\mathcal{V}_1]$$

must be equal to 2^{m-k-1}. This same argument for $k = m$ yields

$$L'_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + a[\mathcal{V}_0] + (1-a)[\mathcal{V}_1],$$

$$L''_m = \sum_{i=0}^{d-1} \left(\sum_{p=0}^{i} \binom{d-k}{p} \right) (-1)^i [\mathcal{O}_X(-i)] + (1-a)[\mathcal{V}_0] + a[\mathcal{V}_1].$$

From the K-cohomology computation, by induction on m, it follows that the classes

$$L'_m \bmod F^{m+1}K_0(X), \quad L''_m \bmod F^{m+1}K_0(X)$$

form a basis of $CH^m(X)$. Since statement ii) of the Theorem holds, the integer a must be 0 or 1 (this also follows from the regularity of the structural sheaves of S'_m and S''_m.) Statements i) and ii) follow from Proposition 7.12. Statements iv) - vii) are easy to see by explicit computations with generating functions for truncated canonical resolution, like in Proposition 7.11. To prove viii) assume, without loss of generality, that L'_m is the class of S'_m and L''_m is the class of S''_m. Consider the class L_m of the subvariety $S_m: \ y_0 = \ldots = y_m = 0$. In the case of even m the classes L''_m and L_m coincide, and S_m, S'_m have no common points, so $L'_m \cdot L''_m = 0$. Moreover, S_m meets S''_m transversally at the rational point S_0, so ${L''_m}^2 = L_0$. Analogously, ${L'_m}^2 = L_0$.
In the case of odd m we have $L'_m = L_m$, so ${L'_m}^2 = {L''_m}^2 = 0$, $L'_m \cdot L''_m = L_0$. ∎

To obtain multiplicative rule in $CH^\bullet(X)$ it is enough to neglect the summand of lower dimension in iv) and vi):

$$\begin{aligned} \text{if } d &= 2m, \text{ then } H^m = L'_m + L''_m \text{ in } CH^m(X), \\ H^{d-k} &= 2L_k \text{ for } k \leq \frac{d-1}{2} \text{ in } CH^{d-k}(X). \end{aligned}$$

Exercise 7.1. *Refine the result to the form*

$$\begin{aligned} CH^\bullet(X) &\cong \mathbb{Z}[x,y]/\left(x^{m+1} - 2y, y^2\right) \ \textit{for } d = 2m+1, \\ CH^\bullet(X) &\cong \mathbb{Z}[x,y]/\left(x^{m+1} - 2xy, y^2\right) \ \textit{for } d = 2m, \ m \textit{ odd}, \\ CH^\bullet(X) &\cong \mathbb{Z}[x,y]/\left(x^{m+1} - 2xy, y^2 - x^m y\right) \ \textit{for } d = 2m, \ m \textit{ even}. \end{aligned}$$

Remark 7.1. *Chow ring of a quadric over an algebraically closed field was computed in 1883 by Segre (see* [25]*) as a cohomology ring of a quadric.*

7.2. **Hilbert 90 for K_2 of fields.** The celebrated Merkurjev Theorem from 1981, asserting that:

$$K_2(F)/2K_2(F) \cong H^2(F, \mu_2)$$

where the latter group is $H^2(Gal(F_s/F), \mu_2) = H^2_{\acute{e}t}(\mathrm{Spec}F, \mu_2)$, and F_s is a separable closure of F, was a prototype of several Theorems proved by Merkurjev and Suslin. The main argument is "Hilbert 90 for K_2". We prove it for quadratic extensions; it remains valid for cyclic extensions of degree n provided F contains a primitive n-th root of unity.

If $E = F[\sqrt{a}]$ is a quadratic extension, we denote by σ the nontrivial automorphism of E/F. The automorphism σ acts on $K_2(E)$ by

$$\sigma\{x,y\} = \{\sigma x, \sigma y\}.$$

There is the canonical homomorphism $r_{E/F} : K_2(F) \to K_2(E)$ induced by the inverse image functor $\mathcal{P}(\mathrm{Spec}F) \to \mathcal{P}(\mathrm{Spec}E)$:

$$r_{E/F}\{b,c\} = \{b,c\}.$$

The transfer map $N_{E/F} : K_2(E) \to K_2(F)$ is induced by direct image functor $\mathcal{P}(\mathrm{Spec}E) \to \mathcal{P}(\mathrm{Spec}F)$. It may be expressed by $\sigma : K_2(E) \to K_2(E)$ as follows.

Lemma 7.14. *The identity*

$$\{b,c\} = \left\{b+c, -\frac{b}{c}\right\}$$

holds in $K_2(F)$ if $b + c \neq 0$.

Proof. The identity $\dfrac{b}{b+c} + \dfrac{c}{b+c} = 1$ implies

$$\begin{aligned}
\left\{\frac{b}{b+c}, \frac{c}{b+c}\right\} &= 0 && \text{by (3.3)}\\
\{b,c\} - \{b+c,c\} - \{b,b+c\} + \{b+c,b+c\} &= 0 && \text{by (3.1)}\\
\{b,c\} - \{b+c,c\} + \{b+c,b\} + \{b+c,-1\} &= 0 && \text{by Exercise 3.1 (4) and (3)}\\
\{b,c\} + \{b+c, -\frac{b}{c}\} &= 0 && \text{by (3.1).} \blacksquare
\end{aligned}$$

Lemma 7.15. *For a quadratic extension $E = F[\sqrt{a}]$, the map*

$$\begin{array}{rcl}
K_1(F) \otimes K_1(E) & \to & K_2(E),\\
\{b\} \otimes \{x\} & \mapsto & \{b,x\}
\end{array}$$

is surjective.

Proof. Consider $\alpha = \{r + s\sqrt{a}, t + u\sqrt{a}\} \in K_2(E)$. If $s = 0$ or $u = 0$, then α is the value of this map. The last case $\{s\sqrt{a}, u\sqrt{a}\}$ is obvious by identity $\{x,x\} = \{-1,x\}$ (Exercise 3.1 (3)). In the remaining case by (3.1)

$$\begin{aligned}
\{r + s\sqrt{a}, t + u\sqrt{a}\} &= \{ru + su\sqrt{a}, t + u\sqrt{a}\} - \{u, t + u\sqrt{a}\}\\
&= \{ru + su\sqrt{a}, -st - su\sqrt{a}\} - \{ru + su\sqrt{a}, -s\} - \{u, t + u\sqrt{a}\}\\
&= \left\{ru - st, -\frac{ru + su\sqrt{a}}{-st - su\sqrt{a}}\right\} + \{-s, ru + su\sqrt{a}\} - \{u, t + u\sqrt{a}\}. \blacksquare
\end{aligned}$$

Proposition 7.16. *For $b \in F^*$, $x \in E^*$*

$$N_{E/F}\{b,x\} = (1+\sigma)\{b,x\} \text{ and } N_{E/F}\{x,b\} = (1+\sigma)\{x,b\}.$$

Proof. It follows from the projection formula (Proposition 5.2) that:

$$\begin{aligned}
N_{E/F}\{b,x\} &= N_{E/F}\left(r_{E/F}\{b\} \cdot \{x\}\right) = \{b\} \cdot N_{E/F}\{x\}\\
&= \{b\} \cdot \{N_{E/F}x\} = \{b, (1+\sigma)x\} = (1+\sigma)\{b,x\}. \blacksquare
\end{aligned}$$

Theorem 7.17 (Hilbert 90 for quadratic extensions). *For a quadratic extension $E = F[\sqrt{a}]$ the sequence*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N_{E/F}} K_2(F)$$

is exact.

It is obvious that the sequence of the Theorem is a complex which functorially depends on F. Denote by $V_a(F)$ the homology group of this complex. To prove that $V_a(F)$ is trivial for all F, we first consider a particular case.

Proposition 7.18. *If the norm map $N_{E/F} : E^* \to F^*$ is surjective, then $V_a(F) = 0$.*

Proof. A map

$$\begin{array}{rcl} F^* \otimes F^* & \to & K_2(E)/(1-\sigma)K_2(E) \\ b \otimes c & \mapsto & \{\beta, c\} \text{ if } N_{E/F}\beta = b \end{array}$$

is well defined by classical Hilbert 90. To show that it factors through $K_2(F)$ providing the inverse for $\overline{N_{E/F}} : K_2(E)/(1-\sigma)K_2(E) \to K_2(F)$, it is enough to show that it is a symbol, i.e. that

$$\text{if } N_{E/F}\beta + c = 1 \text{ then } \{\beta, c\} \in (1-\sigma)K_2(E).$$

- If $N_{E/F}\beta = b = 1 - c$ is a square in F, $b = d^2$, then
$$N_{E/F}\frac{\beta}{d} = \frac{b}{b} = 1,$$
so by classical Hilbert 90 there exists a $\gamma \in E^{\cdot}$ such that
$$\frac{\beta}{b} = \frac{\gamma}{\sigma\gamma},$$
and
$$\{\beta, c\} = \{\beta, 1-b\} = \left\{\frac{\beta}{b}, 1-b\right\}$$
since $\{b, 1-b\} = 0$,
$$\left\{\frac{\beta}{b}, 1-b\right\} = \left\{\frac{\gamma}{\sigma\gamma}, 1-b\right\} = (1-\sigma)\{\gamma, 1-b\} \in (1-\sigma)K_2(E).$$
- If $N_{E/F}\beta = b = 1-c$ is not a square in F, then let $L = F(\sqrt{b})$, $M = E(\sqrt{b})$ and let τ be the nontrivial automorphism of M/E. We denote by σ the nontrivial automorphism of M/L, which is harmless, since its restriction to E is "the old" σ. We have
$$\{\beta, c\} = \{\beta, 1-b\} = \left\{\frac{\beta}{b}, 1-b\right\}$$
since $\{b, 1-b\} = 0$,
$$\left\{\frac{\beta}{b}, 1-b\right\} = 2\left\{\frac{\beta}{\sqrt{b}}, 1-b\right\}$$
by bimultiplicativity, and
$$2\left\{\frac{\beta}{\sqrt{b}}, 1-b\right\} = \left\{\frac{\beta}{\sqrt{b}}, 1-b\right\} + \left\{-\frac{\beta}{\sqrt{b}}, 1-b\right\} + \{-1, 1-b\}$$

by bimultiplicativity again. So

$$\begin{aligned}\left\{-\frac{\beta}{\sqrt{b}},1-b\right\} &= \tau\left\{\frac{\beta}{\sqrt{b}},1-b\right\}\\ \left\{\frac{\beta}{\sqrt{b}},1-b\right\}+\left\{-\frac{\beta}{\sqrt{b}},1-b\right\} &= N_{M/E}\left\{\frac{\beta}{\sqrt{b}},1-b\right\}\end{aligned}$$

by Proposition 7.16, and finally

$$\{\beta,c\}=\{-1,1-b\}+N_{M/E}\left\{\frac{\beta}{\sqrt{b}},1-b\right\}.$$

Now $N_{M/L}\frac{\beta}{\sqrt{b}}=1$ and there exists $\gamma\in M^*$ such that

$$\frac{\beta}{\sqrt{b}}=\frac{\gamma}{\sigma\gamma}.$$

We have

$$\{-1,1-b\}=(1-\sigma)\{\sqrt{a},1-b\}\in(1-\sigma)K_2(E),$$

and

$$\begin{aligned}N_{M/E}\left\{\frac{\beta}{\sqrt{b}},1-b\right\} &= N_{M/E}\left\{\frac{\gamma}{\sigma\gamma},1-b\right\}=N_{M/E}(1-\sigma)\left\{\gamma,1-b\right\}\\ &= (1-\sigma)N_{M/E}\left\{\gamma,1-b\right\}\in(1-\sigma)K_2(E).\end{aligned}$$

Thus there exists a homomorphism $K_2(F)\to K_2(E)/(1-\sigma)K_2(E)$ inverse to $\overline{N_{E/F}}:K_2(E)/(1-\sigma)K_2(E)\to K_2(F)$ and $V_a(F)=0$. ∎

For a generalization to arbitrary cyclic extension of a field F containing appropriate roots of unity, see [26].

Next step shows how to enlarge the image of $N_{E/F}:K_1(E)=E^*\to F^*=K_1(F)$ not affecting $V_a(F)$. Since

$$N_{E/F}\left(x+y\sqrt{a}\right)=x^2-ay^2,$$

any given $b\in F^*$ is a value of $N_{E(X)/F(X)}$ where X is a conic in $\mathbb{P}^2_F$ given by the equation $x^2-ay^2-bz^2=0$.

Proposition 7.19. *If for the projective conic $X=\mathrm{Proj}F[x,y,z]/(x^2-ay^2-bz^2)$ and $X_E=X\times_{\mathrm{Spec}F}\mathrm{Spec}E$, the induced map $H^1(X,\mathcal{K}_2)\to H^1(X_E,\mathcal{K}_2)$ is injective, then the map $V_a(F)\to V_a(F(X))$ is injective.*

Proof. Denote by $F(X)$ and $E(X)$ the function fields of conics X and X_E respectively. Since X_E has rational points, it is isomorphic to $\mathbb{P}^1_E$, which yields exactness of the sequence

$$K_2(E)=H^0(X_E,\mathcal{K}_2)\rightarrowtail K_2(E(X))\xrightarrow{\partial}\coprod_{x\in(X_E)_1}K_1(E(x))\twoheadrightarrow H^1(X_E,\mathcal{K}_2)=K_1(E).$$

A coset $\alpha+(1-\sigma)K_2(E)$ is in the kernel of $V_a(F)\to V_a(F(X))$ iff $N_{E/F}\alpha=0$ and there exists $\beta\in K_2(E(X))$ such that $r_E(\alpha)=(1-\sigma)\beta$. A little chase in the commutative diagram:

$$
\begin{tikzcd}
H^1(X,\mathcal{K}_2) \arrow[r, tail, "\rho"] & H^1(X_E,(\mathcal{K}_2) & & \\
\coprod\limits_{x\in X^1} K_1(F(x)) \arrow[u, two heads, "\kappa_F"] \arrow[r, tail, "\coprod r_x"] & \coprod\limits_{x\in {X_E}^1} K_1(E(x)) \arrow[u, two heads, "\kappa_E"] \arrow[r, tail, "1-\sigma"] & \coprod\limits_{x\in {X_E}^1} K_1(E(x)) & \\
K_2(F(X)) \arrow[u, "\partial"] \arrow[r, "r_{F(X)}"] & K_2(E(X)) \arrow[u, "\partial"] \arrow[r, "1-\sigma"] & K_2(E(X)) \arrow[u, "\partial"] \arrow[r, "N"] & K_2(F(X)) \\
 & K_2(E) \arrow[u, tail, "r_E"] \arrow[r, "1-\sigma"'] & K_2(E) \arrow[u, tail, "r_E"] \arrow[r, "N_{E/F}"'] & K_2(F) \arrow[u, "r_F"']
\end{tikzcd}
$$

- $\partial(1-\sigma)\beta = \partial r_E(\alpha) = 0$;
- $(1-\sigma)\partial\beta = \partial(1-\sigma)\beta = 0$;
- there exists $\gamma \in \coprod\limits_{x\in X^1} K_1(F(x))$ such that $\coprod r_x(\gamma) = \partial\beta$;
- $\rho\kappa_F(\gamma) = \kappa_E\left(\coprod r_x(\gamma)\right) = \kappa_E(\partial\beta) = 0$;
- $\kappa_F(\gamma) = 0$;
- there exists $\delta \in K_2(F(X))$ such that $\partial\delta = \gamma$;
- $\partial\beta = \coprod r_x(\gamma) = \coprod r_x(\partial\delta) = \partial r_X(\delta)$;
- $\partial(\beta - r_X(\delta)) = 0$;
- there exists $\epsilon \in K_2(E)$ such that $\beta - r_X(\delta) = r_E(\epsilon)$;
- $r_E(\alpha) = (1-\sigma)\beta = (1-\sigma)(\beta - r_X(\delta)) = (1-\sigma)\, r_E(\epsilon) = r_E((1-\sigma)\,\epsilon)$;
- $\alpha = (1-\sigma)\,\epsilon$

shows that the coset $\alpha + (1-\sigma)K_2(E)$ is trivial. Thus the map $V_a(F) \to V_a(F(X))$ is injective. ∎

To show that the assumption of injectivity $H^1(X,\mathcal{K}_2) \rightarrowtail H^1(X_E,\mathcal{K}_2)$ is always valid, i.e. to compute $H^1(X,\mathcal{K}_2)$, we need several simple facts on K-theory of central simple algebras. First of all $K_\bullet(X) = K_\bullet(F) \oplus K_\bullet(C_0)$ and C_0 in this case is a quaternion algebra $\left(\frac{a,b}{F}\right)$. This algebra has basis $1, i, j, k$ such that

$$i^2 = a,\ j^2 = b,\ ij = -ji = k,\ k^2 = -ab,$$

and is a division algebra iff its reduced norm form $x^2 - ay^2 - bz^2 + abt^2$ has no nontrivial zeros (x,y,z,t), which is equivalent to the condition that the form $x^2 - ay^2 - bz^2$ has no nontrivial zeros, or that the conic X has no rational points.

Remark 7.2. *This equivalence is not so straightforward. It is so since the quadratic surface $x^2 - ay^2 - bz^2 + abt^2 = 0$ in $\mathbb{P}^3_F$ is isomorphic to $X \times X$. There is also an elementary proof in quadratic form theory, using Witt Cancellation Theorem (*[24, Chapter 1, Corollary 5.8]*) and basic property of Pfister forms (*[24, Chapter 4, Corollary 1.5]*.)*

Any field extension L/F such that $\left(\frac{a,b}{F}\right) \otimes_F L = \left(\frac{a,b}{L}\right)$ splits (i.e. has zero divisors, so is a matrix algebra) is called a *splitting field* of $\left(\frac{a,b}{F}\right)$. Any maximal subfield $F(\alpha)$ of $\left(\frac{a,b}{F}\right)$, where α is a non-central element of the algebra, is its splitting field. The function field $F(X)$ is also a ("generic") splitting field of C_0.

Since modules over a division algebra D are classified up to isomophism by their dimension,

$$K_0(D) = \mathbb{Z} \cdot [D].$$

The forgetful functor $\mathcal{P}\left(\left(\frac{a,b}{F}\right)\right) \to \mathcal{P}(F)$ is exact. The induced homomorphisms are norms $N_{C_0/F} : K_\bullet(C_0) \to K_\bullet(F)$; $N_{C_0/F}(K_0(C_0))$ has index 4 in $K_0(F) = \mathbb{Z}$, and $N_{C_0/F} : K_1(C_0) \to K_1(F) = F^*$ is a polynomial map of degree 4.

If L/F is a splitting field of C_0 which is finite over F, then consider the following composite functor:

$$\begin{tikzcd} \mathcal{P}(L) \arrow[r, "\mathrm{M}orita", "equivalence"'] & \mathcal{P}(M_2(L)) \arrow[r, equal] & \mathcal{P}(C_0 \otimes_F L) \arrow[d, "forgetful"] \\ & & \mathcal{P}(C_0) \end{tikzcd}$$

which is obviously exact. Let $\theta_{L/C_0} : K_\bullet(L) \to K_\bullet(C_0)$ be the induced homomorphism. Another composition of functors for arbitrary splitting field M:

$$\begin{tikzcd} \mathcal{P}(C_0) \arrow[r, "- \otimes_F M"] & \mathcal{P}(C_0 \otimes_F M) \arrow[r, equal] & \mathcal{P}(M_2(M)) \arrow[r, "\mathrm{M}orita", "equivalence"'] & \mathcal{P}(M) \end{tikzcd}$$

induces the homomorphism $p_{M/C_0} : K_\bullet(C_0) \to K_\bullet(L)$. Moreover, there is the following commutative diagram

$$\begin{tikzcd} K_*(L) \arrow[r, "N_{L/F}"] \arrow[d, "\theta_{L/C_0}"'] & K_*(F) \arrow[d] \\ K_*(C_0) \arrow[r, "p_{M/C_0}"'] & K_*(M) \end{tikzcd} .$$

The *reduced norm* $Nrd : K_p(C_0) \to K_p(F)$ is the homomorphism for which all diagrams

$$\begin{tikzcd} K_p(L) \arrow[rr, "\theta_{L/C_0}"] \arrow[dr, "N_{L/F}"'] & & K_p(C_0) \arrow[dl, "Nrd"] \\ & K_p(F) & \end{tikzcd}$$

commute. It is known that the reduced norm exists for $p = 0, 1, 2$, and for division algebras C of square-free degree, $Nrd : K_1(C) \to K_1(F)$ is injective. In the case of quaternion algebra C, every nonzero element of $K_1(C) = C^*/[C^*, C^*]$ is in the image of $\theta_{L/C} : K_1(L) \to K_1(C)$ for some quadratic extension L of F. Thus Nrd for a quaternion algebra is a polynomial mapping of degree 2.

If L is a splitting field of C_0, then X_L has rational points, so it is isomorphic to the projective line. It is easy to check, on the level of functors, that for $h : X_L \to X$

$$h^* = \begin{pmatrix} r_{L/F} & 0 \\ 0 & p_{L/C_0} \end{pmatrix} : K_p(F) \oplus K_p(C_0) \to K_p(L) \oplus K_p(L).$$

If L/F is finite, then

$$h_* = \begin{pmatrix} N_{L/F} & 0 \\ 0 & \theta_{L/C_0} \end{pmatrix} : K_p(L) \oplus K_p(L) \to K_p(F) \oplus K_p(C_0).$$

Moreover, if $i_x : x \to X$ is a closed point, $\eta : \mathrm{Spec} F(X) \to X$ is the generic point, $f : X \to \mathrm{Spec} F$ is the structural map, then

$$\begin{aligned} i_{x*} &= \begin{pmatrix} N_{F(x)/F} \\ -\theta_{F(x)/C_0} \end{pmatrix}, \qquad i^* = \left(r_{F(x)/F}, p_{F(x)/C_0}\right) \\ \eta^* &= \left(r_{F(X)/F}, p_{F(X)/C_0}\right), \qquad f^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

The BGQ spectral sequence for the conic X has two columns, so it is the exact sequence

$$\cdots \xrightarrow{\partial} \coprod_{x \in X_1} K_i(F(x)) \xrightarrow{\sum_x i_{x*}} K_i(F) \oplus K_i(C_0) \xrightarrow{\eta^*} K_i(F(X)) \xrightarrow{\partial} \cdots.$$

Applying the authomorphism $\begin{pmatrix} 1 & Nrd \\ 0 & 1 \end{pmatrix}$ to $K_i(F) \oplus K_i(C_0)$ for $i = 0, 1, 2$, one transforms it to

$$\cdots \xrightarrow{\partial} \coprod_{x \in X_1} K_i(F(x)) \xrightarrow{\sum \begin{pmatrix} 0 \\ -\theta_{F(x)/C_0} \end{pmatrix}} K_i(F) \oplus K_i(C_0) \xrightarrow{(r_{F(X)/F}, 0)} K_i(F(X)) \xrightarrow{\partial} \cdots.$$

It follows that

$$\begin{aligned} H^0(X, \mathcal{K}'_i) &= \ker\left(K_i(F(X)) \xrightarrow{\partial} \coprod_{x \in X_1} K_{i-1}(F(x))\right) = \mathrm{im}(r_{F(X)/F}, 0) = K_i(F) \\ H^1(X, \mathcal{K}'_i) &= \mathrm{coker}\left(K_i(F(X)) \xrightarrow{\partial} \coprod_{x \in X_1} K_{i-1}(F(x))\right) \cong K_{i-1}(C_0). \end{aligned}$$

Thus $H^1(X, \mathcal{K}'_2) = K_1(C_0)$ is identified with the subgroup $Nrd(K_1(C_0))$ of $K_1(F) = F^*$ and maps injectively into $H^1(X_E, \mathcal{K}'_2) = K_1(E) = E^*$.

Now we know that $V_a(F)$ injects into $V_a(F(X))$ and standard arguments show that $V_a(F) = 0$. Namely, put $F_0 = F$, $F_{n+1} =$ compositum of $F_n(X_b)$ for all conics X_b given by $x^2 - ay^2 - bz^2 = 0$ for all $b \in F_n^*$. Thus

- $F_n^* \subset N_{F_{n+1}(\sqrt{a})/F_{n+1}}\left(F_{n+1}(\sqrt{a})^*\right)$
- $V_a(F_n) \hookrightarrow V_a(F_{n+1})$;

and therefore $V_a(F) \hookrightarrow V_a(F_{n+1})$. Next put $F_\infty = \bigcup_n F_n$; the map $N_{F_\infty(\sqrt{a})/F_\infty} : F_\infty(\sqrt{a})^* \to F_\infty^*$ is surjective and $V_a(F) \hookrightarrow V_a(F_\infty)$. But $V_a(F_\infty) = 0$ By Proposition 7.18. So $V_a(F) = 0$ and the Hilbert 90 for K_2 holds. ∎

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References

[1] M. Artin, *Brauer Groups in Ring Theory and Algebraic Geometry*, in: *Brauer groups in ring theory and algebraic geometry*, Proc. Antwerp 1981, Lecture Notes in Math., **917** (1982), 194-210.

[2] P. Berthelot, A. Grothendieck, L. Illusie, *Théorie des intersections et théorème de Riemann-Roch* (SGA 6), Lecture Notes in Math. **225**, Springer, Berlin 1971.

[3] C. Chevalley, *The algebraic theory of spinors*, NY, Columbia Univ. Press 1954.

[4] J. Dieudonné, *La géométrie des groupes classiques*, 3^{rd} ed. Springer, Berlin 1971.

[5] W. Fulton, *Intersection Theory*, MSM F. 3 B. 2, Springer, Berlin 1984.

[6] D.R. Grayson *Higher algebraic K-theory: II*, Lecture Notes in Math., **551** (1976) pp. 217-240.

[7] A. Grothendieck, *Classes de faisceaux et théorème de Riemann-Roch*, mimeographic notes, Princeton 1957 (reproduced in [2] as an appendix).

[8] A. Grothendieck, *Le groupe de Brauer I*, in: J. Giraud, A. Grothendieck, S.L. Kleiman, M. Raynaud, J. Tate, *Dix exposes sur la cohomologie des schemas*, North-Holland, Amsterdam 1968.

[9] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, Berlin 1977.

[10] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), pp. 379-417.

[11] R. Lee, R.H. Szczarba, *The group $K_3(\mathbb{Z})$ is cyclic of order 48*, Annals of Math. **104** (1976), pp. 31-60.

[12] S. Maclane, *Homology*, GMW b. 114, Springer, Berlin 1963.

[13] Y. I. Manin, *Lectures on algebraic K-functor in algebraic geometry* (in Russian), Usp. Mat. Nauk **24** No 5 (1969), pp. 3-86.

[14] A.S. Merkurjev, A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk CCCP Ser. Matem. 46 (1982) N^o 5 pp. 1011-1046 (in Russian); Math. USSR Izvestiya vol. **21** (1983) N^o 2, pp. 307-340 (English translation).

[15] J. Milnor, *Algebraic K-theory and quadratic forms*, Inv. Math. **9** (1970), pp. 318-344.

[16] J. Milnor, *Introduction to algebraic K-theory*, Princeton Univ. Press, Princeton, New Jersey 1971.

[17] D. Mumford, *Lectures on curves on algebraic surface*, Annals of Math. Studies **59**, Princeton Univ. Press, Princeton, New Jersey 1966.

[18] A. Nenashev, *Double short exact sequences produce all elements of Quillens K_1*, Contemp. Math. **199**(1996), pp. 151-160.

[19] A. Nenashev, *K_1 by generators and relations*. J. Pure Appl. Alg. **131** (1998), 195-212.

[20] I.A. Panin, *K-theory of Grassmann fibre bundles and it's twisted forms*, LOMI Preprints E-5-89, Leningrad 1989.

[21] I.A. Panin, *On algebraic K-theory of generalized flag fiber bundles and some their twisted forms*, Adv. in Soviet Math. **4** (1991), p. 21-46.

[22] D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Annals of Math. **96** (1972) No 3, pp. 552-586.

[23] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Math. **341**, Springer, Berlin 1973.

[24] W. Scharlau, *Quadratic and Hermitian forms*, GMW 270, Springer, Berlin 1985.

[25] C. Segre, *Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni*, Mem. Della Reale Academia delle Scienze di Torino (2) **36** (1883), p. 3-86

[26] A.A. Suslin, *Algebraic K-theory and norm residue homomorphism*, Results of science and technique, Contemp. problems of math. **25** (1984), pp. 115-209.

[27] R. G. Swan, *K-theory of quadric hypersurfaces*, Annals of Math. **122** (1985), 113–153.

[28] M. Szyjewski, *An invariant of quadratic forms over schemes*, Documenta Mathematica Journal DMV **1** (1996), 449-478.

[29] C. Weibel, *An Introduction to Algebraic K-theory*, `http://www.math.uiuc.edu/K-theory/0105/`.

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