GROMOV-WITTEN INVARIANTS AND QUANTUM COHOMOLOGY OF GRASSMANNIANS

HARRY TAMVAKIS

ABSTRACT. This is the written version of my five lectures at the Banach Center mini-school on 'Schubert Varieties', in Warsaw, Poland, May 18–22, 2003.

1. Lecture One

The aim of these lectures is to show that three-point genus zero Gromov-Witten invariants on Grassmannians are equal (or related) to classical triple intersection numbers on homogeneous spaces of the same Lie type, and to use this to understand the multiplicative structure of their (small) quantum cohomology rings. This theme will be explained in more detail as the lectures progress. Much of this research is part of a project with Anders S. Buch and Andrew Kresch, presented in the papers [Bu1], [KT1], [KT2], and [BKT1]. I will attempt to give the original references for each result as we discuss the theory.

1.1. **The classical theory.** We begin by reviewing the classical story for the type A Grassmannian. Let $E = \mathbb{C}^N$ and $X = G(m, E) = G(m, N)$ be the Grassmannian of m-dimensional complex linear subspaces of E. One knows that X is a smooth projective algebraic variety of complex dimension mn, where $n = N - m$.

The space X is stratified by Schubert cells; the closures of these cells are the *Schubert varieties* $X_\lambda(F_\bullet)$, where λ is a partition and

$$F_\bullet \ : \ 0 = F_0 \subset F_1 \subset \cdots \subset F_N = E$$

is a complete flag of subspaces of E, with $\dim F_i = i$ for each i. The partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0)$ is a decreasing sequence of nonnegative integers such that $\lambda_1 \leqslant n$. This means that the Young diagram of λ fits inside an $m \times n$ rectangle, which is the diagram of (n^m). We denote this containment relation of diagrams by $\lambda \subset (n^m)$. The diagram shown in Figure 1 corresponds to a Schubert variety in $G(4, 10)$.

The precise definition of $X_\lambda(F_\bullet)$ is

$$X_\lambda(F_\bullet) = \{\, V \in X \mid \dim(V \cap F_{n+i-\lambda_i}) \geqslant i, \ \forall\ 1 \leqslant i \leqslant m \,\}. \tag{1}$$

Each $X_\lambda(F_\bullet)$ is a closed subvariety of X of codimension equal to the weight $|\lambda| = \sum \lambda_i$ of λ. Using the Poincaré duality isomorphism between homology and cohomology, $X_\lambda(F_\bullet)$ defines a *Schubert class* $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $H^{2|\lambda|}(X, \mathbb{Z})$. The algebraic group $GL_N(\mathbb{C})$ acts transitively on X and on the flags in E. The action of an element $g \in GL_N(\mathbb{C})$ on the variety $X_\lambda(F_\bullet)$ is given by $g \cdot X_\lambda(F_\bullet) = X_\lambda(g \cdot F_\bullet)$.

Date: June 27, 2003

2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 14M15, 14N15, 05E15.

Key words and phrases. Gromov-Witten invariants, Grassmannians, Flag varieties, Schubert varieties, Quantum cohomology, Littlewood-Richardson rule.

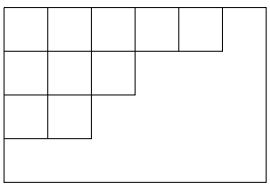


FIGURE 1. The rectangle (6^4) containing $\lambda = (5,3,2)$

It follows that σ_λ does not depend on the choice of flag $F_\bullet$ used to define X_λ. As all cohomology classes in these lectures will occur in even degrees, we will adopt the convention that the degree of a class $\alpha \in H^{2k}(X,\mathbb{Z})$ is equal to k.

We next review the classical facts about the cohomology of $X = G(m,N)$.

1) The additive structure of $H^*(X,\mathbb{Z})$ is given by

$$H^*(X,\mathbb{Z}) = \bigoplus_{\lambda \subset (n^m)} \mathbb{Z} \cdot \sigma_\lambda,$$

that is, $H^*(X,\mathbb{Z})$ is a free abelian group with basis given by the Schubert classes.

2) To describe the cup product in $H^*(X,\mathbb{Z})$, we will use Schubert's Duality Theorem. This states that for any λ and μ with $|\lambda|+|\mu| = mn$, we have $\sigma_\lambda \sigma_\mu = \delta_{\widehat{\lambda}\mu} \cdot [\mathrm{pt}]$, where $[\mathrm{pt}] = \sigma_{(n^m)}$ is the class of a point, and $\widehat{\lambda}$ is the dual partition to λ. The diagram of $\widehat{\lambda}$ is the complement of λ in the rectangle (n^m), rotated by 180°. This is illustrated in Figure 2.

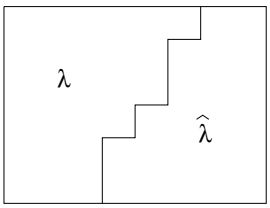


FIGURE 2. Dual Young diagrams

For general products, we have an equation

$$\sigma_\lambda \, \sigma_\mu = \sum_{|\nu|=|\lambda|+|\mu|} c^\nu_{\lambda\mu} \, \sigma_\nu$$

in $H^*(X,\mathbb{Z})$, and the *structure constants* $c^\nu_{\lambda\mu}$ are given by

$$c^\nu_{\lambda\mu} = \int_X \sigma_\lambda \sigma_\mu \sigma_{\widehat{\nu}} = \langle \sigma_\lambda, \sigma_\mu, \sigma_{\widehat{\nu}} \rangle_0 = \#X_\lambda(F_\bullet) \cap X_\mu(G_\bullet) \cap X_{\widehat{\nu}}(H_\bullet),$$

for general full flags $F_\bullet$, $G_\bullet$ and $H_\bullet$ in E. Later, we will discuss a combinatorial formula for these structure constants.

3) The classes $\sigma_1, \ldots, \sigma_n$ are called *special Schubert classes*. Observe that there is a unique Schubert class in codimension one: $H^2(X,\mathbb{Z}) = \mathbb{Z}\,\sigma_1$. If

$$0 \to S \to E_X \to Q \to 0 \tag{2}$$

is the tautological short exact sequence of vector bundles over X, with $E_X = X \times E$, then one can show that σ_i is equal to the ith Chern class $c_i(Q)$ of the quotient bundle Q, for $0 \leqslant i \leqslant n$.

Theorem 1 (Pieri rule, [Pi]). *For $1 \leqslant p \leqslant n$ we have $\sigma_\lambda \, \sigma_p = \sum \sigma_\mu$, where the sum is over all $\mu \subset (n^m)$ obtained from λ by adding p boxes, with no two in the same column.*

Example 1. Suppose $m = n = 2$ and we consider the Grassmannian $X = G(2,4)$ of 2-planes through the origin in $E = \mathbb{C}^4$. Note that X may be identified with the Grassmannian of all lines in projective 3-space $P(E) \cong \mathbb{P}^3$. The list of Schubert classes for X is

$$\sigma_0 = 1, \quad \sigma_1, \quad \sigma_2, \quad \sigma_{1,1}, \quad \sigma_{2,1}, \quad \sigma_{2,2} = [\mathrm{pt}].$$

Observe that the indices of these classes are exactly the six partitions whose diagrams fit inside a 2×2 rectangle. Using the Pieri rule, we compute that

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1}, \ \sigma_1^3 = 2\,\sigma_{2,1}, \ \sigma_1^4 = 2\,\sigma_{2,2} = 2\,[\mathrm{pt}].$$

The last relation means that there are exactly 2 points in the intersection

$$X_1(F_\bullet) \cap X_1(G_\bullet) \cap X_1(H_\bullet) \cap X_1(I_\bullet),$$

for general flags $F_\bullet$, $G_\bullet$, $H_\bullet$, and $I_\bullet$. Since e.g. $X_1(F_\bullet)$ may be identified with the locus of lines in $P(E)$ meeting the fixed line $P(F_2)$, this proves the enumerative fact that there are two lines in $\mathbb{P}^3$ which meet four given lines in general position.

4) Any Schubert class σ_λ may be expressed as a polynomial in the special classes in the following way. Let us agree here and in the sequel that $\sigma_p = 0$ if $p < 0$ or $p > n$.

Theorem 2 (Giambelli formula, [G]). *We have $\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leqslant i,j \leqslant m}$, that is, σ_λ is equal to a Schur determinant in the special classes.*

5) The ring $H^*(X, \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ by the relations

$$D_{m+1} = \cdots = D_N = 0$$

where $D_k = \det(\sigma_{1+j-i})_{1 \leqslant i,j \leqslant k}$. To understand where these relations come from, note that the Whitney sum formula applied to (2) says that $c_t(S) c_t(Q) = 1$, which implies, since $\sigma_i = c_i(Q)$, that $D_k = (-1)^k c_k(S) = c_k(S^*)$. In particular, we see that D_k vanishes for $k > m$, since S^* is a vector bundle of rank m.

1.2. **Gromov-Witten invariants.** Our starting point is the aforementioned fact that the classical structure constant $c_{\lambda\mu}^\nu$ in the cohomology of $X = G(m, N)$ can be realized as a triple intersection number $\# X_\lambda(F_\bullet) \cap X_\mu(G_\bullet) \cap X_{\widehat{\nu}}(H_\bullet)$ on X. The three-point, genus zero Gromov-Witten invariants on X extend these numbers to more general enumerative constants, which are furthermore used to define the 'small quantum cohomology ring' of X.

A *rational map of degree d* to X is a morphism $f \colon \mathbb{P}^1 \to X$ such that

$$\int_X f_*[\mathbb{P}^1] \cdot \sigma_1 = d,$$

i.e. d is the number of points in $f^{-1}(X_1(F_\bullet))$ when $F_\bullet$ is in general position.

Definition 1. Given a degree $d \geqslant 0$ and partitions λ, μ, and ν such that $|\lambda| + |\mu| + |\nu| = mn + dN$, we define the Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ to be the number of rational maps $f\colon \mathbb{P}^1 \to X$ of degree d such that $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$, and $f(\infty) \in X_\nu(H_\bullet)$, for given flags $F_\bullet$, $G_\bullet$, and $H_\bullet$ in general position.

We shall show later that $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ is a well-defined, finite integer. Notice that for the degree zero invariants, we have

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_0 = \int_X \sigma_\lambda \sigma_\mu \sigma_\nu = \#X_\lambda(F_\bullet) \cap X_\mu(G_\bullet) \cap X_\nu(H_\bullet),$$

as a morphism of degree zero is just a constant map to X.

Key example. Consider the Grassmannian $G(d, 2d)$ for any $d \geqslant 0$. We say that two points U, V of $G(d, 2d)$ are in general position if the intersection $U \cap V$ of the corresponding subspaces is the zero subspace.

Proposition 1 ([BKT1])**.** *Let U, V, and W be three points of $\mathcal{Z} = G(d, 2d)$ which are pairwise in general position. Then there is a unique morphism $f\colon \mathbb{P}^1 \to \mathcal{Z}$ of degree d such that $f(0) = U$, $f(1) = V$, and $f(\infty) = W$. In particular, the Gromov-Witten invariant which counts degree d maps to $\mathcal{Z}$ through three general points is equal to* 1.

Proof. Let U, V, and W be given, in pairwise general position. Choose a basis $(v_1, \ldots, v_d)$ of V. Then we can construct a morphism $f\colon \mathbb{P}^1 \to \mathcal{Z}$ of degree d such that $f(0) = U$, $f(1) = V$, and $f(\infty) = W$ as follows. For each i with $1 \leqslant i \leqslant d$, we let u_i and w_i be the projections of v_i onto U and W, respectively. If $(s{:}t)$ are the homogeneous coordinates on $\mathbb{P}^1$, then the morphism

$$f(s{:}t) = \mathrm{Span}\{su_1 + tw_1, \ldots, su_d + tw_d\}$$

satisfies the required conditions. Observe that f does not depend on the chosen basis for V. Indeed, if $v_i' = \sum a_{ij} v_j$, then $u_i' = \sum a_{ij} u_j$, $w_i' = \sum a_{ij} w_j$ and one checks easily that

$$\mathrm{Span}\{su_1 + tw_1, \ldots, su_d + tw_d\} = \mathrm{Span}\{su_1' + tw_1', \ldots, su_d' + tw_d'\}.$$

Exercise. Show that the map f is an embedding of $\mathbb{P}^1$ into $\mathcal{Z}$ such that $f(p_1)$ and $f(p_2)$ are in general position, for all points p_1, p_2 in $\mathbb{P}^1$ with $p_1 \neq p_2$. Show also that f has degree d.

Next, suppose that $f\colon \mathbb{P}^1 \to \mathcal{Z}$ is any morphism of degree d which sends 0, 1, ∞ to U, V, W, respectively. Let $S \subset \mathbb{C}^{2d} \otimes \mathcal{O}_\mathcal{Z}$ be the tautological rank d vector bundle over $\mathcal{Z}$, and consider the pullback $f^*S \to \mathbb{P}^1$. The morphism $f\colon \mathbb{P}^1 \to G(d, 2d)$ is determined by the inclusion of f^*S in $\mathbb{C}^{2d} \otimes \mathcal{O}_{\mathbb{P}^1}$, i.e., a point $p \in \mathbb{P}^1$ is mapped by f to the fiber over p of the image of this inclusion.

Every vector bundle over $\mathbb{P}^1$ splits as a direct sum of line bundles, so $f^*S \cong \oplus_{i=1}^d \mathcal{O}(a_i)$. Each $\mathcal{O}(a_i)$ is a subbundle of a trivial bundle, hence $a_i \leqslant 0$, and $\sum a_i = -d$ as f has degree d. We deduce that $a_i = -1$ for each i, since otherwise f^*S would have a trivial summand, and this contradicts the general position hypothesis. It follows that we can write $f(s{:}t) = \mathrm{Span}\{su_1 + tw_1, \ldots, su_d + tw_d\}$ for suitable vectors $u_i, w_i \in \mathbb{C}^{2d}$, which depend on the chosen identification of f^*S with $\oplus_{i=1}^d \mathcal{O}(-1)$. We conclude that f is the map constructed as above from the basis $(v_1, \ldots, v_d)$, where $v_i = u_i + w_i$. $\square$

We now introduce the key definition upon which the subsequent analysis depends.

Definition 2 ([Bu1])**.** For any morphism $f\colon \mathbb{P}^1 \to G(m,N)$, define the *kernel* of f to be the intersection of all the subspaces $V \subset E$ corresponding to image points of f. Similarly, the *span* of f is the linear span of these subspaces.

$$\mathrm{Ker}(f) = \bigcap_{p\in\mathbb{P}^1} f(p); \qquad \mathrm{Span}(f) = \sum_{p\in\mathbb{P}^1} f(p).$$

Note that for each $f\colon \mathbb{P}^1 \to X$, we have $\mathrm{Ker}(f) \subset \mathrm{Span}(f) \subset E$.

Lemma 1 ([Bu1])**.** *If $f\colon \mathbb{P}^1 \to G(m,N)$ is a morphism of degree d, then* $\dim \mathrm{Ker}(f) \geqslant m-d$ *and* $\dim \mathrm{Span}(f) \leqslant m+d$.

Proof. Let $S \to X$ be the rank m tautological bundle over $X = G(m,N)$. Given any morphism $f\colon \mathbb{P}^1 \to X$ of degree d, we have that $f^*S \cong \oplus_{i=1}^m \mathcal{O}(a_i)$, where $a_i \leqslant 0$ and $\sum a_i = -d$. Moreover, the map f is induced by the inclusion $f^*S \subset E \otimes \mathcal{O}_{\mathbb{P}^1}$. There are at least $m-d$ zeroes among the integers a_i, hence f^*S contains a trivial summand of rank at least $m-d$. But this corresponds to a fixed subspace of E of the same dimension which is contained in $\mathrm{Ker}(f)$, hence $\dim \mathrm{Ker}(f) \geqslant m-d$.

Similar reasoning shows that if $Q \to X$ is the rank n universal quotient bundle over X, then f^*Q has a trivial summand of rank at least $n-d$. It follows that the image of the map $f^*S \to E \otimes \mathcal{O}_{\mathbb{P}^1}$ factors through a subspace of E of codimension at least $n-d$, and hence of dimension at most $m+d$. □

In the next lecture, we will see that for those maps f which are counted by a degree d Gromov-Witten invariant for X, we have $\dim \mathrm{Ker}(f) = m-d$ and $\dim \mathrm{Span}(f) = m+d$. In fact, it will turn out that the pair $(\mathrm{Ker}(f), \mathrm{Span}(f))$ determines f completely!

2. Lecture Two

2.1. **The main theorem.** Given integers a and b, we let $F(a,b;E) = F(a,b;N)$ denote the two-step flag variety parametrizing pairs of subspaces (A,B) with $A \subset B \subset E$, $\dim A = a$ and $\dim B = b$. We agree that $F(a,b;N)$ is empty unless $0 \leqslant a \leqslant b \leqslant N$; when the latter condition holds then $F(a,b;N)$ is a projective complex manifold of dimension $(N-b)b + (b-a)a$. For any non-negative integer d we set $Y_d = F(m-d, m+d; E)$; this will be the parameter space of the pairs $(\mathrm{Ker}(f), \mathrm{Span}(f))$ for the relevant morphisms $f\colon \mathbb{P}^1 \to X$. Our main theorem will be used to identify Gromov-Witten invariants on $X = G(m,E)$ with classical triple intersection numbers on the flag varieties Y_d.

To any subvariety $\mathcal{W} \subset X$ we associate the subvariety $W^{(d)}$ in Y_d defined by

$$\mathcal{W}^{(d)} = \{\, (A,B) \in Y_d \mid \exists\, V \in \mathcal{W} : A \subset V \subset B \,\}. \tag{3}$$

Let $F(m-d, m, m+d; E)$ denote the variety of three-step flags in E of dimensions $m-d$, m, and $m+d$. There are natural projection maps

$$\pi_1\colon F(m-d,m,m+d;E) \to X \quad \text{and} \quad \pi_2\colon F(m-d,m,m+d;E) \to Y_d.$$

We then have $\mathcal{W}^{(d)} = \pi_2(\pi_1^{-1}(\mathcal{W}))$. Moreover, as the maps π_i are GL_N-equivariant, if $\mathcal{W} = X_\lambda(F_\bullet)$ is a Schubert variety in X, then $\mathcal{W}^{(d)} = X_\lambda^{(d)}(F_\bullet)$ is a Schubert variety in Y_d. We will describe this Schubert variety in more detail after we prove the main theorem.

Remarks. 1) One computes that $\dim Y_d = mn + dN - 3d^2$.

2) Since the fibers of π_2 are isomorphic to $G(d,2d)$, the codimension of $X_\lambda^{(d)}(F_\bullet)$ in Y_d is at least $|\lambda| - d^2$.

Theorem 3 ([BKT1])**.** *Let λ, μ, and ν be partitions and d be an integer such that $|\lambda| + |\mu| + |\nu| = mn + dN$, and let $F_\bullet$, $G_\bullet$, and $H_\bullet$ be complete flags of $E = \mathbb{C}^N$ in general position. Then the map $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ gives a bijection of the set of rational maps $f\colon \mathbb{P}^1 \to G(m,N)$ of degree d satisfying $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$, and $f(\infty) \in X_\nu(H_\bullet)$, with the set of points in the intersection $X_\lambda^{(d)}(F_\bullet) \cap X_\mu^{(d)}(G_\bullet) \cap X_\nu^{(d)}(H_\bullet)$ in $Y_d = F(m-d, m+d; N)$.*

It follows from Theorem 3 that we can express any Gromov-Witten invariant of degree d on $G(m,N)$ as a classical intersection number on Y_d. Let $[X_\lambda^{(d)}]$ denote the cohomology class of $X_\lambda^{(d)}(F_\bullet)$ in $H^*(Y_d, \mathbb{Z})$.

Corollary 1. *Let λ, μ, and ν be partitions and $d \geqslant 0$ an integer such that $|\lambda| + |\mu| + |\nu| = mn + dN$. We then have*

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{F(m-d,m+d;N)} [X_\lambda^{(d)}] \cdot [X_\mu^{(d)}] \cdot [X_\nu^{(d)}].$$

Proof of Theorem 3. Let $f\colon \mathbb{P}^1 \to X$ be a rational map as in the statement of the theorem.

Claim 1. We have $d \leqslant \min(m,n)$, $\dim \mathrm{Ker}(f) = m-d$ and $\dim \mathrm{Span}(f) = m+d$.

Indeed, let $a = \dim \mathrm{Ker}(f)$ and $b = \dim \mathrm{Span}(f)$. In the two-step flag variety $Y' = F(a,b;E)$ there are associated Schubert varieties $X'_\lambda(F_\bullet)$, $X'_\mu(G_\bullet)$, and $X'_\nu(H_\bullet)$, defined as in (3). Writing $e_1 = m-a$ and $e_2 = b-m$, we see that the codimension of $X'_\lambda(F_\bullet)$ in Y' is at least $|\lambda| - e_1 e_2$, and similar inequalities hold with

μ and ν in place of λ. Since $(\mathrm{Ker}(f), \mathrm{Span}(f)) \in X'_\lambda(F_\bullet) \cap X'_\mu(G_\bullet) \cap X'_\nu(H_\bullet)$ and the three flags $F_\bullet$, $G_\bullet$ and $H_\bullet$ are in general position, we obtain

$$mn + dN - 3e_1e_2 \leqslant \dim F(a, b; E) = (N-b)(m+e_2) + (e_1+e_2)a,$$

and hence, by a short computation,

$$dN \leqslant 2e_1e_2 + e_2(N-b) + ae_1. \tag{4}$$

Lemma 1 says that $e_1 \leqslant d$ and $e_2 \leqslant d$, and therefore that the right-hand side of (4) is at most $2e_1e_2 + d(N-b+a)$. Since $b-a = e_1+e_2$, it follows that

$$(e_1+e_2)^2 \leqslant 2d(e_1+e_2) \leqslant 4e_1e_2,$$

and hence $e_1 = e_2 = d$. This proves Claim 1.

Let $\mathcal{M}$ denote the set of rational maps in the statement of the theorem, and set $\mathcal{I} = X^{(d)}_\lambda(F_\bullet) \cap X^{(d)}_\mu(G_\bullet) \cap X^{(d)}_\nu(H_\bullet)$. If $f \in \mathcal{M}$ then Claim 1 shows that $(\mathrm{Ker}(f), \mathrm{Span}(f)) \in \mathcal{I}$. We next describe the inverse of the resulting map $\mathcal{M} \to \mathcal{I}$.

Given $(A, B) \in \mathcal{I}$, we let $\mathcal{Z} = G(d, B/A) \subset X$ be the set of m-dimensional subspaces of E between A and B. Observe that $\mathcal{Z} \cong G(d, 2d)$, and that $X_\lambda(F_\bullet) \cap \mathcal{Z}$, $X_\mu(G_\bullet) \cap \mathcal{Z}$, and $X_\nu(H_\bullet) \cap \mathcal{Z}$ are non-empty Schubert varieties in $\mathcal{Z}$. (Indeed, e.g. $X_\lambda(F_\bullet) \cap \mathcal{Z}$ is defined by the attitude of V/A with respect to the flag $\widetilde{F}_\bullet$ in B/A with $\widetilde{F}_i = ((F_i + A) \cap B)/A$ for each i.) We assert that each of $X_\lambda(F_\bullet) \cap \mathcal{Z}$, $X_\mu(G_\bullet) \cap \mathcal{Z}$, and $X_\nu(H_\bullet) \cap \mathcal{Z}$ must be a single point, and that these three points are subspaces of B/A in pairwise general position. Proposition 1 then provides the unique $f\colon \mathbb{P}^1 \to X$ in $\mathcal{M}$ with $\mathrm{Ker}(f) = A$ and $\mathrm{Span}(f) = B$.

Claim 2. Let U, V, and W be three points in $\mathcal{Z}$, one from each intersection. Then the subspaces U, V and W are in pairwise general position.

Assuming this claim, we can finish the proof as follows. Observe that any positive dimensional Schubert variety in $\mathcal{Z}$ must contain a point U' which meets U non-trivially, and similarly for V and W. Indeed, on $G(d, 2d)$, the locus of d-dimensional subspaces Σ with $\Sigma \cap U \neq \{0\}$ is, up to a general translate, the unique Schubert variety in codimension 1. It follows that this locus must meet any other Schubert variety non-trivially, unless the latter is zero dimensional, in other words, a point. Therefore Claim 2 implies that $X_\lambda(F_\bullet) \cap \mathcal{Z}$, $X_\mu(G_\bullet) \cap \mathcal{Z}$, and $X_\nu(H_\bullet) \cap \mathcal{Z}$ are three points on $\mathcal{Z}$ in pairwise general position.

To prove Claim 2, we again use a dimension counting argument to show that if the three reference flags are chosen generically, no two subspaces among U, V, W can have non-trivial intersection. Consider the three-step flag variety $Y'' = F(m-d, m-d+1, m+d; E)$ and the projection $\pi\colon Y'' \to Y_d$. Note that $\dim Y'' = \dim Y_d + 2d - 1$, as Y'' is a $\mathbb{P}^{2d-1}$-bundle over Y_d.

To each subvariety $\mathcal{W} \subset G(m, E)$ we associate $\mathcal{W}'' \subset Y''$ defined by

$$\mathcal{W}'' = \{\, (A, A', B) \in Y'' \mid \exists\, V \in \mathcal{W} : A' \subset V \subset B \,\}.$$

We find that the codimension of $X''_\mu(G_\bullet)$ in Y'' is at least $|\mu| - d^2 + d$, and similarly for $X''_\nu(H_\bullet)$. Since the three flags are in general position, and $\pi^{-1}(X^{(d)}_\lambda(F_\bullet))$ has codimension at least $|\lambda| - d^2$ in Y'', we must have

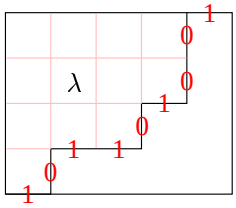
$$\pi^{-1}(X^{(d)}_\lambda(F_\bullet)) \cap X''_\mu(G_\bullet) \cap X''_\nu(H_\bullet) = \emptyset,$$

and the same is true for the other two analogous triple intersections. This completes the proof of Claim 2, and of the theorem. □

It is worth pointing out that we may rephrase Theorem 3 using rational curves in X, instead of rational maps to X. For this, recall from the Exercise given in the first lecture that every rational map f that is counted in Theorem 3 is an embedding of $\mathbb{P}^1$ into X of degree equal to the degree of the curve $\mathrm{Im}(f)$. Moreover, the bijection of the theorem shows that all of these maps have different images.

2.2. **Parametrizations of Schubert varieties.** We now describe an alternative way to parametrize the Schubert varieties on $G(m,N)$, by replacing each partition λ by a 01-string $I(\lambda)$ of length N, with m zeroes. Begin by drawing the Young diagram of the partition λ in the upper-left corner of an $m \times n$ rectangle. We then put a label on each step of the path from the lower-left to the upper-right corner of this rectangle which follows the border of λ. Each vertical step is labeled "0", while the remaining n horizontal steps are labeled "1". The string $I(\lambda)$ then consists of these labels in lower-left to upper-right order.

Example 2. On the Grassmannian $G(4,9)$, the 01-string of the partition $\lambda = (4,4,3,1)$ is $I(\lambda) = 101101001$. This is illustrated below.



Alternatively, each partition $\lambda \subset (n^m)$ corresponds to a Grassmannian permutation w_λ in the symmetric group S_N, which is a minimal length representative in the coset space $S_N/(S_m \times S_n)$. The element $w = w_\lambda$ is such that the positions of the "0"s (respectively, the "1"s) in the 01-string $I(\lambda)$ are given by $w(1), \ldots, w(m)$ (respectively, by $w(m+1), \ldots, w(N)$). In Example 2, we have $w_\lambda = 257813469 \in S_9$.

In a similar fashion, the Schubert varieties on the two-step flag variety $F(a,b;N)$ are parametrized by permutations $w \in S_N$ with $w(i) < w(i+1)$ for $i \notin \{a,b\}$. For each such permutation w and fixed full flag $F_\bullet$ in E, the Schubert variety $X_w(F_\bullet) \subset F(a,b;N)$ is defined as the locus of flags $A \subset B \subset E$ such that

$$\dim(A \cap F_i) \geqslant \#\{p \leqslant a \mid w(p) > N-i\} \quad \text{and} \quad \dim(B \cap F_i) \geqslant \#\{p \leqslant b \mid w(p) > N-i\}$$

for each i. The codimension of $X_w(F_\bullet)$ in $F(a,b;N)$ is equal to the *length* $\ell(w)$ of the permutation w. Furthermore, these indexing permutations w correspond to 012-strings $J(w)$ of length N with a "0"s and $b-a$ "1"s. The positions of the "0"s (respectively, the "1"s) in $J(w)$ are recorded by $w(1), \ldots, w(a)$ (respectively, by $w(a+1), \ldots, w(b)$).

Finally, we describe the 012-string $J^d(\lambda)$ associated to the modified Schubert variety $X_\lambda^{(d)}(F_\bullet)$ in $Y_d = F(m-d, m+d; N)$. This string is obtained by first multiplying each number in the 01-string $I(\lambda)$ by 2, to get a 02-string $2I(\lambda)$. We then get the 012-string $J^d(\lambda)$ by changing the first d "2"s and the last d "0"s of $2I(\lambda)$ to "1"s. Taking $d = 2$ in Example 2, we get $J^2(4,4,3,1) = 101202112$, which corresponds to a Schubert variety in $F(2,6;9)$.

Corollary 2 ([Y]). *Let λ, μ, and ν be partitions and $d \geqslant 0$ be such that $|\lambda| + |\mu| + |\nu| = mn + dN$. If any of λ_d, μ_d, and ν_d is less than d, then $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = 0$.*

Proof. By computing the length of the permutation corresponding to $J^d(\lambda)$, one checks easily that when $\lambda_d < d$, the codimension of $X_\lambda^{(d)}(F_\bullet)$ in Y_d is strictly greater than $|\lambda| - d^2$. Therefore, when any of λ_d, μ_d, or ν_d is less than d, the sum of the codimensions of the three Schubert varieties $X_\lambda^{(d)}(F_\bullet)$, $X_\mu^{(d)}(G_\bullet)$, and $X_\nu^{(d)}(H_\bullet)$ which appear in the statement of Theorem 3 is strictly greater than the dimension of $Y_d = F(m-d, m+d; N)$. We deduce that

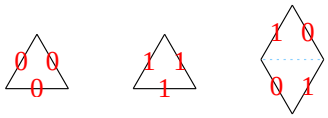
$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{Y_d} [X_\lambda^{(d)}] \cdot [X_\mu^{(d)}] \cdot [X_\nu^{(d)}] = 0.$$

□

3. Lecture Three

3.1. **Classical and quantum Littlewood-Richardson rules.** The problem we turn to now is that of finding a positive combinatorial formula for the Gromov-Witten invariants $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$. For the classical structure constants (the case $d = 0$), this problem was solved in the 1930's by Littlewood and Richardson, although complete proofs only appeared in the early 1970's. In the past few years, there has been a resurgence of interest in this question (see, for example, [F2]), which has led to a new formulation of the rule in terms of 'puzzles' (due to Knutson, Tao, and Woodward).

Define a *puzzle* to be a triangle decomposed into *puzzle pieces* of the three types displayed below.



A puzzle piece may be rotated but not reflected when used in a puzzle. Furthermore, the common edges of two puzzle pieces next to each other must have the same labels. Recall from the last lecture that a Schubert class σ_λ in $H^*(G(m,N),\mathbb{Z})$ may also be indexed by a 01-string $I(\lambda)$ with m "0"s and n "1"s.

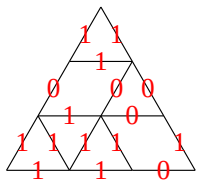
Theorem 4 ([KTW])**.** *For any three Schubert classes σ_λ, σ_μ, and σ_ν in the cohomology of $X = G(m,N)$, the integral $\int_X \sigma_\lambda \sigma_\mu \sigma_\nu$ is equal to the number of puzzles such that $I(\lambda)$, $I(\mu)$, and $I(\nu)$ are the labels on the north-west, north-east, and south sides when read in clockwise order.*

The formula in Theorem 4 is bijectively equivalent to the classical Littlewood-Richardson rule, which describes the same numbers as the cardinality of a certain set of Young tableaux (see [V, §4.1]).

Example 3. In the projective plane $\mathbb{P}^2 = G(1,3)$, two general lines intersect in a single point. This corresponds to the structure constant

$$\langle \sigma_1, \sigma_1, \sigma_0 \rangle_0 = \int_{\mathbb{P}^2} \sigma_1^2 = 1.$$

The figure below displays the unique puzzle with the corresponding three strings 101, 101, and 011 on its north-west, north-east, and south sides.



We suggest that the reader works out the puzzle which corresponds to the intersection $\sigma_k \sigma_\ell = \sigma_{k+\ell}$ on $\mathbb{P}^n$, for $k + \ell \leqslant n$.

It is certainly tempting to try to generalize Theorem 4 to a result that would hold for the flag variety SL_N/B. This time the three sides of the puzzle would be labeled by permutations, and one has to specify the correct set of puzzle pieces to make the rule work. In the fall of 1999, Knutson proposed such a general

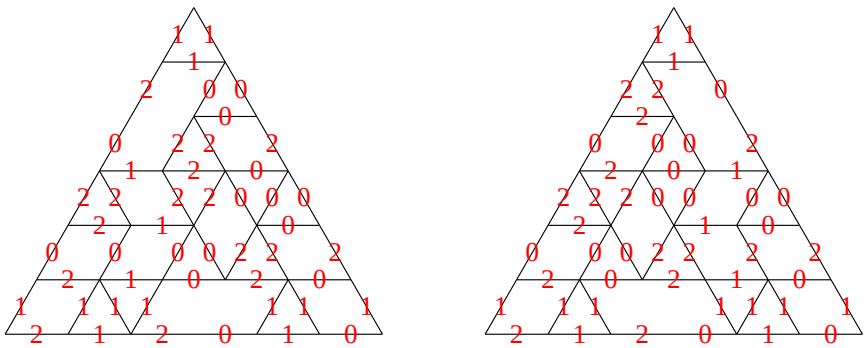
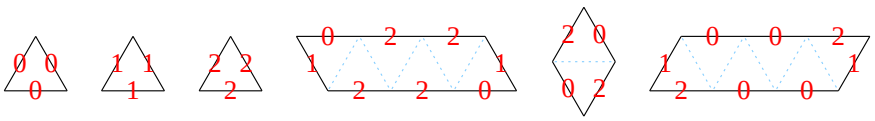


FIGURE 3. Two puzzles with the same boundary labels

conjecture for the Schubert structure constants on all partial flag varieties, which specialized to Theorem 4 in the Grassmannian case. However, he soon discovered counterexamples to this conjecture (in fact, it fails for the three-step flag variety $F(1,3,4;5)$).

Motivated by Theorem 3, Buch, Kresch and the author were especially interested in a combinatorial rule for the structure constants on two-step flag varieties. Surprisingly, there is extensive computer evidence which suggests that Knutson's conjecture is true in this special case. Recall from the last lecture that the Schubert classes on two-step flag varieties are indexed by the 012-strings $J(w)$, for permutations $w \in S_N$. In this setting we have the following six different types of puzzle pieces.



The length of the fourth and of the sixth piece above may vary. The fourth piece can have any number of "2"s (including none) to the right of the "0" on the top edge and equally many to the left of the "0" on the bottom edge. Similarly the sixth piece can have an arbitrary number of "0"s on the top and bottom edges. Again each puzzle piece may be rotated but not reflected. Figure 3 shows two examples of such puzzles.

We can now state Knutson's conjecture in the case of two-step flag varieties. This conjecture has been verified by computer for all two-step flag varieties $F(a,b;N)$ for which $N \leqslant 16$.

Conjecture 1 (Knutson)**.** *For any three Schubert varieties X_u, X_v, and X_w in the flag variety $F(a,b;N)$, the integral $\int_{F(a,b;N)} [X_u] \cdot [X_v] \cdot [X_w]$ is equal to the number of puzzles such that $J(u)$, $J(v)$, and $J(w)$ are the labels on the north-west, north-east, and south sides when read in clockwise order.*

By combining Theorem 3 with Conjecture 1, we arrive at a conjectural 'quantum Littlewood-Richardson rule' for the Gromov-Witten invariants $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$. This

time we use the 012-string $J^d(\lambda)$ associated to the Schubert variety $X_\lambda^{(d)}(F_\bullet)$ in $F(m-d, m+d; N)$.

Conjecture 2 ([BKT1]). *For partitions λ, μ, ν such that $|\lambda|+|\mu|+|\nu| = mn+dN$ the Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ is equal to the number of puzzles such that $J^d(\lambda)$, $J^d(\mu)$, and $J^d(\nu)$ are the labels on the north-west, north-east, and south sides when read in clockwise order.*

The verified cases of Conjecture 1 imply that Conjecture 2 holds for all Grassmannians $G(m,N)$ for which $N \leqslant 16$. It has also been proved in some special cases including when λ has length at most 2 or when m is at most 3.

Example 4. On the Grassmannian $G(3,6)$, the Gromov-Witten invariant

$$\langle \sigma_{3,2,1},\, \sigma_{3,2,1},\, \sigma_{2,1} \rangle_1$$

is equal to 2. We have $J^1(3,2,1) = 102021$ and $J^1(2,1) = 010212$. Figure 3 displays the two puzzles with the labels $J^1(3,2,1)$, $J^1(3,2,1)$, and $J^1(2,1)$ on their sides.

3.2. **Quantum cohomology of $G(m,N)$.** As was alluded to in the first lecture and also by the phrase 'quantum Littlewood-Richardson rule', the above Gromov-Witten invariants are the structure constants in a deformation of the cohomology ring of $X = G(m,N)$. This (small) *quantum cohomology ring* $QH^*(X)$ was introduced by string theorists, and is a $\mathbb{Z}[q]$-algebra which is isomorphic to $H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. Here q is a formal variable of degree $N = m+n$. The ring structure on $QH^*(X)$ is determined by the relation

$$\sigma_\lambda \cdot \sigma_\mu = \sum \langle \sigma_\lambda, \sigma_\mu, \sigma_{\widehat{\nu}} \rangle_d \, \sigma_\nu \, q^d, \tag{5}$$

the sum over $d \geqslant 0$ and partitions ν with $|\nu| = |\lambda| + |\mu| - dN$. Note that the terms corresponding to $d = 0$ just give the classical cup product in $H^*(X,\mathbb{Z})$. We will need to use the hard fact that equation (5) defines an *associative* product, which turns $QH^*(X)$ into a commutative ring with unit. The reader can find a proof of this basic result in the expository paper [FP].

We now prove, following [Bu1], analogues of the basic structure theorems about $H^*(X,\mathbb{Z})$ for the quantum cohomology ring $QH^*(X)$. For any Young diagram $\lambda \subset (n^m)$, let $\overline{\lambda}$ denote the diagram obtained by removing the leftmost d columns of λ. In terms of partitions, we have $\overline{\lambda}_i = \max\{\lambda_i - d, 0\}$. For any Schubert variety $X_\lambda(F_\bullet)$ in $G(m,E)$, we consider an associated Schubert variety $X_{\overline{\lambda}}(F_\bullet)$ in $G(m+d,E)$. It is easy to see that if $\pi : F(m-d, m+d; E) \to G(m+d, E)$ is the projection map, then $\pi(X_\lambda^{(d)}(F_\bullet)) = X_{\overline{\lambda}}(F_\bullet)$.

Corollary 3. *If $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d \neq 0$, then $[X_{\overline{\lambda}}] \cdot [X_{\overline{\mu}}] \cdot [X_{\overline{\nu}}] \neq 0$ in $H^*(G(m+d,E),\mathbb{Z})$.*

Corollary 4. *If $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d \neq 0$ and $\ell(\lambda) + \ell(\mu) \leqslant m$, then $d = 0$.*

Proof. We know a priori that $|\lambda| + |\mu| + |\nu| = mn + dN$. The assumption on the lengths of λ and μ implies that

$$|\overline{\lambda}| + |\overline{\mu}| + |\overline{\nu}| \geqslant |\lambda| + |\mu| + |\nu| - 2md = \dim G(m+d,E) + d^2.$$

By Corollary 3, we must have $d = 0$. □

Corollary 4 implies that if $\ell(\lambda) + \ell(\mu) \leqslant m$, then

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{d,\nu} \langle \sigma_\lambda, \sigma_\mu, \sigma_{\widehat{\nu}} \rangle_d \, \sigma_\nu \, q^d = \sum_{|\nu| = |\lambda|+|\mu|} \langle \sigma_\lambda, \sigma_\mu, \sigma_{\widehat{\nu}} \rangle_0 \, \sigma_\nu,$$

that is, there are no quantum correction terms in the product $\sigma_\lambda \cdot \sigma_\mu$.

Theorem 5 (Quantum Giambelli, [Be]). *We have $\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{1\leqslant i,j\leqslant m}$ in $QH^*(X)$. That is, the classical Giambelli and quantum Giambelli formulas coincide for $G(m,N)$.*

Proof. Define a linear map $\phi : H^*(X,\mathbb{Z}) \to QH^*(X)$ by $\phi([X_\lambda]) = \sigma_\lambda$. It follows from Corollary 4 that $\sigma_p \cdot \sigma_\mu = \phi([X_p][X_\mu])$ whenever $\ell(\mu) \leqslant m-1$. Using the classical Pieri rule and induction, we see that

$$\sigma_{p_1} \cdots \sigma_{p_m} = \phi([X_{p_1}] \cdots [X_{p_m}]),$$

for any m special Schubert classes $\sigma_{p_1}, \ldots, \sigma_{p_m}$. This implies that

$$\det(\sigma_{\lambda_i+j-i})_{1\leqslant i,j\leqslant m} = \phi(\det([X_{\lambda_i+j-i}])_{1\leqslant i,j\leqslant m}) = \phi([X_\lambda]) = \sigma_\lambda.$$

□

Theorem 6 (Quantum Pieri, [Be]). *For $1 \leqslant p \leqslant n$, we have*

$$\sigma_\lambda \cdot \sigma_p = \sum_\mu \sigma_\mu + q \sum_\nu \sigma_\nu, \tag{6}$$

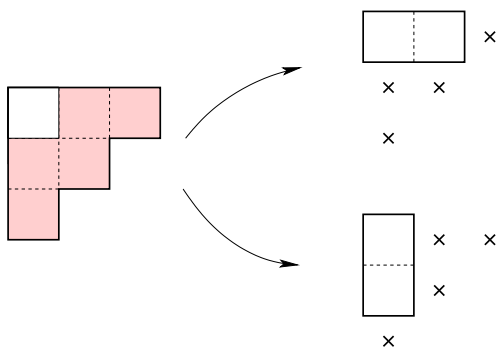
where the first sum is over diagrams μ obtained from λ by adding p boxes, no two in the same column, and the second sum is over all ν obtained from λ by removing $N-p$ boxes from the 'rim' of λ, at least one from each row.

Here the 'rim' of a diagram λ is the rim hook (or 'border strip') contained in λ whose south-east border follows the path we used earlier to define the 01-string corresponding to λ.

Example 5. For the Grassmannian $G(3,6)$, we have

$$\sigma_{3,2,1} \cdot \sigma_2 = \sigma_{3,3,2} + q(\sigma_2 + \sigma_{1,1})$$

in $QH^*(G(3,6))$. The rule for obtaining the two q-terms is illustrated below.



Proof of Theorem 6. By applying the vanishing Corollary 2, we see that it will suffice to check that the line numbers (that is, the Gromov-Witten invariants for $d=1$) agree with the second sum in (6). We will sketch the steps in this argument, and leave the omitted details as an exercise for the reader.

Let $\sigma_{\overline{\lambda}} = [X_{\overline{\lambda}}]$ denote the cohomology class in $H^*(G(m+1,E),\mathbb{Z})$ associated to σ_λ for $d=1$, and define $\sigma_{\overline{\mu}}$ and $\sigma_{\overline{p}}$ in a similar way. One then uses the classical

Pieri rule to show that the prescription for the line numbers in $\sigma_\lambda \cdot \sigma_p$ given in (6) is equivalent to the identity

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_p \rangle_1 = \langle \sigma_{\overline{\lambda}}, \sigma_{\overline{\mu}}, \sigma_{\overline{p}} \rangle_0, \tag{7}$$

where the right hand side of (7) is a classical intersection number on $G(m+1, E)$.

To prove (7), observe that the right hand side is given by the classical Pieri rule on $G(m+1, N)$, and so equals 0 or 1. If $\langle \sigma_{\overline{\lambda}}, \sigma_{\overline{\mu}}, \sigma_{\overline{p}} \rangle_0 = 0$, then Corollary 3 shows that $\langle \sigma_\lambda, \sigma_\mu, \sigma_p \rangle_1 = 0$ as well.

Next, assume that $\langle \sigma_{\overline{\lambda}}, \sigma_{\overline{\mu}}, \sigma_{\overline{p}} \rangle_0 = 1$, so that there is a unique $(m+1)$-dimensional subspace B in the intersection $X_{\overline{\lambda}}(F_\bullet) \cap X_{\overline{\mu}}(G_\bullet) \cap X_{\overline{p}}(H_\bullet)$, for generally chosen reference flags. Note that the construction of B ensures that it lies in the intersection of the corresponding three Schubert *cells* in $G(m+1, E)$, where the defining inequalities in (1) are all equalities. It follows that the two subspaces

$$V_m = B \cap F_{N-\lambda_m} \quad \text{and} \quad V'_m = B \cap G_{N-\mu_m}$$

each have dimension m, and in fact $V_m \in X_\lambda(F_\bullet)$ and $V'_m \in X_\mu(G_\bullet)$. Since

$$|\lambda| + |\mu| = mn + N - p > \dim G(m, N),$$

we see that $X_\lambda(F_\bullet) \cap X_\mu(G_\bullet) = \emptyset$, and hence $V_m \neq V'_m$. As V_m and V'_m are both codimension one subspaces of B, this proves that $A = V_m \cap V'_m$ has dimension $m-1$. We deduce that the only line (corresponding to the required map $f : \mathbb{P}^1 \to X$ of degree one) meeting the three Schubert varieties $X_\lambda(F_\bullet)$, $X_\mu(G_\bullet)$, and $X_p(H_\bullet)$ is the locus $\{V \in X \mid A \subset V \subset B\}$. □

We conclude with Siebert and Tian's presentation of $QH^*(G(m, N))$ in terms of generators and relations.

Theorem 7 (Ring presentation, [ST]). *The ring $QH^*(X)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, q]$ by the relations*

$$D_{m+1} = \cdots = D_{N-1} = 0 \quad \text{and} \quad D_N + (-1)^n q = 0,$$

where $D_k = \det(\sigma_{1+j-i})_{1 \leqslant i,j \leqslant k}$ for each k.

Proof. We will justify why the above relations hold in $QH^*(X)$, and then sketch the rest of the argument. Since the degree of q is N, the relations $D_k = 0$ for $k < N$, which hold in $H^*(X, \mathbb{Z})$, remain true in $QH^*(X)$. For the last relation we use the formal identity of Schur determinants

$$D_N - \sigma_1 D_{N-1} + \sigma_2 D_{N-2} - \cdots + (-1)^n \sigma_n D_m = 0$$

to deduce that $D_N = (-1)^n \sigma_n D_m = (-1)^n \sigma_n \sigma_{(1^m)}$. Therefore it will suffice to show that $\sigma_n \sigma_{(1^m)} = q$; but this is a consequence of Theorem 6.

With a bit more work, one can show that the quotient ring in the theorem is in fact isomorphic to $QH^*(X)$ (see e.g. [Bu1]). Alternatively, one may use an algebraic result of Siebert and Tian [ST]. This states that for a homogeneous space X, given a presentation

$$H^*(X, \mathbb{Z}) = \mathbb{Z}[u_1, \ldots, u_r]/(f_1, \ldots, f_t)$$

of $H^*(X, \mathbb{Z})$ in terms of homogeneous generators and relations, if $f'_1, \ldots, f'_t$ are homogeneous elements in $\mathbb{Z}[u_1, \ldots, u_r, q]$ such that $f'_i(u_1, \ldots, u_r, 0) = f_i(u_1, \ldots, u_r)$ in $\mathbb{Z}[u_1, \ldots, u_r, q]$ and $f'_i(u_1, \ldots, u_r, q) = 0$ in $QH^*(X)$, then the canonical map

$$\mathbb{Z}[u_1, \ldots, u_r, q]/(f'_1, \ldots, f'_t) \to QH^*(X)$$

is an isomorphism. For a proof of this, see [FP, Prop. 11]. □

Remark. The proofs of Theorems 5, 6, and 7 do not require the full force of our main Theorem 3. Indeed, the notion of the kernel and span of a map to X together with Lemma 1 suffice to obtain the simple proofs presented here. For instance, to prove Corollary 3 one can check directly that the span of a rational map which contributes to the Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ must lie in the intersection $X_{\overline{\lambda}}(F_\bullet) \cap X_{\overline{\mu}}(G_\bullet) \cap X_{\overline{\nu}}(H_\bullet)$ in $G(m+d, E)$. This was the original approach in [Bu1].

The proof of Theorem 5 used the surprising fact that in the expansion of the Schur determinant in the quantum Giambelli formula, each individual monomial is purely classical, that is, has no q correction terms. This was also observed and generalized to partial flag varieties by Ciocan-Fontanine [C-F, Thm. 3.14].

4. Lecture Four

4.1. **Schur polynomials.** People have known for a long time about the relation between the product of Schubert classes in the cohomology ring of $G(m,N)$ and the multiplication of *Schur polynomials*, which are the characters of irreducible polynomial representations of GL_n. Recall that if Q denotes the universal (or tautological) quotient bundle of rank n over X, then the special Schubert class σ_i is just the ith Chern class $c_i(Q)$. If the variables $x_1,\ldots,x_n$ are the Chern roots of Q, then the Giambelli formula implies that for any partition λ,

$$\sigma_\lambda = \det(c_{\lambda_i+j-i}(Q)) = \det(e_{\lambda_i+j-i}(x_1,\ldots,x_n)) = s_{\lambda'}(x_1,\ldots,x_n), \tag{8}$$

where λ' is the conjugate partition to λ (whose diagram is the transpose of the diagram of λ), and $s_{\lambda'}(x_1,\ldots,x_n)$ is a Schur S-polynomial in the variables $x_1,\ldots,x_n$. The Schur polynomials $s_\lambda(x_1,\ldots,x_n)$ for λ of length at most n form a $\mathbb{Z}$-basis for the ring $\Lambda_n = \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$ of symmetric polynomials in n variables. It follows that the structure constants $N^\nu_{\lambda\mu}$ for Schur polynomials

$$s_\lambda s_\mu = \sum_\nu N^\nu_{\lambda\mu} s_\nu$$

agree with the Schubert structure constants $c^\nu_{\lambda\mu}$ in $H^*(X,\mathbb{Z})$.

Our main theorem implies that the Gromov-Witten invariants $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ are structure constants in the product of the two *Schubert polynomials* indexed by the permutations for the modified Schubert varieties $X_\lambda^{(d)}$ and $X_\mu^{(d)}$. Postnikov [P] has shown how one may obtain the same numbers as the coefficients when certain 'toric Schur polynomials' are expanded in the basis of the regular Schur polynomials.

For the rest of these lectures, we will present the analogue of the theory developed thus far in the other classical Lie types. To save time, there will be very little discussion of proofs, but only an exposition of the main results. The arguments are often analogous to the ones in type A, but there are also significant differences. For instance, we shall see that in the case of maximal isotropic Grassmannians, the equation directly analogous to (8) defines a family of '$\widetilde{Q}$-polynomials'. The latter polynomials have the property that the structure constants in their product expansions contain both the classical and quantum invariants for these varieties.

4.2. **The Lagrangian Grassmannian $LG(n,2n)$.** We begin with the symplectic case and work with the Lagrangian Grassmannian $LG = LG(n,2n)$ parametrizing Lagrangian subspaces of $E = \mathbb{C}^{2n}$ equipped with a symplectic form $\langle\ ,\ \rangle$. Recall that a subspace V of E is *isotropic* if the restriction of the form to V vanishes. The maximal possible dimension of an isotropic subspace is n, and in this case V is called a *Lagrangian* subspace. The variety LG is the projective complex manifold of dimension $n(n+1)/2$ which parametrizes Lagrangian subspaces in E.

The Schubert varieties $X_\lambda(F_\bullet)$ in $LG(n,2n)$ now depend on a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leqslant n$; we let $\mathcal{D}_n$ denote the parameter space of all such λ (a partition is *strict* if all its parts are distinct). We also require a *complete isotropic flag* of subspaces of E:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n \subset E$$

where $\dim(F_i) = i$ for each i, and F_n is Lagrangian. The codimension $|\lambda|$ Schubert variety $X_\lambda(F_\bullet) \subset LG$ is defined as the locus of $V \in LG$ such that

$$\dim(V \cap F_{n+1-\lambda_i}) \geqslant i, \quad \text{for} \quad i = 1,\ldots,\ell(\lambda). \tag{9}$$

Let σ_λ be the class of $X_\lambda(F_\bullet)$ in the cohomology group $H^{2|\lambda|}(LG,\mathbb{Z})$. We then have a similar list of classical facts, analogous to those for the type A Grassmannian. However, these results were obtained much more recently than the theorems of Pieri and Giambelli.

1) We have $H^*(LG,\mathbb{Z}) \cong \bigoplus_{\lambda\in\mathcal{D}_n} \mathbb{Z}\,\sigma_\lambda$, that is, the cohomology group of LG is free abelian with basis given by the Schubert classes σ_λ.

2) There is an equation $\sigma_\lambda\sigma_\mu = \sum_\nu e^\nu_{\lambda\mu}\sigma_\nu$ in $H^*(LG,\mathbb{Z})$, with

$$e^\nu_{\lambda\mu} = \int_{LG} \sigma_\lambda\sigma_\mu\sigma_{\nu^\vee} = \#X_\lambda(F_\bullet)\cap X_\mu(G_\bullet)\cap X_{\nu^\vee}(H_\bullet), \tag{10}$$

for general complete isotropic flags $F_\bullet$, $G_\bullet$ and $H_\bullet$ in E. Here the 'dual' partition $\nu^\vee$ is again defined so that $\int_{LG}\sigma_\lambda\sigma_\mu = \delta_{\lambda^\vee\mu}$, and it has the property that the parts of $\nu^\vee$ are the complement of the parts of ν in the set $\{1,\dots,n\}$. For example, the partitions $(4,2,1)$ and $(5,3)$ form a dual pair in $\mathcal{D}_5$.

Stembridge [Ste] has given a combinatorial rule similar to the classical Littlewood-Richardson rule, which expresses the structure constants $e^\nu_{\lambda\mu}$ in terms of certain sets of shifted Young tableaux. It would be interesting to find an analogue of the 'puzzle rule' of Theorem 4 that works in this setting.

3) The classes $\sigma_1,\dots,\sigma_n$ are called *special Schubert classes*, and again we have $H^2(LG,\mathbb{Z}) = \mathbb{Z}\,\sigma_1$. If

$$0 \to S \to E_X \to Q \to 0$$

denotes the tautological short exact sequence of vector bundles over LG, then we can use the symplectic form on E to identify Q with the dual of the vector bundle S, and we have $\sigma_i = c_i(S^*)$, for $0 \leqslant i \leqslant n$.

Let us say that two boxes in a (skew) diagram α are connected if they share a vertex or an edge; this defines the connected components of α. We now have the following Pieri rule for LG, due to Hiller and Boe.

Theorem 8 (Pieri rule for LG, [HB]). *For any $\lambda\in\mathcal{D}_n$ and $p \geqslant 0$ we have*

$$\sigma_\lambda\,\sigma_p = \sum_\mu 2^{N(\lambda,\mu)}\sigma_\mu \tag{11}$$

in $H^*(LG,\mathbb{Z})$, where the sum is over all strict partitions μ obtained from λ by adding p boxes, with no two in the same column, and $N(\lambda,\mu)$ is the number of connected components of μ/λ which do not meet the first column.

4) The Pieri rule (11) agrees with the analogous product of Schur Q-functions. This was used by Pragacz to obtain a Giambelli formula for LG, which expresses each Schubert class as a polynomial in the special Schubert classes.

Theorem 9 (Giambelli formula for LG, [P]). *For $i > j > 0$, we have*

$$\sigma_{i,j} = \sigma_i\sigma_j + 2\sum_{k=1}^{n-i}(-1)^k\sigma_{i+k}\sigma_{j-k},$$

while for λ of length greater than two,

$$\sigma_\lambda = \mathrm{Pfaffian}[\sigma_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}, \tag{12}$$

where r is the smallest even integer such that $r \geqslant \ell(\lambda)$.

For those who are not so familiar with Pfaffians, we recall that they are analogous to (and in fact, square roots of) determinants; see e.g. [FPr, Appendix D] for more information. The Pfaffian formula (12) is equivalent to the Laplace-type expansion for Pfaffians

$$\sigma_\lambda = \sum_{j=1}^{r-1} (-1)^{j-1} \sigma_{\lambda_j,\lambda_r} \sigma_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}.$$

5) The ring $H^*(LG,\mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[c(S^*)] = \mathbb{Z}[\sigma_1,\ldots,\sigma_n]$ modulo the relations coming from the Whitney sum formula

$$c_t(S)c_t(S^*) = (1-\sigma_1 t+\sigma_2 t^2-\cdots)(1+\sigma_1 t+\sigma_2 t^2+\cdots) = 1. \tag{13}$$

By equating the coefficients of like powers of t in (13), we see that the relations are given by

$$\sigma_i^2 + 2\sum_{k=1}^{n-i} (-1)^k \sigma_{i+k}\sigma_{i-k} = 0$$

for $1 \leqslant i \leqslant n$, where it is understood that $\sigma_0 = 1$ and $\sigma_j = 0$ for $j < 0$. In terms of the Chern roots $x_1,\ldots,x_n$ of S^*, the equations (13) may be written as

$$\prod_i (1-x_i t) \prod_i (1+x_i t) = \prod_i (1-x_i^2 t^2) = 1.$$

We thus see that $H^*(LG,\mathbb{Z})$ is isomorphic to the ring $\Lambda_n = \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$ modulo the relations $e_i(x_1^2,\ldots,x_n^2) = 0$, for $1 \leqslant i \leqslant n$.

4.3. **$\widetilde{Q}$-polynomials.** We turn now to the analogue of Schur's S-polynomials in type C, as suggested by the discussion in §4.1. These are a family of polynomials symmetric in the variables $X = (x_1,\ldots,x_n)$, which are modelled on Schur's Q-polynomials. They were defined by Pragacz and Ratajski [PR] in the course of their work on degeneracy loci.

For strict partitions $\lambda \in \mathcal{D}_n$, the polynomials $\widetilde{Q}_\lambda(X)$ are obtained by writing

$$\sigma_\lambda = \widetilde{Q}_\lambda(S^*) = \widetilde{Q}_\lambda(x_1,\ldots,x_n)$$

as a polynomial in the Chern roots of S^*, as we did in (8). So $\widetilde{Q}_i(X) = e_i(X)$ for $0 \leqslant i \leqslant n$,

$$\widetilde{Q}_{i,j}(X) = \widetilde{Q}_i(X)\widetilde{Q}_j(X) + 2\sum_{k=1}^{n-i} (-1)^k \widetilde{Q}_{i+k}(X)\widetilde{Q}_{j-k}(X),$$

for $i > j > 0$, and for $\ell(\lambda) \geqslant 3$,

$$\widetilde{Q}_\lambda(X) = \text{Pfaffian}[\widetilde{Q}_{\lambda_i,\lambda_j}(X)]_{1\leqslant i<j\leqslant r}.$$

Pragacz and Ratajski noticed that this definition also makes sense for *non-strict* partitions λ. For $\lambda_1 > n$, one checks easily that $\widetilde{Q}_\lambda(X) = 0$. Let $\mathcal{E}_n$ denote the parameter space of all partitions λ with $\lambda_1 \leqslant n$. We then obtain polynomials $\widetilde{Q}_\lambda(X)$ for $\lambda \in \mathcal{E}_n$ with the following properties:

a) The set $\{\widetilde{Q}_\lambda(X) \mid \lambda \in \mathcal{E}_n\}$ is a free $\mathbb{Z}$-basis for Λ_n.

b) $\widetilde{Q}_{i,i}(X) = e_i(x_1^2,\ldots,x_n^2)$, for $1 \leqslant i \leqslant n$.

c) (Factorization Property) If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and λ^+ is defined by $\lambda^+ = \lambda \cup (i,i) = (\lambda_1, \ldots, i, i, \ldots, \lambda_\ell)$, then

$$\widetilde{Q}_{\lambda^+}(X) = \widetilde{Q}_\lambda(X) \cdot \widetilde{Q}_{i,i}(X).$$

d) For strict λ, the $\widetilde{Q}_\lambda(X)$ enjoy the same Pieri rule as in (11)

$$\widetilde{Q}_\lambda(X) \cdot \widetilde{Q}_p(X) = \sum_\mu 2^{N(\lambda,\mu)} \widetilde{Q}_\mu(X), \tag{14}$$

only this time the sum in (14) is over all partitions $\mu \in \mathcal{E}_n$ (possibly not strict) obtained from λ by adding p boxes, with no two in the same column. In particular, it follows that

$$\widetilde{Q}_n(X) \cdot \widetilde{Q}_\lambda(X) = \widetilde{Q}_{(n,\lambda)}(X)$$

for all $\lambda \in \mathcal{E}_n$.

e) There are structure constants $e^\nu_{\lambda\mu}$ such that

$$\widetilde{Q}_\lambda(X) \cdot \widetilde{Q}_\mu(X) = \sum_\nu e^\nu_{\lambda\mu}\, \widetilde{Q}_\nu(X),$$

defined for $\lambda, \mu, \nu \in \mathcal{E}_n$ with $|\nu| = |\lambda| + |\mu|$. These agree with the integers in (10) if λ, μ, and ν are strict. In general, however, these integers can be negative, for example

$$e^{(4,4,2,2)}_{(3,2,1),(3,2,1)} = -4.$$

In the next lecture, we will see that some of the constants $e^\nu_{\lambda\mu}$ for non-strict ν must be positive, as they are equal to three-point Gromov-Witten invariants, up to a power of 2.

Finally, observe that the above properties allow us to present the cohomology ring of $LG(n,2n)$ as the quotient of the ring $\Lambda_n = \mathbb{Z}[X]^{S_n}$ of $\widetilde{Q}$-polynomials in X modulo the relations $\widetilde{Q}_{i,i}(X) = 0$, for $1 \leqslant i \leqslant n$.

5. Lecture Five

5.1. **Gromov-Witten invariants on LG.** As in the first lecture, by a rational map to LG we mean a morphism $f\colon \mathbb{P}^1 \to LG$, and its degree is the degree of $f_*[\mathbb{P}^1]\cdot\sigma_1$. The Gromov-Witten invariant $\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_d$ is defined for $|\lambda|+|\mu|+|\nu| = n(n+1)/2+d(n+1)$ and counts the number of rational maps $f\colon \mathbb{P}^1 \to LG(n,2n)$ of degree d such that $f(0)\in X_\lambda(F_\bullet)$, $f(1)\in X_\mu(G_\bullet)$, and $f(\infty)\in X_\nu(H_\bullet)$, for given flags $F_\bullet$, $G_\bullet$, and $H_\bullet$ in general position.

We also define the kernel of a map $f : \mathbb{P}^1 \to LG$ as the intersection of the subspaces $f(p)$ for all $p\in\mathbb{P}^1$. In the symplectic case it happens that the span of f is the orthogonal complement of the kernel of f, and hence is not necessary. Therefore the relevant parameter space of kernels that replaces the two-step flag variety is the isotropic Grassmannian $IG(n-d,2n)$, whose points correspond to isotropic subspaces of E of dimension $n-d$.

If $d\geqslant 0$ is an integer, λ, μ, $\nu\in\mathcal{D}_n$ are such that $|\lambda|+|\mu|+|\nu| = n(n+1)/2 + d(n+1)$, and $F_\bullet$, $G_\bullet$, and $H_\bullet$ are complete isotropic flags of $E=\mathbb{C}^{2n}$ in general position, then similar arguments to the ones discussed earlier show that the map $f\mapsto \mathrm{Ker}(f)$ gives a bijection of the set of rational maps $f\colon\mathbb{P}^1\to LG$ of degree d satisfying $f(0)\in X_\lambda(F_\bullet)$, $f(1)\in X_\mu(G_\bullet)$, and $f(\infty)\in X_\nu(H_\bullet)$, with the set of points in the intersection $X_\lambda^{(d)}(F_\bullet)\cap X_\mu^{(d)}(G_\bullet)\cap X_\nu^{(d)}(H_\bullet)$ in $Y_d = IG(n-d,2n)$. We therefore get

Corollary 5 ([BKT1]). *Let $d\geqslant 0$ and λ, μ, $\nu\in\mathcal{D}_n$ be chosen as above. Then*

$$\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_d = \int_{IG(n-d,2n)} [X_\lambda^{(d)}]\cdot[X_\mu^{(d)}]\cdot[X_\nu^{(d)}].$$

The *line numbers* $\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_1$ satisfy an additional relation, which is an extra ingredient needed to complete the analysis for $LG(n,2n)$.

Proposition 2 ([KT1]). *For λ, μ, $\nu\in\mathcal{D}_n$ we have*

$$\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_1 = \frac{1}{2}\int_{LG(n+1,2n+2)} [X_\lambda^+]\cdot[X_\mu^+]\cdot[X_\nu^+],$$

where X_λ^+, X_μ^+, X_ν^+ denote Schubert varieties in $LG(n+1,2n+2)$.

The proof of Proposition 2 in [KT1] proceeds geometrically, by using a correspondence between lines on $LG(n,2n)$ (which are parametrized by points of $IG(n-1,2n)$) and points on $LG(n+1,2n+2)$.

5.2. **Quantum cohomology of $LG(n,2n)$.** The quantum cohomology ring of LG is a $\mathbb{Z}[q]$-algebra isomorphic to $H^*(LG,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$, but here q is a formal variable of degree $n+1$. The product in $QH^*(LG)$ is defined by the same equation (5) as before, but as $\deg(q) = n+1 < 2n$, we expect different behavior than what we have seen for $G(m,N)$. The previous results allow one to prove the following theorem (the original proofs in [KT1] were more involved).

Theorem 10 (Ring presentation and quantum Giambelli, [KT1]). *The ring $QH^*(LG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1,\ldots,\sigma_n,q]$ by the relations*

$$\sigma_i^2 + 2\sum_{k=1}^{n-i}(-1)^k\sigma_{i+k}\sigma_{i-k} = (-1)^{n-i}\sigma_{2i-n-1}\,q$$

for $1 \leqslant i \leqslant n$. The Schubert class σ_λ in this presentation is given by the Giambelli formulas

$$\sigma_{i,j} = \sigma_i \sigma_j + 2 \sum_{k=1}^{n-i} (-1)^k \sigma_{i+k} \sigma_{j-k} + (-1)^{n+1-i} \sigma_{i+j-n-1}\, q$$

for $i > j > 0$, and for $\ell(\lambda) \geqslant 3$,

$$\sigma_\lambda = \text{Pfaffian}[\sigma_{\lambda_i, \lambda_j}]_{1 \leqslant i < j \leqslant r}. \tag{15}$$

The key observation here is that the quantum Giambelli formulas for $LG(n, 2n)$ coincide with the classical Giambelli formulas for $LG(n+1, 2n+2)$, when the class $2\sigma_{n+1}$ is identified with q. Note that this does not imply that the cohomology ring of $LG(n+1, 2n+2)$ is isomorphic to $QH^*(LG(n, 2n))$, because the relation $\sigma_{n+1}^2 = 0$, which holds in the former ring, does not hold in the latter (as $q^2 \neq 0$).

Using the $\widetilde{Q}$-polynomials, we can write the presentation of $QH^*(LG)$ as follows. Let $X^+ = (x_1, \ldots, x_{n+1})$ and let $\widetilde{\Lambda}_{n+1}$ be the subring of Λ_{n+1} generated by the polynomials $\widetilde{Q}_i(X^+)$ for $1 \leqslant i \leqslant n$ together with $2\,\widetilde{Q}_{n+1}(X^+)$. Then the map $\widetilde{\Lambda}_{n+1} \to QH^*(LG)$ which sends $\widetilde{Q}_\lambda(X^+)$ to σ_λ for $\lambda \in \mathcal{D}_n$ and $2\,\widetilde{Q}_{n+1}(X^+)$ to q extends to a surjective ring homomorphism, whose kernel is generated by the relations $\widetilde{Q}_{i,i}(X^+) = 0$ for $1 \leqslant i \leqslant n$.

Theorem 10 therefore implies that the algebra in $QH^*(LG)$ is controlled by the multiplication of $\widetilde{Q}$-polynomials. In particular, the quantum Pieri rule for LG is a specialization of the Pieri rule for $\widetilde{Q}$-polynomials.

Theorem 11 (Quantum Pieri rule for LG, [KT1])**.** *For any $\lambda \in \mathcal{D}_n$ and $p \geqslant 0$ we have*

$$\sigma_\lambda \cdot \sigma_p = \sum_\mu 2^{N(\lambda,\mu)} \sigma_\mu + \sum_\nu 2^{N'(\nu,\lambda)} \sigma_\nu\, q \tag{16}$$

in $QH^*(LG(n, 2n))$, where the first sum is classical, as in (11), while the second is over all strict ν obtained from λ by subtracting $n+1-p$ boxes, no two in the same column, and $N'(\nu, \lambda)$ is one less than the number of connected components of λ/ν.

The informed reader will notice that the exponents $N'(\nu, \lambda)$ in the multiplicities of the quantum correction terms in (16) are of the kind encountered in the classical Pieri rule for *orthogonal* Grassmannians. This was the first indication of a more general phenomenon, which we will discuss at the end of this lecture.

For arbitrary products in $QH^*(LG)$, we have

Corollary 6. *In the relation*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{d \geqslant 0 \\ |\nu| = |\lambda| + |\mu| - d(n+1)}} \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee} \rangle_d\, \sigma_\nu\, q^d,$$

the quantum structure constant $\langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee} \rangle_d$ is equal to $2^{-d} e_{\lambda,\mu}^{((n+1)^d, \nu)}$.

Corollary 6 follows immediately from Theorem 10 together with the identity

$$\widetilde{Q}_{((n+1)^d, \nu)}(X^+) = \widetilde{Q}_{n+1}(X^+)^d \cdot \widetilde{Q}_\lambda(X^+)$$

of $\widetilde{Q}$-polynomials. We deduce that the $\widetilde{Q}$-polynomial structure constants of the form $e_{\lambda,\mu}^{(n^d,\nu)}$ (for $\lambda, \mu, \nu \in \mathcal{D}_{n-1}$) are nonnegative integers, divisible by 2^d. A combinatorial rule for these numbers would give a quantum Littlewood-Richardson rule for LG.

The proofs of Theorems 10 and 11, as compared to those for the type A Grassmannian $G(m,N)$, are complicated by two facts. First, a different argument is needed to establish the quantum Pieri rule, which is related by Proposition 2 to the classical Pieri rule on $LG(n+1,2n+2)$. Second, in the quantum Giambelli Pfaffian expansion $\sigma_\lambda = \text{Pfaffian}[\sigma_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}$, there are terms which do involve q-corrections, and these extra q-terms cancel each other out in the end. Thus more combinatorial work is required to prove that the Pfaffian formula (15) holds in $QH^*(LG)$.

5.3. **The orthogonal Grassmannian $OG(n+1,2n+2)$.** We now turn to the analogue of the above theory in the orthogonal Lie types. We will work with the even orthogonal Grassmannian $OG = OG(n+1,2n+2) = SO_{2n+2}/P_{n+1}$. This variety parametrizes (one component of) the locus of maximal isotropic subspaces of a $(2n+2)$-dimensional vector space E, equipped with a nondegenerate symmetric form. Note that there are two families of such subspaces; by convention, given a fixed isotropic flag $F_\bullet$ in E, we consider only those isotropic V in E such that $V \cap F_{n+1}$ has even codimension in F_{n+1}. We remark that OG is isomorphic to the odd orthogonal Grassmannian $OG(n,2n+1) = SO_{2n+1}/P_n$, hence our analysis (for the maximal isotropic case) will include both the Lie types B and D.

The Schubert varieties $X_\lambda(F_\bullet)$ in OG are again parametrized by partitions $\lambda \in \mathcal{D}_n$ and are defined by the same equations (9) as before, with respect to a complete isotropic flag $F_\bullet$ in E. Let τ_λ be the cohomology class of $X_\lambda(F_\bullet)$; the set $\{\tau_\lambda \,|\, \lambda \in \mathcal{D}_n\}$ is a $\mathbb{Z}$-basis of $H^*(OG,\mathbb{Z})$. Now much of the theory for OG is similar to that for $LG(n,2n)$. To save time, we will pass immediately to the results about the quantum cohomology ring of OG. We again have an isomorphism of $\mathbb{Z}[q]$-modules $QH^*(OG) \cong H^*(OG,\mathbb{Z}) \otimes \mathbb{Z}[q]$, but this time the variable q has degree $2n$.

Another difference between the symplectic and orthogonal case is that the natural embedding of $OG(n+1,2n+2)$ into the type A Grassmannian $G(n+1,2n+2)$ is degree doubling. This means that for every degree d map $f : \mathbb{P}^1 \to OG$, the pullback of the tautological quotient bundle over OG has degree $2d$. It follows that the relevant parameter space of kernels of the maps counted by a Gromov-Witten invariant is the sub-maximal isotropic Grassmannian $OG(n+1-2d,2n+2)$. We pass directly to the corollary of the corresponding 'main theorem':

Corollary 7 ([BKT1]). *Let $d \geqslant 0$ and λ, μ, $\nu \in \mathcal{D}_n$ be such that $|\lambda| + |\mu| + |\nu| = n(n+1)/2 + 2nd$. Then*

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \int_{OG(n+1-2d,2n+2)} [X_\lambda^{(d)}] \cdot [X_\mu^{(d)}] \cdot [X_\nu^{(d)}].$$

This result may be used to obtain analogous structure theorems for $QH^*(OG)$.

Theorem 12 (Ring presentation and quantum Giambelli,[KT2]). *The ring $QH^*(OG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1,\ldots,\tau_n,q]$ modulo the relations*

$$\tau_i^2 + 2\sum_{k=1}^{i-1}(-1)^k \tau_{i+k}\,\tau_{i-k} + (-1)^i \tau_{2i} = 0$$

for all $i < n$, together with the quantum relation

$$\tau_n^2 = q.$$

The Schubert class τ_λ in this presentation is given by the Giambelli formulas

$$\tau_{i,j} = \tau_i\tau_j + 2\sum_{k=1}^{j-1}(-1)^k\tau_{i+k}\tau_{j-k} + (-1)^j\tau_{i+j}$$

for $i > j > 0$, and for $\ell(\lambda) \geqslant 3$,

$$\tau_\lambda = \text{Pfaffian}[\tau_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}.$$

It follows from this that the quantum Giambelli formula for OG coincides with the classical Giambelli formula, and indeed the results in the orthogonal case are formally closer to those in type A.

One can use the $\widetilde{P}$-polynomials, defined by $\widetilde{P}_\lambda = 2^{-\ell(\lambda)}\widetilde{Q}_\lambda$ for each λ, to describe the multiplicative structure of $QH^*(OG)$. Let Λ'_n denote the $\mathbb{Z}$-algebra generated by the polynomials $\widetilde{P}_\lambda(X)$, for $\lambda \in \mathcal{E}_n$, where $X = (x_1, \ldots, x_n)$. Then the map which sends $\widetilde{P}_\lambda(X)$ to τ_λ for all $\lambda \in \mathcal{D}_n$ and $\widetilde{P}_{n,n}(X)$ to q extends to a surjective ring homomorphism $\Lambda'_n \to QH^*(OG)$ with kernel generated by the relations $\widetilde{P}_{i,i}(X) = 0$, for all $i < n$. Define the structure constants $f^\nu_{\lambda\mu}$ by the relation

$$\widetilde{P}_\lambda(X)\cdot\widetilde{P}_\mu(X) = \sum_{|\nu|=|\lambda|+|\mu|} f^\nu_{\lambda\mu}\,\widetilde{P}_\nu(X).$$

Corollary 8. *The Gromov-Witten invariant (and quantum structure constant) $\langle\tau_\lambda, \tau_\mu, \tau_{\nu^\vee}\rangle_d$ is equal to $f^{(n^{2d},\nu)}_{\lambda,\mu}$.*

Theorem 13 (Quantum Pieri rule for OG, [KT2])**.** *For any $\lambda \in \mathcal{D}_n$ and $p \geqslant 0$ we have*

$$\tau_\lambda\cdot\tau_p = \sum_\mu 2^{N'(\lambda,\mu)}\tau_\mu + \sum_\nu 2^{N'(\lambda,\nu)}\tau_{\nu\smallsetminus(n,n)}\,q,$$

where the first sum is over strict μ and the second over partitions $\nu = (n, n, \overline{\nu})$ with $\overline{\nu}$ strict, such that both μ and ν are obtained from λ by adding p boxes, with no two in the same column.

The above quantum Pieri rule implies that

$$\tau_\lambda\cdot\tau_n = \begin{cases}\tau_{(n,\lambda)} & \text{if } \lambda_1 < n,\\ \tau_{\lambda\smallsetminus(n)}\,q & \text{if } \lambda_1 = n\end{cases}$$

in the quantum cohomology ring of $OG(n+1, 2n+2)$. We thus see that multiplication by τ_n is straightforward; it follows that to compute all the Gromov–Witten invariants for OG, it suffices to evaluate the $\langle\tau_\lambda, \tau_\mu, \tau_\nu\rangle_d$ for $\mu, \nu \in \mathcal{D}_{n-1}$. Define a map $\widehat{\ }\colon \mathcal{D}_n \to \mathcal{D}_{n-1}$ by setting $\widehat{\lambda} = (n-\lambda_\ell, \ldots, n-\lambda_1)$ for any partition λ of length ℓ. Notice that $\widehat{\ }$ is essentially a type A Poincaré duality map (for the type A Grassmannian $G(\ell, n+\ell)$).

Partitions in $\mathcal{D}_{n-1}$ also parametrize the Schubert classes σ_λ in the cohomology of $LG(n-1, 2n-2)$. The two spaces $OG(n+1, 2n+2)$ and $LG(n-1, 2n-2)$ have different dimensions and seemingly little in common besides perhaps the fact the the degree of q in the quantum cohomology of the former is twice the degree of q in the latter. However, if $^\vee : \mathcal{D}_{n-1} \to \mathcal{D}_{n-1}$ denotes the Poincaré duality involution on $\mathcal{D}_{n-1}$, we have the following result.

Theorem 14 ([KT2]). *Suppose that $\lambda \in \mathcal{D}_n$ is a non-zero partition with $\ell(\lambda) = 2d + e + 1$ for some nonnegative integers d and e. For any $\mu, \nu \in \mathcal{D}_{n-1}$, we have an equality*

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \langle \sigma_{\widehat{\lambda}}, \sigma_{\mu^\vee}, \sigma_{\nu^\vee} \rangle_e \tag{17}$$

of Gromov–Witten invariants for $OG(n+1, 2n+2)$ and $LG(n-1, 2n-2)$, respectively. If λ is zero or $\ell(\lambda) < 2d+1$, then $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = 0$.

We remark that the left hand side of (17) is symmetric in λ, μ, and ν, unlike the right hand side. This reflects a $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry shared by the Gromov-Witten invariants for both LG and OG. In fact, Theorem 14 is essentially equivalent to this symmetry. The proof in [KT1][KT2] proceeds by first establishing the symmetry by a clever use of the quantum Pieri rule, and then using the relation between the structure constants of $\widetilde{Q}$- and $\widetilde{P}$-polynomials to put everything together. As of this writing, we lack a purely geometric result that would explain Theorem 14.

5.4. **Concluding remarks.** The kernel and span ideas in these notes have been used by Buch in [Bu2] and [Bu3] to obtain simple proofs of the main structure theorems regarding the quantum cohomology of any partial flag manifold SL_N/P, where P is a parabolic subgroup of SL_N (the arguments assume the associativity of the quantum product). However, in [BKT1] it is shown that there is no direct analogue of Theorem 3 in this generality, at least not for the complete flag manifold SL_N/B.

In recent work with Buch and Kresch [BKT2], we present similar results to the ones described here (including an analogue of Theorem 3) for any homogeneous space of the form G/P, were G is a classical Lie group and P is a maximal parabolic subgroup of G. These manifolds include the Grassmannians parametrizing non-maximal isotropic subspaces which were mentioned earlier.

For the reader who is interested in learning more about the classical and quantum cohomology of homogeneous spaces, we recommend the texts by Fulton [F1] and Manivel [M] and the expository article [FP]. The latter reference features the general approach to Gromov-Witten theory using Kontsevich's moduli space of stable maps.

Acknowledgements. It is a pleasure to thank Piotr Pragacz and Andrzej Weber for their efforts in organizing the mini-school on 'Schubert Varieties' at the Banach Center in Warsaw. I am also grateful to Marc Levine and the Universität Essen for their hospitality and the Wolfgang Paul program of the Humbolt Foundation for support during the period when these notes were written. Finally, I thank my collaborators Anders Buch and Andrew Kresch for many enlightening discussions, and for all the fun we've had working together on this project.

References

[Be] A. Bertram : *Quantum Schubert calculus*, Adv. Math. **128** (1997), no. 2, 289–305.

[Bu1] A. S. Buch : *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235.

[Bu2] A. S. Buch : *A direct proof of the quantum version of Monk's formula*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2037–2042.

[Bu3] A. S. Buch : *Quantum cohomology of partial flag manifolds*, preprint (2003), math.AG/0303245.

[BKT1] A. S. Buch, A. Kresch, and H. Tamvakis : *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc., to appear.

[BKT2] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, in preparation.

[C-F] I. Ciocan-Fontanine : *On quantum cohomology rings of partial flag varieties*, Duke Math. J. **98** (1999), no. 3, 485–524.

[F1] W. Fulton : *Young tableaux*, L.M.S. Student Texts **35**, Cambridge Univ. Press, Cambridge, 1997.

[F2] W. Fulton : *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249.

[FP] W. Fulton and R. Pandharipande : *Notes on stable maps and quantum cohomology*, in: Algebraic geometry (Santa Cruz, 1995), Proc. Sympos. Pure. Math. **62**, Part 2, Amer. Math. Soc., Providence (1997), pp. 45–96.

[FPr] W. Fulton and P. Pragacz : *Schubert varieties and degeneracy loci*, Lecture Notes in Math. **1689**, Springer-Verlag, Berlin, 1998.

[G] G. Z. Giambelli : *Risoluzione del problema degli spazi secanti*, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.

[HB] H. Hiller and B. Boe : *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n*, Adv. Math. **62** (1986), 49–67.

[KTW] A. Knutson, T. Tao and C. Woodward : *The honeycomb model of $GL(n)$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc., to appear.

[KT1] A. Kresch and H. Tamvakis : *Quantum cohomology of the Lagrangian Grassmannian*, J. Algebraic Geom., to appear.

[KT2] A. Kresch and H. Tamvakis : *Quantum cohomology of orthogonal Grassmannians*, Compositio Math., to appear.

[M] L. Manivel : *Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence*, Cours Spécialisés **3**, Société Mathématique de France, Paris, 1998.

[Pi] M. Pieri : *Sul problema degli spazi secanti. Nota 1^a*, Rend. Ist. Lombardo (2) **26** (1893), 534–546.

[Po] A. Postnikov : *Affine approach to quantum Schubert calculus*, preprint (2002), math.CO/0205165.

[P] P. Pragacz : *Algebro-geometric applications of Schur S- and Q-polynomials*, Séminare d'Algèbre Dubreil-Malliavin 1989–1990, Lecture Notes in Math. **1478**, 130–191, Springer-Verlag, Berlin, 1991.

[PR] P. Pragacz and J. Ratajski : *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach*, Compositio Math. **107** (1997), no. 1, 11–87.

[ST] B. Siebert and G. Tian : *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), no. 4, 679–695.

[Ste] J. R. Stembridge : *Shifted tableaux and the projective representations of symmetric groups*, Adv. Math. **74** (1989), 87–134.

[V] R. Vakil : *A geometric Littlewood-Richardson rule*, preprint (2003), math.AG/0302294.

[Y] A. Yong : *Degree bounds in quantum Schubert calculus*, Proc. Amer. Math. Soc., to appear.

Department of Mathematics, Brandeis University - MS 050, P. O. Box 9110, Waltham, MA 02454-9110, USA

E-mail address: harryt@brandeis.edu