Positivity in global singularity theory

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red herring: it was thought that $c_1^2 - 2c_2$ is positive but is not.

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Whenever we speak about the classes of algebraic cycles, we always mean their *Poincaré dual classes* in cohomology.

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If a *singularity class* Σ is "stable" (e.g. closed under the contact equivalence), then $\mathcal{T}^\Sigma$ depends on $c_i(TM - f^*TN)$.

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(R_m "parametrizes" TM for $\dim M = m$, similarly for R_n.)

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Well defined up to conjugacy; it can be chosen so that the images of its projections to the factors are *linear*. Its representations on the source and target will be denoted by

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Fix a singularity η. Assume that the number of singularities of codimension $\leq \operatorname{codim} \eta$ *is finite*. Suppose that the Euler classes of all singularities of smaller codimension than $\operatorname{codim}(\eta)$, are not zero-divisors. Then

(i) if $\xi \neq \eta$ and $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$, then $\mathcal{T}^{\eta}(c(\xi)) = 0$;

(ii) $\mathcal{T}^{\eta}(c(\eta)) = e(\eta)$.

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^{\eta}$ in a unique way.

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Since $\mathcal{J}(E,F) = F^N$ is ample, the latter polynomial is positive for ample v.b., so is a positive combination of Schur polynomials.

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Theorem. *(PP, 1988) Let η be of Thom-Boardman type $\Sigma^{i,\cdots}$. Then all summands in the Schur function expansion of $\mathcal{T}_r^{\eta}$ are indexed by partitions containing the rectangle partition $(r+i-1, \ldots, r+i-1)$ (i times).*

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Every germ of a Lagrangian submanifold of V is the image of W via a certain germ symplectomorphism.

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A Lagrange singularity class is any closed pure dimensional algebraic subset of $\mathcal{J}^k(V)$ which is invariant w.r.t. the action of H.

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Suppose that a general flag $V_\bullet : \; V_1 \subset V_2 \subset \cdots \subset V_n \subset V$ of isotropic subspaces with $\dim \; V_i = i$, is given.

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Suppose that a general flag $V_\bullet :\ V_1 \subset V_2 \subset \cdots \subset V_n \subset V$ of isotropic subspaces with $\dim\ V_i = i$, is given.
Given a strict partition $I \subset \rho$, i.e.
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Theorem. *(P, 1986)* $\Omega_I = \widetilde{Q}_I(R^*)$, *where* R *is the tautological subbundle on* $LG(V)$.

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Theorem. *(MM+PP+AW, 2007) For any Lagrange singularity class Σ, the Thom polynomial $\mathcal{T}^\Sigma$ is a nonnegative combination of $\widetilde{Q}$-functions.*

Proposition. *For a strict partition $I \subset \rho$, there exists only one strict partition $I' \subset \rho$ and $|I'| = \dim LG(V) - |I|$, for which $\widetilde{Q}_I(R^*) \cdot \Omega_{I'} \neq 0$. ($I'$ complements I in ρ).*

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Lemma. *Let $\pi : E \to X$ be a globally generated bundle on a proper homogeneous variety X. Let C be a cone in E, and let Z be any algebraic cycle in X of the complementary dimension. Then the intersection $[C] \cdot [Z]$ is nonnegative.*

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Lemma. *We have a natural isomorphism*

$$N_G \mathcal{J}^k \cong \bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(R^*)\,.$$

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Let

$$C = C_{G \cap \Sigma} \Sigma \subset N_G \mathcal{J}$$

be the *normal cone* of $G \cap \Sigma$ in Σ. Denote by $j : G \hookrightarrow N_G \mathcal{J}$ the zero-section inclusion.

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The Lagrangian Grassmannian $G = LG(V)$ is a homogeneous space with respect to the action of the symplectic group $Sp(V)$. The lemma applied to the bundle $N_G\mathcal{J} \to G$, entails $[C] \cdot \Omega_{I'}$ nonnegative.

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Get 2 types of submanifolds: *linear subspaces*, *the submanifolds which have the tangent space at the origin equal to W*; they are the graphs of the differentials of the functions $f : W \to \xi$ satisfying $df(0) = 0$ and $d^2f(0) = 0$

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Additionally, we assume that Σ is stable with respect to enlarging the dimension of W.

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Since any changes of coordinates of W and ξ induce holomorphic contactomorphisms of $V \oplus \xi$, any Legendre singularity class Σ defines $\Sigma(W,\xi) \subset \mathcal{J}^k(W,\xi)$.

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The Chern classes $a_i = c_i(A)$ generate the cohomology $H^*(LG(V,\omega),\mathbf{Z}) \cong H^*(\mathcal{J}^k(W,\xi),\mathbf{Z})$ as an algebra over $H^*(X,\mathbf{Z})$.

Let us fix an approximation of $BU(1) = \bigcup_{n \in \mathbf{N}} \mathbf{P}^n$, that is we set $X = \mathbf{P}^n$, $\xi = \mathcal{O}(1)$. Let $W = \mathbf{1}^n$ be the trivial bundle of rank n.

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and is often denoted by $\mathcal{T}^\Sigma$. It is written in terms of the generators a_i and $s = c_1(\xi)$.

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Fix two "opposite" standard isotropic flags in V:

$$F_h^+ := \bigoplus_{i=1}^{h} \alpha_i\,, \qquad F_h^- := \bigoplus_{i=1}^{h} \alpha_{n-i+1}^* \otimes \xi\,, \qquad (h = 1, 2, \ldots, n)$$

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Consider two Borel groups $B^\pm \subset Sp(V, \omega)$, preserving the flags $F_\bullet^\pm$. The orbits of $B^\pm$ in $LG(V, \omega)$ form two "opposite" cell decompositions $\{\Omega_I(F_\bullet^\pm, \xi)\}$ of $LG(V, \omega)$.

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$LG(V, \omega)$ admits two (relative) stratifications

$$\{\Omega_I(F_\bullet^{\pm}, \xi) \to X\}_I$$

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The classes of their closures give a basis of homology, called Z^--basis. Note that each $Z^-_{I\lambda}$ is transverse to each stratum $\Omega_J(F^+_\bullet, \xi)$, where $J \subset \rho$ is a strict partition.

Assume that $X = G/P$ is a compact manifold, homogeneous with respect to an action of a linear group G. Then X admits an algebraic cell decomposition $\{\sigma_\lambda\}$.

The subsets

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We pass now to a nonnegativity result on the Legendrian Thom polynomials and the Z^--decomposition.

Recall that our goal is to study cycles $\Sigma(W,\xi)$ in:

$$\mathcal{J} = \mathcal{J}^k(W,\xi) = \tau^* \left(\bigoplus_{i=3}^{k+1} \mathrm{Sym}^i(W^*) \otimes \xi \right).$$

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$$\begin{array}{ccccccc}
 & & & \pi & & \tau & \\
\Sigma(W,\xi) & \subset & \mathcal{J} & \twoheadrightarrow & LG(V,\omega) & \twoheadrightarrow & X \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\
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\end{array}$$

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Theorem. *Fix $I \subset \rho$ and λ. Suppose that the vector bundle $\mathcal{J}$ is globally generated. Then, in $\mathcal{J}$, the intersection of $\Sigma(W,\xi)$ with the closure of any $\pi^{-1}(Z^-_{I\lambda})$ is represented by a nonnegative cycle.*

We shall apply the Theorem in the situation when all α_i are equal to the same line bundle α (i.e. $W = \alpha^{\oplus n}$) and $\alpha^{-m} \otimes \xi$ is globally generated for $m \geq 3$.

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Consider the following three cases: the base is always $X = \mathbf{P}^n$ and

$$\xi_1 = \mathcal{O}(-2)\,, \qquad \alpha_1 = \mathcal{O}(-1)\,,$$

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We obtain symplectic bundles $V_i = \alpha_i^{\oplus n} \oplus (\alpha_i^* \otimes \xi_i)^{\oplus n}$ with twisted symplectic forms ω_i for $i = 1, 2, 3$.

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Some bases giving positivity properties in these 3 cases were known formerly.

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The space $LG(V, \omega)$ has a distinguished cell decomposition $Z^-_{I\lambda}$ where I runs over strict partitions contained in ρ, and $\lambda = (a, b)$ with a and b natural numbers $\leq n$.

The classes of closures of the cells of this decomposition give a basis of the homology of $LG(V,\omega)$. The dual basis of cohomology (in the sense of linear algebra) is denoted by

$$e_{I,a,b} = [\overline{Z^-_{I,a,b}}]^*.$$

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Theorem. *(MM+PP+AW 2010) Let* Σ *be a Legendre singularity class. Then* $[\Sigma(W,\xi)]$ *has nonnegative coefficients in the basis* $\{e_{I,a,b}\}$.

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Theorem. *(MM+PP+AW 2010) Let Σ be a Legendre singularity class. Then $[\Sigma(W,\xi)]$ has nonnegative coefficients in the basis $\{e_{I,a,b}\}$.*

The bundle $\mathcal{J}$ here is gg (hence desired intersections in $\mathcal{J}$ are nonnegative):

$$\tau^*\left(\bigoplus_{j=3}^{k+1} \operatorname{Sym}^j(W^*) \otimes \xi\right) =$$

$$\tau^*\left(\bigoplus_{j=3}^{k+1} \operatorname{Sym}^j(\mathbf{1}^n) \otimes p_1^*\mathcal{O}(j-3) \otimes p_2^*\mathcal{O}(1)\right).$$

Let Σ be a Legendre singularity class. Thom polynomial of Σ, evaluated at the Chern classes of $A = W^* \otimes \xi - R$ and $c_1(\xi) = v_2 - 3v_1$, is a nonnegative $\mathbf{Z}$-linear combination:

$$\mathcal{T}^\Sigma = \sum_{I,a,b} \gamma_{I,a,b} \, e_{I,a,b} = \sum_{I,a,b} \gamma_{I,a,b} [\Omega_I(F_\bullet^+, \xi)] v_1^a v_2^b \, .$$

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The family $[\Omega_I(F_\bullet^+,\xi)]\, t^i$ is a one-parameter family of bases depending on the parameter p/q.

Case 1. $\xi_1 = \mathcal{O}(-2), \alpha_1 = \mathcal{O}(-1)$. This corresponds to fixing the parameter to be 1; $p = 1$ and $q = 1$; $v_1 = v_2 = t$. Geometrically, this means that we study the restriction of the bundles W and ξ to the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$.

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In the next theorem A is a virtual bundle $W^* \otimes \xi - R$, and t is half the first Chern class of ξ^*.

Theorem. *The Thom polynomial of a Legendre singularity class Σ is a combination:*

$$\mathcal{T}^{\Sigma} = \sum_{j \geq 0} \sum_{I} \alpha_{I,j} \ \widetilde{Q}_I(A \otimes \xi^{-\frac{1}{2}}) \cdot t^j \,.$$

Here I runs over strict partitions in ρ, and $\alpha_{I,j}$ are nonnegative integers.

Legendrian vs. classical

$t = v_1 = v_2$

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$$Tp^\Sigma = \mathcal{T}^\Sigma \cdot c_n(T^*M \otimes f^*TC)\,.$$

We know that Tp^Σ is nonzero. One shows that Tp^Σ, specialized with $f^*TC = \mathbf{1}$ i.e. $t = 0$, is also nonzero. The assertion follows from the equation.

Green-Griffiths conjecture: Every projective algebraic variety of general type contains a proper subvariety $Y \subset X$ such that all nonconstant entire holomorphic curves $f : \mathbf{C} \to X$ must necessarily lie in Y.

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Theorem of Berczi: Assume that the Rimanyi conjecture holds. Then for a general hypersurface $X \subset \mathbf{P}^{n+1}$, the Green-Griffiths conjecture is true if $\deg(X) > n^6$.

THE END