# On certain family of B-modules

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# Schur functors

**Issai Schur's dissertation (Berlin, 1901)**: classification of irreducible polynomial representations of $GL_n$:

Homomorphisms $GL_n \to GL_N$ sending $X$ to a matrix $[P_{ij}(X)]$, where $P_{ij}$ is a polynomial in the entries of $X$.

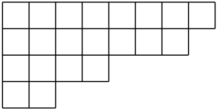
Two actions on $E^{\otimes n}$ ($E$ vector space over a field $K$ of char. 0).

– of the symmetric group $S_n$ via permutations of the factors,

– the diagonal action of $GL(E)$.

Irreductible representations $S^\lambda$ of the symmetric group $S_n$ are labeled by partitions of $n$.

*Partition* of $n$: $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ s.t. $\lambda_1 + \cdots + \lambda_k = n$. Graphical presentation for 8742:



*Schur module*:

$$V_\lambda(E) := \mathrm{Hom}_{\mathbb{Z}[S_n]}(S^\lambda, E^{\otimes n})$$

$V_\lambda(-)$ is a *functor*: if $E, F$ are $R$-modules, $f : E \to F$ is an $R$-homomorphism, then $f$ induces an $R$-homomorphism $V_\lambda(E) \to V_\lambda(F)$. In this way, we get all irreducible polynomial representations of $GL_n$.

Let us label the boxes of the diagram with $1, \ldots, n$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 15 | 19 | 3 | 10 | 5 | 21 | 13 |
| 11 | 8 | 18 | 9 | 6 | 17 | 4 | |
| 7 | 20 | 12 | 16 | | | | |
| 16 | 2 | | | | | | |

$P$:= sum of the permutations preserving the rows ($P \in \mathbb{Z}[S_n]$),

$N$:= sum of the permutations with their signs, preserving the columns.

$$e(\lambda) := N \circ P - \text{the Young idempotent};$$

$$V_\lambda(E) = e(\lambda) E^{\otimes n}.$$

Example: $V_{(n)}(E) = S^n(E)$, $V_{(1^n)}(E) = \wedge^n(E)$.

Let $T$ be the subgroup of diagonal matrices in $GL_n$:

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & 0 \\ & & x_3 & \\ & 0 & & \ddots \end{pmatrix}$$

Consider the action of $T$ on $V_\lambda(E)$ induced from the action of $GL_n$ via restriction.

**Main result of Schur's Thesis:**

*The trace of the action of $T$ on $V_\lambda(E)$ is equal to the Schur function:*

$$s_\lambda(x_1, \dots, x_n) = \det\left(s_{\lambda_p - p + q}(x_1, \dots, x_n)\right)_{1 \le p,q \le k},$$

where $s_i(x_1, \dots, x_n)$ is the $i$th complete symmetric function.

# Schubert polynomials

Permutation: bijection $\mathbb{N} \to \mathbb{N}$, which is the identity off a finite set.

$A := \mathbb{Z}[x_1, x_2, \ldots]$.

We define $\partial_i : A \to A$

$$\partial_i(f) := \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

For a simple reflection $s_i = 1, \ldots, i-1, i+1, i, i+2, \ldots$ , we put $\partial_{s_i} := \partial_i$.

Let $w = s_1 \cdots s_k = t_1 \cdots t_k$ be two reduced words of $w$. Then

$$\partial_{s_1} \circ \cdots \circ \partial_{s_k} = \partial_{t_1} \circ \cdots \circ \partial_{t_k}.$$

Thus for any permutation $w$, we can define $\partial_w$ as $\partial_{s_1} \circ \cdots \circ \partial_{s_k}$ independently of a reduced word of $w$. Let $n$ be a natural number such that $w(k) = k$ for $k > n$.

*Schubert polynomial* (Lascoux-Schützenberger 1982):

$$\mathfrak{S}_w := \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1 x_n^0)$$

where $w_0$ is the permutation $(n, n-1, \ldots, 2, 1), n+1, n+2, \ldots$.

We define the $k$th *inversion set* of $w$:

$$I_k(w) := \{l : l > k,\ w(k) > w(l)\} \quad k = 1, 2, \ldots$$

*Code* of $w$ ($c(w)$): sequence $i_k = |I_k(w)|$, $k = 1, 2, \ldots$.

$c(5,2,1,6,4,3,7,8,\ldots)$ is equal to $(4,1,0,2,1,0,\ldots)$.

– Schubert polynomial $\mathfrak{S}_w$ is symmetric in $x_k$ i $x_{k+1}$ if and only if $w(k) < w(k+1)$ (or equivalently if $i_k \le i_{k+1}$).

– If $w(1) < w(2) < \cdots < w(k) > w(k+1) < w(k+2) < \cdots$ (or $i_1 \le i_2 \le \cdots \le i_k$, $0 = i_{k+1} = i_{k+2} = \cdots$), then $\mathfrak{S}_w$ is equal to $s_{i_k,\ldots,i_2,i_1}(x_1,\ldots,x_k)$.

– If $i_1 \ge i_2 \ge \cdots$ , then $\mathfrak{S}_w = x_1^{i_1} x_2^{i_2} \cdots$ is a monomial.

If the sets $I_k(w)$ form a chain (w.r.t. inclusion), then $w$ is called a *vexillary* permutation.

## Theorem

(Lascoux-Schützenberger, Wachs) *If $w$ is a vexillary permutation with code $(i_1, i_2, \ldots, i_n > 0, 0 \ldots)$, then* $\mathfrak{S}_w = s_{(i_1,\ldots,i_n)^{\geq}}(\min I_1(w) - 1, \ldots, \min I_n(w) - 1)^{\leq}$.

*Flag Schur function:* For two sequences of natural numbers $i_1 \geq \cdots \geq i_k$ and $0 < b_1 \leq \cdots \leq b_k$,

$$s_{i_1,\ldots,i_k}(b_1, \ldots, b_k) := \det\left(s_{i_p - p + q}(x_1, \ldots, x_{b_p})\right)_{1 \leq p,q \leq k}$$

# Functors asked by Lascoux

$R$ – commutative $\mathbb{Q}$-algebra, $E_\cdot : E_1 \subset E_2 \subset \cdots$ a flag of $R$-modules. Suppose that $\mathcal{I} = [i_{k,l}]$, $k, l = 1, 2, \ldots$, is a matrix of 0's and 1's s.t.

– $i_{k,l} = 0$ for $k \geq l$;

– $\sum_l i_{k,l}$ is finite for any $k$;

– $\mathcal{I}$ has a finite number of nonzero rows.

Such a matrix $\mathcal{I}$ is called a *shape*:

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
 & & & & \ldots\ldots & & & &
\end{array}
=
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & & \times & 0 & \times & 0 \\
 & & & \times & 0 & \times \\
 & & & & \times & 0 \\
 & & & & & 0
\end{array}$$

*Shape of permutation* $w$ is the matrix:

$$\mathcal{I}_w = [i_{k,l}] := [\chi_k(l)]\,,\ \ k,l = 1,2,\ldots$$

where $\chi_k$ is the characteristic function of $I_k(w)$. For $w = 5,2,1,6,4,3,7,8,\ldots$, the shape $\mathcal{I}_w$ is equal to

$$\begin{array}{ccccc} \times & \times & 0 & \times & \times \\ & \times & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & \times & \times \\ & & & & \times \end{array}$$

We define a module $S_w(E.)$, associated with a permutation $w$ and a flag $E.$ as $S_{\mathcal{I}_w}(E.)$; this leads to a functor $S_w(-)$.

From now on, let $E.$ be a flag of $K$-vector spaces with $\dim E_i = i$ . Let $B$ be the Borel group of linear endomorphisms of $E := \bigcup E_i$, which preserve $E.$. The modules used in the definition of $S_w(E.)$ are $\mathbb{Z}[B]$-modules, and maps are homomorphisms of $\mathbb{Z}[B]$-modules. Let $\{u_i : i = 1, 2, \ldots\}$ be a basis of $E$ such that $u_1, u_2, \ldots, u_k$ span $E_k$. Then $S_w(E)$ as a cyclic $\mathbb{Z}[B]$-submodule in $\bigotimes_l \bigwedge^{\tilde{i}_l} E_l$, generated by the element

$$u_w := \otimes_l u_{k_{1,l}} \wedge u_{k_{2,l}} \wedge \cdots \wedge u_{k_{i_l,l}}$$

where $k_{1,l} < k_{2,l} < \cdots < k_{i_l,l}$ are precisely those indices for which $i_{k_{r,l},l} = 1$.

E.g. $S_{5,2,1,6,4,3,7,\ldots}(E.)$ is generated over $\mathbb{Z}[B]$ by

$$u_1 \otimes u_1 \wedge u_2 \otimes u_1 \wedge u_4 \otimes u_1 \wedge u_4 \wedge u_5 \,.$$

# Theorem

(K-P) *The trace of the action of a maximal torus $T \subset B$ on $S_w(E.)$ is equal to the Schubert polynomial $\mathfrak{S}_w$.*

About the proof: we study multiplicative properties of $S_w(E.)$.

$t_{p,q}(\dots w(p) \dots w(q) \dots) = (\dots w(q) \dots w(p) \dots)$

*Chevalley-Monk formula* for multiplication by $\mathfrak{S}_{s_k}$:

$$\mathfrak{S}_w \cdot (x_1 + \cdots + x_k) = \sum \mathfrak{S}_{w \circ t_{p,q}},$$

the sum over $p, q$ s.t. $p \leq k$, $q > k$ and $l(w \circ t_{p,q}) = l(w) + 1$. For example

$\mathfrak{S}_{246315879\dots} \cdot (x_1 + x_2) = \mathfrak{S}_{346215879\dots} + \mathfrak{S}_{264315879\dots} + \mathfrak{S}_{256314879\dots}$.

*Transition formula*: Let $(j, s)$ be a pair of positive integers s.t.

– $j < s$ and $w(j) > w(s)$,

– for any $i \in ]j, s[$, $w(i) \notin [w(s), w(j)]$,

– for any $r > j$, if $w(s) < w(r)$ then there exists $i \in ]j, r[$ s.t. $w(i) \in [w(s), w(r)]$.

Then $\quad \mathfrak{S}_w = \mathfrak{S}_v \cdot x_j + \sum_{p=1}^m \mathfrak{S}_{v_p}$ ,

where $v = w \circ t_{j,s}$, $v_p = w \circ t_{j,s} \circ t_{k_p,j}$ , the sum over $k_p$ s.t.

– $k_p < j$ and $w(k_p) < w(s)$,

– if $i \in ]k_p, j[$ then $w(i) \notin [w(k_p), w(s)]$.

Such a pair $(j, s)$ always exists for a nontrivial permutation: it suffices to take the maximal pair in the lexicographical order s.t. $w(j) > w(s)$.

E.g. $\mathfrak{S}_{521863479} =$

$= \mathfrak{S}_{521843679...} \cdot x_5 + \mathfrak{S}_{524813679...} + \mathfrak{S}_{541823679...}$

– maximal transition

$= \mathfrak{S}_{521763489...} \cdot x_4 + \mathfrak{S}_{527163489...} + \mathfrak{S}_{571263489...} + \mathfrak{S}_{721563489...}$

$= \mathfrak{S}_{512864379...} \cdot x_2.$

We prove that for the maximal transition, there exists a filtration of $\mathbb{Z}[B]$-modules

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \mathcal{F} = S_w(E.)$$

and isomorphisms $\mathcal{F}/\mathcal{F}_k \simeq S_v(E.) \otimes E_j/E_{j-1}$ and $\mathcal{F}_p/\mathcal{F}_{p-1} \simeq S_{v_p}(E.)$ for $p = 1, \ldots, m$. $\square$

There exist *flag Schur functors* $S_\lambda(-)$, for which we have

## Theorem

(K-P) *If* $w$ *is a vexillary permutation with code* $(i_1, i_2, \ldots, i_n > 0, 0 \ldots)$, *then* $S_w(E.) = S_{(i_1,\ldots,i_n)^{\geq}}\left(E_{\min I_1(w)-1}, \ldots, E_{\min I_n(w)-1}\right)^{\leq}$.

# Filtrations of weight modules

Let $\mathfrak{b}$ be the Lie algebra of $n \times n$ upper matrices, $\mathfrak{t}$ that of diagonal matrices, and $U(\mathfrak{b})$ the enveloping algebra of $\mathfrak{b}$.

$M$ a $U(\mathfrak{b})$-module, $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$,

$M_\lambda = \{m \in M : hm =< \lambda, h > m\}$ weight space of $\lambda$,
$< \lambda, h >= \sum \lambda_i h_i$

If $M$ is a direct sum of its weight spaces and each weight space has finite dimension, then $M$ is called a *weight module*

$ch(M) := \sum_\lambda \dim M_\lambda x^\lambda$, where $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$

Let $e_{ij}$ be the matrix with 1 at the $(i,j)$-position and 0 elsewhere.

Let $K_\lambda$ be a 1-dim'l $U(\mathfrak{b})$-module, where $h$ acts by $< \lambda, h >$ and the matrices $e_{ij}$, where $i < j$, acts by zero. Any finite dim'l weight module admits a filtration by these 1 dim'l modules.

$w \in S_\infty^{(n)} := \{w : w(n+1) < w(n+2) < ...\}$.

$E = \bigoplus_{1 \le i \le n} K u_i$.

For each $j \in \mathbb{N}$, let $\{i < j : w(i) > w(j)\} = \{i_1 < \ldots < i_{l_j}\}$
$u_w^{(j)} = u_{i_1} \wedge \cdots \wedge u_{i_{l_j}} \in \Lambda^{l_j} E$

$u_w = u_w^1 \otimes u_w^2 \otimes \cdots$

$S_w = U(\mathfrak{b}) u_w$ The weight of $u_w$ is $c(w)$ .

Thm (K-P) For any $w \in S_\infty^{(n)}$, $S_w$ is a weight module and $ch(S_w) = \mathfrak{S}_w$.

What is the annihilator of $u_w$?

$$1 \le i < j \le n \to m_{ij}(w) = \#\{k > j : w(i) < w(k) < w(j)\}$$

$e_{ij}^{m_{ij}+1}$ annihilates $u_w$.

Let $I_w \subset U(\mathfrak{b})$ be the ideal generated by $h- < c(w), h >$ , $h \in \mathfrak{t}$ and $e_{ij}^{m_{ij}(w)+1}$, $i < j$.

There exists $U(\mathfrak{b})/I_w \twoheadrightarrow S_w$ s.t. 1 mod $I_w \mapsto u_w$.

## Theorem

*(W) This surjection is an isomorphism.*

For $\lambda \in \mathbb{Z}^n_{\geq 0}$ we set $S_\lambda := S_w$ where $c(w) = \lambda$. For $\lambda \in \mathbb{Z}^n$ take $k$ s.t. $\lambda + k\mathbf{1} \in \mathbb{Z}^n_{\geq 0}$ ($\mathbf{1} = (1, \ldots, 1)$ $n$ times), and set $S_\lambda = K_{-k\mathbf{1}} \otimes S_{\lambda + k\mathbf{1}}$. Similarly for $\mathfrak{S}_\lambda$.

QUESTIONS: 1. When a weighted module admits a filtration with subquotients isomorphic to some $S_\lambda$'s?

2. Does $S_\lambda \otimes S_\mu$ have such a filtration?

$\rho = (n-1, n-2, \ldots, 2, 1, 0)$, $\quad K_\rho$ "dualizing module"

$\mathcal{C}$ category of all weight modules, for $\Lambda \subset \mathbb{Z}^n$, $\mathcal{C}_\Lambda$ is the full subcategory of $\mathcal{C}$ consisting of all weight modules whose weights are in $\Lambda$.

$|\Lambda| < \infty \quad \Lambda' = \{\rho - \lambda : \lambda \in \Lambda\} \quad C_{\Lambda'} \cong \mathcal{C}^{op}_{\Lambda} \quad M \mapsto M^* \otimes K_\rho$

Lemma For any $\Lambda \subset \mathbb{Z}^n$, $\mathcal{C}_\Lambda$ has enough projectives.

Orders: $w, v \in S_\infty$ $w \leq_{lex} v$ if $w = v$ or there exists $i > 0$ s.t. $w(j) = v(j)$ for $j < i$ and $w(i) < v(i)$.

For $\lambda \in \mathbb{Z}^n$, define $|\lambda| = \sum \lambda_i$. If $\lambda = c(w), \mu = c(v)$, we write $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $w^{-1} \leq_{lex} v^{-1}$. For general $\lambda, \mu \in \mathbb{Z}^n$ take $k$ s.t. $\lambda + k\mathbf{1}, \mu + k\mathbf{1} \in \mathbb{Z}^n_{\geq 0}$, and define $\lambda \geq \mu$ iff $\lambda + k\mathbf{1} \geq \mu + k\mathbf{1}$.

For $\lambda \in \mathbb{Z}^n$, set $\mathcal{C}_{\leq\lambda} := \mathcal{C}_{\{\nu : \nu \leq \lambda\}}$. All Ext's over $U(\mathfrak{b})$, in $\mathcal{C}_{\leq\lambda}$.

Prop. For $\lambda \in \mathbb{Z}^n$ the modules $S_\lambda$ and $S^*_{\rho-\lambda} \otimes K_\rho$ are in $\mathcal{C}_{\leq\lambda}$. Moreover $S_\lambda$ is projective and $S^*_{\rho-\lambda} \otimes K_\rho$ is injective.

## Theorem

*(W) For $\mu, \nu \leq \lambda$, $Ext^i(S_\mu, S^*_{\rho-\nu} \otimes K_\rho) = 0$, $i \geq 1$.*

# Theorem

*(W) Let* $M \in \mathcal{C}_{\leq\lambda}$*. If* $Ext^1(M, S^*_{\rho-\mu} \otimes K_\rho) = 0$ *for all* $\mu \leq \lambda$*, then* $M$ *has a filtration s.t. each of its subquotients is isomorphic to some* $S_\nu$ $(\nu \leq \lambda)$*.*

Cor. (1) If $M = M_1 \oplus \ldots \oplus M_r$, then $M$ has such a filtration iff each $M_i$ has.

(2) If $0 \to L \to M \to N \to 0$ is exact and $M, N$ have such filtrations, then $L$ also has.

Proof (1) $Ext^1(M, N) = \oplus Ext^1(M_i, N)$ for any $N$.

(2) $Ext^1(M, A) \to Ext^1(L, A) \to Ext^2(N, A)$ exact for any $A$.

Prop. $w \in S_\infty^{(n)}$, $1 \leq k \leq n-1$. Then $S_w \otimes S_{s_k}$ has such a filtration. (KP for $k=1$, W in general)

## Theorem

*(W) $S_w \otimes S_v$ has such a filtration for $w, v \in S_\infty^{(n)}$.*

Consider a $B$-module $T_w = \otimes_{2 \leq i \leq n} (\Lambda^{l_i(w)} K^{i-1})$.

$T_w$ is a direct sum component of $\otimes_{2 \leq i \leq n} S_{s_{i-1}} \otimes \cdots \otimes S_{s_{i-1}}$ , $l_i(w)$ times

Prop. $w \in S_n$. Then there is an exact sequence $0 \to S_w \to T_w \to N \to 0$, where $N$ has a filtration whose subquotients are $S_u$ with $u^{-1} >_{lex} w^{-1}$.

– from Cauchy formula $\prod_{i+j \leq n} (x_i + y_j) = \sum_w \mathfrak{S}_w(x) \mathfrak{S}_{ww_0}(y)$

## Proof of the thm

Exact sequence:

$$0 \quad \to \quad S_w \otimes S_v \quad \to \quad T_w \otimes S_v \quad \to \quad N \otimes S_v \quad \to \quad 0$$

filtr. by Cor.(2) filtr. by Prop. filtr. by ind. on lex($w$)

## Theorem

*(W) Let $\lambda \in \mathbb{Z}^n$ and $M \in \mathcal{C}_{\leq\lambda}$. Then we have*

$$ch(M) \leq \sum_{\nu\leq\lambda} \dim_K(\mathrm{Hom}_{\mathfrak{b}}(M, S_{\rho-\nu} \otimes K_\rho))\mathfrak{S}_\nu$$

*The equality holds if and only if $M$ has a filtration with all subquotients isomorphic to $S_\mu$, where $\mu \leq \lambda$.*

As a corollary, we get a formula for the coefficient of $\mathfrak{S}_w$ in $\mathfrak{S}_u\mathfrak{S}_v$:

Cor. This coefficient is equal to the dimension of

$$\mathrm{Hom}_{\mathfrak{b}}(S_u \otimes S_v, S_{w_0 w} \otimes K_\rho) = \mathrm{Hom}_{\mathfrak{b}}(S_u \otimes S_v \otimes S_{w_o w}, K_\rho).$$

Proof We use *ch*:

$$\mathfrak{S}_u\mathfrak{S}_v = ch(S_u \otimes S_v) = \sum_w (S_u \otimes S_v, S^*_{\rho-\lambda} \otimes K_\rho)\mathfrak{S}_w\,.$$

## Some plethysm

Let $s_\sigma$ denote the Schur functor associated to a partition $\sigma$

Prop. $s_\sigma(S_\lambda)$ has a filtration with its subquotients isomorphic to some $S_\nu$.

Proof $(S_\lambda)^{\otimes k}$ has such a filtration for any $\lambda$ and any $k$.

Hence $Ext^1((S_\lambda)^{\otimes k}, S_\nu^* \otimes K_\rho) = 0$ for any $\nu$.

$s_\sigma(S_\lambda)$ is a direct sum factor of $(S_\lambda)^{|\sigma|}$.

Hence $Ext^1(s_\sigma(S_\lambda), S_\nu^* \otimes K_\rho) = 0$ for any $\nu$, and $s_\sigma(S_\lambda)$ has the desired filtration.

Cor. If $\mathfrak{S}_w$ is a sum of monomials $x^\alpha + x^\beta + \cdots$, then $s_\sigma(x^\alpha, x^\beta, \ldots)$ is a positive sum of Schubert polynomials.

The End

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