Thom polynomials and Schur functions: towards the singularities $A_i(-)$

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Abstract

We develop algebro-combinatorial tools for computing the Thom polynomials for the Morin singularities $A_i(-)$ $(i \geq 0)$. The main tool is the function $F_r^{(i)}$ defined as a combination of Schur functions with certain numerical specializations of Schur polynomials as their coefficients. We show that the Thom polynomial $\mathcal{T}^{A_i}$ for the singularity A_i (any i) associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, with any parameter $k \geq 0$, under the assumption that $\Sigma^j = \emptyset$ for all $j \geq 2$, is given by $F_{k+1}^{(i)}$. Equivalently, this says that "the 1-part" of $\mathcal{T}^{A_i}$ equals $F_{k+1}^{(i)}$. We investigate 2 examples when $\mathcal{T}^{A_i}$ apart from its 1-part consists also of the 2-part being a single Schur function with some multiplicity. Our computations combine the characterization of Thom polynomials via the "method of restriction equations" of Rimányi et al. with the techniques of Schur functions.

1 Introduction

The global behavior of singularities is governed by their *Thom polynomials* (cf. [40], [19], [1], [14], [36], [16]). Knowing the Thom polynomial of a singularity η, denoted $\mathcal{T}^\eta$, one can compute the cohomology class represented by the η-points of a map.

In the present paper, following a series of papers by Rimányi et al. [37], [35], [36], [8], [2], we study the Thom polynomials for the singularities A_i associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$.

The way of obtaining the thought Thom polynomial is through the solution of a system of linear equations, which is fine when we want to find one concrete Thom polynomial, say, for a fixed k. However, if we want to find the Thom polynomials for a series of singularities, associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$

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with k as a parameter, we have to solve *simultaneously* a countable family of systems of linear equations. We do it here for the restriction equations for the above mentioned singularities. Instead of using *Chern monomial expansions* (as the authors of previous papers constantly did), we use *Schur function expansions.* This puts a more transparent structure on computations of Thom polynomials (cf. also [7], [29]).

Another feature of using the Schur function expansions for Thom polynomials is that all the coefficients are *nonnegative.* This has been recently proved by A. Weber and the author in [33] (see also [34]).

To be more precise, we use here (the specializations of) *supersymmetric* Schur functions also called "Schur functions in difference of alphabets" together with their three basic properties: *vanishing*, *cancellation* and *factorization*, (cf. [39], [4], [23], [28], [32], [24], [10], and [21]). These functions contain resultants among themselves. Their geometric role was illuminated, e.g., in the study of $\mathcal{P}$*-ideals of singularities* Σ^i (cf. [30, end of Sect. 2 and Theorem 11]) which is based on the enumerative geometry of degeneracy loci of [27]. In fact, in the present paper (and in [31]), we use the point of view of this last paper to some extent. We know by the Thom-Damon theorem that $\mathcal{T}^{A_i}$ is a $\mathbf{Z}$-linear combination of Schur functions in $TX^*-f^*(TY^*)$. Given a positive integer h, we shall say that a $\mathbf{Z}$-linear combination

$$\sum_I \alpha_I S_I$$

is an *h-combination* if for any partition I appearing nontrivially the following condition $(*)_h$ holds[1]: I contains the rectangle partition

$$(k+h, \ldots, k+h)$$

(h times), but it does not contain the larger Young diagram

$$(k+h+1, \ldots, k+h+1)$$

($h+1$ times). For example, a 1-combination consists of Schur functions containing a single row $(k+1)$ but not containing $(k+2, k+2)$; a 2-combination consists of Schur functions containing $(k+2, k+2)$ but not containing $(k+3, k+3, k+3)$ etc. (An h-combination, with the argument "$TX^*-f^*(TY^*)$", is a typical universal polynomial supported on the $(\bullet - h)$th degeneracy locus of the derivative morphism of the tangent vector bundles.) Since the singularity A_i is of Thom-Boardman type Σ^1, we have by [28, Theorem 10] (based on the structure of the $\mathcal{P}$-ideal of the singularity Σ^1) that all partitions in the Schur expansion of $\mathcal{T}^{A_i}$ contain a single row $(k+1)$. For a fixed h, let us consider the sum of all Schur functions appearing nontrivially in $\mathcal{T}^{A_i}$ (multiplied by their coefficients) corresponding to partitions satisfying $(*)_h$. This h-combination will be called the *h-part of* $\mathcal{T}^{A_i}$. Of course, $\mathcal{T}^{A_i}$ is a sum of its h-parts.

The main body of this paper is devoted to study the 1-part of the Thom polynomial for the singularities A_i associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$. We introduce, via its Schur function expansion, the basic functions $F(\mathbb{A}, -)$ and $F^{(i)}$. Using the properties of these functions (Proposition

[1]We say that one partition *is contained* in another if this holds for their Young diagrams (cf. [21]).

10 and Corollary 11), we show (Theorem 12) that it gives the Thom polynomial for A_i when $\Sigma^j = \emptyset$ for all $j \geq 2$. Equivalently, it says that the 1-part of the Thom polynomial for a generic singularity A_i is equal to $F^{(i)}_{k+1}$. For $k = 0$, this polynomial was given in [26] in the Chern monomial basis.

With the help of $F^{(1)}$ and $F^{(2)}$, we reprove the formulas of Thom [40] and Ronga [38] for A_1, A_2 and for any parameter $k \geq 0$.

We give also computations of two Thom polynomials having apart from their 1-parts also the nontrivial 2-parts (consisting of single Schur functions with certain multiplicities). We first reprove the result of Gaffney [11] for A_4 and $k = 0$. This was also done by Rimányi [35]; our approach uses Schur functions. Then we do the computations for A_3 and $k = 1$; this, in turn, can be considered as an introduction to the general case A_3 (any k) in [31].

In our calculations, we use extensively the functorial λ-ring approach to symmetric functions developed mainly in Lascoux's book [21].

Main results of the present paper were announced in [29].

Inspired by the present article, [29], [30], and [31], Özer Öztürk [25] computed the Thom polynomials for A_4 and $k = 2, 3$.

2 Recollections on Thom polynomials

Our main reference for this section is [36]. We start with recalling what we shall mean by a "singularity". Let $k \geq 0$ be a fixed integer. By a *singularity* we shall mean an equivalence class of stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension.

We recall[2] that the *Thom polynomial* $\mathcal{T}^\eta$ of a singularity η is a polynomial in the formal variables $c_1, c_2, \ldots$ that after the substitution

$$c_i = c_i(f^*TY - TX) = [c(f^*TY)/c(TX)]_i\,, \tag{1}$$

for a general map $f : X \to Y$ between complex analytic manifolds, evaluates the Poincaré dual of $[V^\eta(f)]$, where $V^\eta(f)$ is the cycle carried by the closure of the set

$$\{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\}\,. \tag{2}$$

By *codimension of a singularity* η, $\operatorname{codim}(\eta)$, we shall mean $\operatorname{codim}_X(V^\eta(f))$ for such an f. The concept of the polynomial $\mathcal{T}^\eta$ comes from Thom's fundamental paper [40]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied by Kazarian in [14], [15], [16].

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional $\mathbf{C}$-algebras. We shall use the following notation:

– A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;

– $I_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^2+y^2)$;

[2] This statement is usually called the Thom-Damon theorem [40], [5].

– $III_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x,y]]/(xy, x^2, y^2)$ (here $k \geq 1$).

In the present article, the computations of Thom polynomials shall use the method which stems from a sequence of papers by Rimányi et al. [37], [35], [36], [8], [2]. We sketch briefly this approach, refering the interested reader for more details to these papers, the main references being the last three mentioned items.

Let $k \geq 0$ be a fixed integer, and let $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ be a stable singularity with a prototype $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\varphi, \psi) \in \operatorname{Diff}(\mathbf{C}^n, 0) \times \operatorname{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \varphi^{-1} = \kappa\} \tag{3}$$

of κ will be denoted by G_η. Even if $\operatorname{Aut}\kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way (cf. [12] and [41]). In fact, G_η can be chosen so that the images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$ will be denoted by $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called E'_η and E_η. The *total Chern class of the singularity* η is defined in $H^*(BG_\eta, \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}\,. \tag{4}$$

The *Euler class* of η is defined in $H^{2\operatorname{codim}(\eta)}(BG_\eta, \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta)\,. \tag{5}$$

Sometimes, it will be convenient not to work with the whole maximal compact subgroup G_η but with its suitable subgroup; this subgroup should be, however, as "close" to G_η as possible (cf. [36], p. 502). We shall denote this subgroup by the same symbol G_η.

In the following theorem, we collect information from [36], Theorem 2.4 and [8], Theorem 3.5, needed for the calculations in the present paper.

Theorem 1 *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\operatorname{codim}(\eta)$*, are not zero-divisors* [3]*. Then we have*
(i) if $\xi \neq \eta$ and $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$*, then $\mathcal{T}^\eta(c(\xi)) = 0$;*
(ii) $\mathcal{T}^\eta(c(\eta)) = e(\eta)$.
This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way. [4]

To use this method of determining the Thom polynomials for singularities, one needs their classification, see, e.g., [6].

[3] This is the so-called "Euler condition" (*loc.cit.*).

[4] To make it precise, we need one more condition that the number of singularities(=contact orbits) of smaller codimension is finite: we may assume that η is a *simple* singularity type, i.e., there is no moduli adjacent to η.

To effectively use Theorem 1, we need to study the maximal compact subgroups of singularities. We recall the following recipe from [36] pp. 505–507. Let η be a singularity whose prototype is $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbf{C}^m, 0) \to (\mathbf{C}^{m+k}, 0)$ with $d\beta = 0$. The group G_η is a subgroup of the maximal compact subgroup of the algebraic automorphism group of the local algebra Q_η of η times the unitary group $U(k{-}d)$, where d is the difference between the minimal number of relations and the number of generators of Q_η. With β well chosen, G_η acts as right-left symmetry group on β with representations μ_1 and μ_2. The representations λ_1 and λ_2 are

$$\lambda_1 = \mu_1 \oplus \mu_V \quad \text{and} \quad \lambda_2 = \mu_2 \oplus \mu_V\,, \tag{6}$$

where μ_V is the representation of G_η on the unfolding space $V = \mathbf{C}^{n-m}$ given, for $\alpha \in V$ and $(\varphi, \psi) \in G_\eta$, by

$$(\varphi, \psi)\ \alpha = \psi \circ \alpha \circ \varphi^{-1}\,. \tag{7}$$

For example, for the singularity of type A_i: $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, we have $G_{A_i} = U(1) \times U(k)$ with

$$\mu_1 = \rho_1, \quad \mu_2 = \rho_1^{i+1} \oplus \rho_k, \quad \mu_V = \oplus_{j=2}^i\ \rho_1^j \oplus \oplus_{j=1}^i (\rho_k \otimes \rho_1^{-j})\,, \tag{8}$$

where ρ_j denotes the standard representation of the unitary group $U(j)$. Hence, we obtain assertion (i) of the following

Proposition 2 *(i) Let $\eta = A_i$; for any k, writing x and $y_1, \ldots, y_k$ for the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$,*

$$c(A_i) = \frac{1 + (i+1)x}{1+x} \prod_{j=1}^{k} (1 + y_j)\,, \tag{9}$$

$$e(A_i) = i!\ x^i \prod_{j=1}^{k} (y_j - ix) \cdots (y_j - 2x)(y_j - x)\,. \tag{10}$$

(ii) Let $\eta = I_{2,2}$. Denote by H the extension of $U(1) \times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$ ("the group generated by multiplication on the coordinates and their exchange"). For $k = 0$, we have $G_\eta = H$. Hence, for the purpose of our computations we can use $G_\eta = U(1) \times U(1)$. Writing x_1, x_2 for the Chern roots of the universal bundles on two copies of $BU(1)$,

$$c(I_{2,2}) = \frac{(1 + 2x_1)(1 + 2x_2)}{(1 + x_1)(1 + x_2)}\,. \tag{11}$$

(iii) Let $\eta = III_{2,2}$; for $k = 1$, $G_\eta = U(2)$, and writing x_1, x_2 for the Chern roots of the universal bundles on $BU(2)$, we have

$$c(III_{2,2}) = \frac{(1{+}2x_1)(1{+}2x_2)(1{+}x_1{+}x_2)}{(1{+}x_1)(1{+}x_2)}\,. \tag{12}$$

(Assertions (ii) and (iii) are obtained, in a standard way, following the instructions of [36], Sect. 4. Assertion (ii) is proved in [36, pp. 506–507], whereas assertion (iii) stems from [2, p. 65].)

3 Recollections on Schur functions

In this section, we collect needed notions related to symmetric functions. We adopt a functorial point of view of [21]. Namely, given a commutative ring, we treat symmetric functions as operators acting on the ring. We shall give here only a very brief summary of the corresponding material from our previous paper [30].

For $m \in \mathbf{N}$, by "an alphabet $\mathbb{A}_m$" we shall mean an alphabet $\mathbb{A} = (a_1, \ldots, a_m)$ (of cardinality m); ditto for $\mathbb{B}_n = (b_1, \ldots, b_n)$, $\mathbb{Y}_k = (y_1, \ldots, y_k)$, and $\mathbb{X}_2 = (x_1, x_2)$.

Definition 3 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}-\mathbb{B})$ are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i = \prod_{b\in\mathbb{B}}(1-bz) / \prod_{a\in\mathbb{A}}(1-az)\,. \tag{13}$$

Convention 4 *We shall often identify an alphabet $\mathbb{A} = \{a_1, \ldots, a_m\}$ with the sum $a_1 + \cdots + a_m$ and perform usual algebraic operations on such elements. For example, $\mathbb{A}b$ will denote the alphabet $(a_1 b, \ldots, a_m b)$. We will give priority to the algebraic notation over the set-theoretic one.*

Definition 5 *Given a partition*[5] $I = (0 \le i_1 \le i_2 \le \ldots \le i_s) \in \mathbf{N}^s$*, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function $S_I(\mathbb{A}-\mathbb{B})$ is*

$$S_I(\mathbb{A}-\mathbb{B}) := \Big| S_{i_p+p-q}(\mathbb{A}-\mathbb{B}) \Big|_{1\le p,q\le s}\,. \tag{14}$$

These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. Their properties were studied, among others, in [4], [23], [28], [32], [24], [10], and [21]. From the last item, we borrow increasing "French" partitions and the determinant of the form (14) evaluating a Schur function. We shall use the the simplified notation $i_1 i_2 \cdots i_s$ for a partition $(i_1, \ldots, i_s)$.

We have the following *cancellation property*:

$$S_I((\mathbb{A}+\mathbb{C}) - (\mathbb{B}+\mathbb{C})) = S_I(\mathbb{A}-\mathbb{B})\,. \tag{15}$$

We identify partitions with their Young diagrams, as is customary.

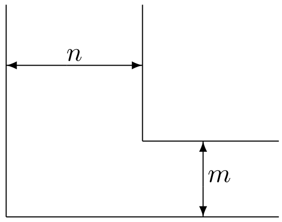
We record the following property (*loc.cit.*), justifying the notational remark from the end of Section 2; for a partition I,

$$S_I(\mathbb{A}-\mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B}-\mathbb{A}) = S_J(\mathbb{B}^*-\mathbb{A}^*)\,, \tag{16}$$

where J is the conjugate partition of I (i.e. the consecutive rows of the diagram of J are the transposed columns of the diagram of I), and $\mathbb{A}^*$ denotes the alphabet $\{-a_1, -a_2, \ldots\}$.

Fix two positive integers m and n. We shall say that a partition $I = (0 < i_1 \le i_2 \le \cdots \le i_s)$ *is contained in* the (m,n)-hook if either $s \le m$, or $s > m$ and $i_{s-m} \le n$. Pictorially, this means that the Young diagram of I is contained in the "tickened" hook:

[5] We identify partitions with their Young diagrams, as is customary.



We record the following *vanishing property*. Given alphabets $\mathbb{A}$ and $\mathbb{B}$ of cardinalities m and n, if a partition I is not contained in the (m, n)-hook, then (*loc.cit.*):

$$S_I(\mathbb{A} - \mathbb{B}) = 0\,. \tag{17}$$

In the present paper, by a *symmetric function* we shall mean a $\mathbf{Z}$-linear combination of the operators $S_I(-)$.

We shall use the following convention from [22].

Convention 6 *We may need to specialize a letter to* 4*, but this must not be confused with taking four copies of* 1*. To allow one, nevertheless, specializing a letter to an (integer, or even complex) number r inside a symmetric function, without introducing intermediate variables, we write* $\boxed{r}$ *for this specialization. Boxes have to be treated as single variables. For example,*

$$S_i(2) = i + 1 \quad \textit{but} \quad S_i(\boxed{2}) = 2^i\,.$$

A similar remark applies to $\mathbf{Z}$*-linear combinations of variables. We have*

$$S_2(\mathbb{X}_2) = x_1^2 + x_1x_2 + x_2^2 \quad \textit{but} \quad S_2(\boxed{x_1{+}x_2}) = x_1^2 + 2x_1x_2 + x_2^2\,.$$

Definition 7 *Given two alphabets $\mathbb{A}, \mathbb{B}$, we define their resultant:*

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a{-}b)\,. \tag{18}$$

For example, we have the following formal identity:

$$i!(-x)^i \prod_{j=1}^{k} (ix{-}y_j)\cdots(2x{-}y_j)(x{-}y_j) = R\big(x{+}\boxed{2x}{+}\cdots{+}\boxed{ix}, \mathbb{Y}_k{+}\boxed{(i{+}1)x}\,\big)\,. \tag{19}$$

We have (cf. [21])

$$R(\mathbb{A}_m, \mathbb{B}_n) = S_{(n^m)}(\mathbb{A}{-}\mathbb{B}) = \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B})\,, \tag{20}$$

where the sum is over all partitions $I \subset (n^m)$.

When a partition is contained in the (m, n)-hook and at the same time it contains the rectangle (n^m), then we have the following *factorization property* (*loc.cit.*): for partitions $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_s)$,

$$S_{(j_1,\ldots,j_s,i_1+n,\ldots,i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A})\; R(\mathbb{A}, \mathbb{B})\; S_J(-\mathbb{B})\,. \tag{21}$$

Rather than the Chern classes

$$c_i(f^*TY - TX) = [f^*c(TY)/c(TX)]_i\,,$$

we shall use *Segre classes* S_i of the virtual bundle $TX^* - f^*(TY^*)$, i.e. complete symmetric functions $S_i(\mathbb{A}-\mathbb{B})$ for the alphabets of the *Chern roots* $\mathbb{A}, \mathbb{B}$ of TX^* and TY^*.

In the present paper, it will be more handy to use, instead of k, a "shifted" parameter

$$r := k+1\,. \tag{22}$$

Sometimes, we shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+r-1}, 0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}^\eta_r$ – to emphasize the dependence of both items on r.

Note that in our notation, the Thom polynomial for the singularity $A_1(r)$ for $r \geq 1$, is: $\mathcal{T}^{A_1}_r = S_r$, instead of c_{k+1} in [36]. In general, a Thom polynomial in terms of the c_i's (in those papers) will be written here as a linear combination of Schur functions obtained by changing each c_i to S_i and expanding in the Schur function basis. Another example is the Thom polynomial for $A_2(1)$: $c_1^2 + c_2$ rewritten in our notation as $\mathcal{T}^{A_2}_1 = S_{11} + 2S_2$.

Recall (from the Introduction) that the *h-part* of $\mathcal{T}^{A_i}_r$ is the sum of all Schur functions appearing nontrivially in $\mathcal{T}^{A_i}_r$ (multiplied by their coefficients) such that the corresponding partitions satisfy the following condition: I contains the rectangle partition $\big((r+h-1)^h\big)$, but it does not contain the larger Young diagram $\big((r+h)^{h+1}\big)$. The polynomial $\mathcal{T}^{A_i}_r$ is a sum of its h-parts, $h = 1, 2, \ldots$.

4 Functions $F(\mathbb{A}, -)$ and $F^{(i)}_r$

We now pass to the following function F which will give rise to the 1-part of $\mathcal{T}^{A_i}_r$, i.e. to the function $F^{(i)}_r$ that will be studied in this section. Fix positive integers m and n. For an alphabet $\mathbb{A}$ of cardinality m, we define

$$F(\mathbb{A}, -) := \sum_I S_I(\mathbb{A}) S_{n-i_m,\ldots,n-i_1,n+|I|}(-)\,, \tag{23}$$

where the sum is over partitions $I = (i_1 \leq i_2 \leq \cdots \leq i_m \leq n)$, i.e. over $I \subset (n^m)$.

Lemma 8 *For a variable x and an alphabet $\mathbb{B}$ of cardinality n,*

$$F(\mathbb{A}, x - \mathbb{B}) = R(x + \mathbb{A}x, \mathbb{B})\,. \tag{24}$$

Proof. For a fixed partition $I = (i_1 \leq i_2 \leq \cdots \leq i_m \leq n)$, it follows from the factorization property (21) that

$$S_{n-i_m,\ldots,n-i_1,n+|I|}(x-\mathbb{B}) = S_{(n^m)/I}(-\mathbb{B})\ R(x,\mathbb{B})\ x^{|I|}\,.$$

Hence, using $S_I(\mathbb{A}x) = S_I(\mathbb{A})x^{|I|}$, Eq. (20) and Eq. (18), we have

$$\begin{aligned} F(\mathbb{A}, x-\mathbb{B}) &= \sum_I S_I(\mathbb{A}) S_{(n^m)/I}(-\mathbb{B})\ R(x,\mathbb{B})\ x^{|I|} \\ &= \sum_I S_I(\mathbb{A}x)\ S_{(n^m)/I}(-\mathbb{B})\ R(x,\mathbb{B}) \\ &= R(\mathbb{A}x, \mathbb{B})\ R(x,\mathbb{B}) = R(x+\mathbb{A}x, \mathbb{B})\,. \end{aligned}$$

The lemma has been proved. $\square$

The following function $F_r^{(i)}$ will be basic for computing the Thom polynomials for A_i ($i \ge 1$). We set

$$F_r^{(i)}(-) := \sum_J S_J(\boxed{2} + \boxed{3} + \cdots + \boxed{i}) S_{r-j_{i-1},\dots,r-j_1,r+|J|}(-)\,, \qquad (25)$$

where the sum is over partitions $J \subset (r^{i-1})$, and for $i = 1$ we understand $F_r^{(1)}(-) = S_r(-)$.

Example 9 *We have*

$$F_r^{(2)} = \sum_{j\le r} S_j(\boxed{2}) S_{r-j,r+j} = \sum_{j\le r} 2^j S_{r-j,r+j}\,;$$

$$F_r^{(3)} = \sum_{j_1\le j_2\le r} S_{j_1,j_2}(\boxed{2} + \boxed{3}) S_{r-j_2,r-j_1,r+j_1+j_2}\,;$$

in particular,

$$F_1^{(3)} = S_{111} + 5S_{12} + 6S_3$$

and

$$F_2^{(3)} = S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6;$$

$$F_r^{(4)} = \sum_{j_1\le j_2\le j_3\le r} S_{j_1,j_2,j_3}(\boxed{2} + \boxed{3} + \boxed{4}) S_{r-j_3,r-j_2,r-j_1,r+j_1+j_2+j_3}\,;$$

in particular,

$$F_1^{(4)} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4$$

and

$$\begin{aligned} F_2^{(4)} =& S_{2222} + 9S_{1223} + 26S_{1124} + 24S_{1115} + 55S_{224} + 210S_{125} + 216S_{116} \\ &+ 391S_{26} + 555S_{17} + 507S_8\,; \end{aligned}$$

$$F_1^{(i)} = \sum_{j\le i-1} S_{1^j}(\boxed{2} + \boxed{3} + \cdots + \boxed{i}) S_{1^{i-j-1},j+1}\,.$$

In the following, we shall tacitly assume that x, x_1, x_2, and $\mathbb{B}_r$ are variables[6] (though many results remain valid without this assumption).

The following result gives the key algebraic property of $F_r^{(i)}$.

Proposition 10 *We have*

$$F_r^{(i)}(x - \mathbb{B}_r) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}\,, \mathbb{B}_r)\,. \qquad (26)$$

[6]Note that these variables will correspond in the following to the Chern roots of the *cotangent* bundles. On the contrary, in Proposition 2 the Chern roots of the *tangent* bundles were used. This causes some differences of signs in several formulas. The same remark applies to our former paper [30].

Proof. The assertion follows from Lemma 8 with $m = i - 1$, $n = r$, and

$$\mathbb{A} = \boxed{2} + \boxed{3} + \cdots + \boxed{i} \,. \; \square$$

Corollary 11 *Fix an integer $i \geq 1$.*
(i) For an integer $p \leq i$, we have

$$F_r^{(i)}(x - \mathbb{B}_{r-1} - \boxed{px}) = 0\,. \tag{27}$$

(ii) Moreover, we have

$$F_r^{(i)}(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_{r-1} + \boxed{(i+1)x})\,. \tag{28}$$

Proof. Substituting in Eq. (26):

$$\mathbb{B}_r = \mathbb{B}_{r-1} + \boxed{px}$$

for $p \leq i$, and, respectively,

$$\mathbb{B}_r = \mathbb{B}_{r-1} + \boxed{(i+1)x}\,,$$

we get the assertions. $\square$

5 Towards Thom polynomials for $A_i(r)$

In the following theorem, we shall consider maps $f : X \to Y$ with degeneracies.

Theorem 12 *Suppose that $\Sigma^j(f) = \emptyset$ for $j \geq 2$* [7]*. Then, for any $r \geq 1$, we have*

$$\mathcal{T}_r^{A_i} = F_r^{(i)}\,. \tag{29}$$

Proof. By the assumption $\Sigma^j(f) = \emptyset$ for $j \geq 2$, the Euler condition (needed in Theorem 1) is satisfied here for any $i \geq 0$ and $r \geq 1$. The equations characterizing $\mathcal{T}_r^{A_i}$ in the sense of Theorem 1 are, for $p \leq i$,

$$P(x - \mathbb{B}_{r-1} - \boxed{px}) = 0\,, \tag{30}$$

and additionally, invoking Eq. (19),

$$P(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_{r-1} + \boxed{(i+1)x})\,. \tag{31}$$

It follows from Corollary 11 that $P = F_r^{(i)}$ satisfies these equations. The theorem has been proved. $\square$

Corollary 13 *For any singularity $A_i(r)$, the first part of its Thom polynomial is equal to $F_r^{(i)}$.*

In the special case $r = 1$, Porteous [26] gave an expression for the Thom polynomial from the theorem in terms of the Chern monomial basis (see also [20]).

The functions $F_r^{(1)}$, $F_r^{(2)}$ give the Thom polynomials for A_1, A_2 (any r) for a general map $f : X \to Y$.

[7] This says that the kernel of the derivative map $df : TX \to f^*TY$ of f is a line bundle.

Theorem 14 *([40], [38]) The polynomials* S_r *and* $\sum_{j\le r} 2^j S_{r-j,r+j}$ *are Thom polynomials for the singularities* $A_1(r)$ *and* $A_2(r)$*.*

Proof. Since only A_0 has smaller codimension than A_1, and only A_0, A_1 are of smaller codimension than A_2, the Euler conditions hold, and the equations from Theorem 1 characterizing these Thom polynomials are:

$$P(-\mathbb{B}_{r-1}) = 0, \quad P(x - \mathbb{B}_{r-1} - \boxed{2x}) = R(x, \mathbb{B}_{r-1} + \boxed{2x}) \tag{32}$$

for A_1, and

$$\begin{aligned} P(-\mathbb{B}_{r-1}) = P(x - \mathbb{B}_{r-1} - \boxed{2x}) = 0, \\ P(x - \mathbb{B}_{r-1} - \boxed{3x}) = R(x + \boxed{2x}, \mathbb{B}_{r-1} + \boxed{3x}) \end{aligned} \tag{33}$$

for A_2. Hence the assertion follows from Corollary 11.[8] $\square$

6 Two examples

In the present section, we show two (relatively simple) examples of Schur function expansions of Thom polynomials for A_i, where two h-parts appear. The method used will be applied in [31] to more complicated singularities. Recall that the Thom polynomial $\mathcal{T}_r^{A_i}$ is a sum of its h-parts, the 1-part being $F_r^{(i)}$. To get the correct Thom polynomial, one must add to $F_r^{(i)}$ the h-parts of $\mathcal{T}_r^{A_i}$ for $h = 2, 3, \ldots$.

Let us discuss first A_4 for $r = 1$ (its codimension is 4). Then the singularities $\neq A_4$, whose codimension is $\le \operatorname{codim}(A_4)$ are: A_0, A_1, A_2, A_3, $I_{2,2}$. The Thom polynomial[9] is

$$\mathcal{T}_1^{A_4} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4 + 10S_{22}\,. \tag{34}$$

We have

$$F_1^{(4)} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4\,. \tag{35}$$

By Corollary 11, this function satisfies the following equations imposed by A_0, A_1, A_2, A_3, A_4:

$$F_1^{(4)}(0) = F_1^{(4)}(x - \boxed{2x}) = F_1^{(4)}(x - \boxed{3x}) = F_1^{(4)}(x - \boxed{4x}) = 0\,, \tag{36}$$

$$F_1^{(4)}(x - \boxed{5x}) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \boxed{5x})\,. \tag{37}$$

However, $F_1^{(4)}$ does not satisfy the vanishing imposed by $I_{2,2}$. Namely, we have

$$F_1^{(4)}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = (-10)x_1x_2(x_1 - 2x_2)(x_2 - 2x_1)\,. \tag{38}$$

To see this, invoke Proposition 10:

$$F_1^{(4)}(x - \mathbb{B}_1) = R(x + \boxed{2x} + \boxed{3x} + \boxed{4x}, \mathbb{B}_1)\,. \tag{39}$$

[8] Or, as the referee points out, it is simpler to say that this follows from Theorem 12 since $\operatorname{codim} \overline{\Sigma^2}$ is greater than $\operatorname{codim} A_i$ $(i = 1, 2)$.

[9] This Thom polynomial was originally computed by Gaffney in [11] via the desingularization method. Its alternative derivation via solving equations imposed by the above singularities was done by Rimányi in [35]). Both authors used Chern monomial expansions.

Substituting to the LHS of Eq. (38) $x_1 = 0$, we get by this proposition

$$F_1^{(4)}(x_2 - \boxed{2x_2}) = R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{2x_2}) = 0\,,$$

and substituting $x_1 = 2x_2$,

$$\begin{aligned} F_1^{(4)}(x_2 - \boxed{2x_1}) &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{2x_1}) \\ &= R(x_2 + \boxed{2x_2} + \boxed{3x_2} + \boxed{4x_2}, \boxed{4x_2}) = 0\,. \end{aligned}$$

Therefore

$$x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1)$$

divides this LHS.

To compute the resulting factor we use the specialization $x_1 = x_2 = 1$. We then have

$$x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1) = 1\,,$$

and $S_{1111} = 28$, $S_{112} = -4$, $S_{13} = -1$, $S_4 = 1$. Hence the factor is

$$F_1^{(4)} = 1 \cdot 28 + 9 \cdot 4 + 26 \cdot (-1) + 24 \cdot 1 = -10\,, \tag{40}$$

and Eq. (38) is now proved.

On the other hand, the Schur function S_{22} satisfies Eqs. (36):

$$S_{22}(0) = S_{22}(x - \boxed{2x}) = S_{22}(x - \boxed{3x}) = S_{22}(x - \boxed{4x})$$

because the partition 22 is not contained in the $(1,1)$-hook. By the same reason, S_{22} satisfies Eq. (37) with its RHS replaced by zero:

$$S_{22}(x - \boxed{5x}) = 0\,.$$

Moreover, we have

$$S_{22}(\mathbb{X}_2 - \boxed{2x_1} - \boxed{2x_2}) = R(\mathbb{X}_2, \boxed{2x_1} + \boxed{2x_2}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1)\,. \tag{41}$$

Combining Eq. (38) with Eq. (41), the desired expression (34) follows.

We now pass to the second example: A_3 and $r = 2$. The Thom polynomial in this case was computed originally by Rimányi [36]. We shall now give its Schur function expansion. (It is easy to see that the Thom polynomial for A_3 and $r = 1$ is just equal to $F_1^{(3)}$.)

Since the singularities $\neq A_3$, whose codimension is $\leq \operatorname{codim}(A_3)$ are: A_0, A_1, A_2 and $III_{2,2}$ (cf. [6]), Theorem 1 yields the following equations characterizing $\mathcal{T}_2^{A_3}$, where b is a variable:

$$P(-b) = P(x - b - \boxed{2x}) = P(x - b - \boxed{3x}) = 0\,, \tag{42}$$

$$P(x - b - \boxed{4x}) = R(x + \boxed{2x} + \boxed{3x}, b + \boxed{4x}) \tag{43}$$

$$P(\mathbb{X}_2 - \mathbb{D}) = 0\,. \tag{44}$$

Here,

$$\mathbb{D} = \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2}\,.$$

By Corollary 11, the first four equations are satisfied by the function $F_2^{(3)}$. However $F_2^{(3)}$ does not satisfy the last vanishing, imposed by $III_{2,2}$. We shall "modify" $F_2^{(3)}$ in order to obtain the Thom polynomial for $A_3(2)$.

We claim that this Thom polynomial is equal to

$$S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33}\,, \tag{45}$$

and it differs from its 1-part $F_2^{(3)}$ by $5S_{33}$ which is its 2-part. Indeed, arguing similarly as in the previous example, we have

$$F_2^{(3)}(\mathbb{X}_2 - \mathbb{D}) = (-5)(x_1x_2)^2(x_1 - 2x_2)(x_2 - 2x_1)\,.$$

On the other hand, the Schur function S_{33} satisfies Eqs. (42):

$$S_{33}(0) = S_{33}(x - b - \boxed{2x}) = S_{33}(x - b - \boxed{3x}) = 0$$

because the partition 33 is not contained in the $(1,2)$-hook. By the same reason, S_{33} satisfies Eq. (43) with its RHS replaced by zero:

$$S_{33}(x - b - \boxed{4x}) = 0\,.$$

Moreover, we have

$$S_{33}(\mathbb{X}_2 - \mathbb{D}) = R(\mathbb{X}_2, \mathbb{D}) = (x_1x_2)^2(x_1 - 2x_2)(x_2 - 2x_1)\,. \tag{46}$$

Summing up, we get that the Thom polynomial for $A_3(2)$ has Schur function expansion (45) indeed.

In [31], we shall give a *parametric* Schur function expansion of the Thom polynomials for the singularities $A_3(r)$ with parameter $r \geq 1$.

Remark 15 Let $\mathrm{rank}(\mathcal{T}_r^{A_i})$ be the largest h such that there exists a nontrivial h-part in $\mathcal{T}_r^{A_i}$. By the results of the present paper, we have

- $\mathrm{rank}(\mathcal{T}_r^{A_i}) = 1$ for $i = 1, 2$ and any r;
- $\mathrm{rank}(\mathcal{T}_1^{A_3}) = 1$, $\mathrm{rank}(\mathcal{T}_2^{A_3}) = 2$, and $\mathrm{rank}(\mathcal{T}_1^{A_4}) = 2$.

Moreover, we have

- $\mathrm{rank}(\mathcal{T}_r^{A_3}) = 2$ for $r \geq 2$ ([30]);
- $\mathrm{rank}(\mathcal{T}_2^{A_4}) = 2$ ([36], [33]);
- $\mathrm{rank}(\mathcal{T}_r^{A_4}) = 2$ for $r = 3, 4$ ([25]).

Since $\mathrm{codim}(A_i(r)) = ir$, for $i \geq 2$ and $r \geq 1$, we clearly have

$$\mathrm{rank}(\mathcal{T}_r^{A_i}) \leq i - 1.$$

This invariant (also for other singularities) will be discussed in a subsequent paper.

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Notes 1. After the appearance of the first version [29] of the present paper, we received a letter from Kazarian [17] informing us that he has found another formula for $\mathcal{T}_r^{A_i}$ under the assumptions of Theorem 12, but modulo a certain ideal (cf. [18]).
2. As the referee points out, the Thom polynomials for Morin singularities have been recently also studied – using quite different methods – by Fehér and Rimányi in [9], and by Bérczi and Szenes in [3].

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