Moduli of twisted K3 surfaces and cubic fourfolds

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Overview

Goal:
Construct moduli spaces of twisted K3 surfaces, suitable for studying their Hodge-theoretic relation with cubic fourfolds (everything over $\mathbb{C}$).

Plan:

1. Twisted K3 surfaces and their Hodge structures
2. (Twisted) K3 surfaces and cubic fourfolds
3. Non-existence result for moduli spaces
4. Modification of Brauer group & relation to cubic fourfolds
5. Construction of moduli spaces
6. Back to cubics once more.

Reminder on K3 surfaces

Definition

A *complex algebraic K3 surface* is a 2-dimensional smooth complete variety S over $\mathbb{C}$ such that $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ and $\Omega_S^2 \cong \mathcal{O}_S$.

E.g. quartic hypersurfaces in $\mathbb{P}^3$, Kummer surfaces, ...

Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \end{array}$$

Cohomology lattice: $\mathrm{H}^*(S, \mathbb{Z})$ + pairing $(\alpha, \beta) := \int_S \alpha \smile \beta \in 2\mathbb{Z}$, isomorphic to a fixed lattice of rank 24.

Picard group: The map $c_1 \colon \operatorname{Pic} S \to \mathrm{H}^2(S, \mathbb{Z})$ is injective. Hence, $\operatorname{Pic}(S) \cong \mathbb{Z}^{\oplus \rho(S)}$ with $1 \leq \rho(S) \leq 20$; all numbers occur.

A *polarization* on a K3 surface S is a primitive ample class $L \in \mathrm{H}^2(S, \mathbb{Z})$. Its *degree* is $d = (L, L) \in 2\mathbb{Z}$.

Moduli of polarized K3 surfaces

Let $\mathcal{M}_d$ be the moduli functor for polarized K3s of degree d:

$$\mathcal{M}_d \colon (\mathrm{Sch}/\mathbb{C})^{\mathrm{op}} \to \mathrm{Sets}$$
$$T \mapsto \{(f \colon S \to T, L)\}/\cong$$

with f a smooth proper morphism and $L \in \mathrm{H}^0(T, R^1 f_* \mathbb{G}_{\mathrm{m}})$ s.t. (S_t, L_t) is a polarized K3 of degree d for all $t \in T(\mathbb{C})$.

Facts:

- $\mathcal{M}_d$ has a coarse moduli space $\Phi \colon \mathcal{M}_d \to \mathrm{M}_d$. That is, $\Phi(\mathbb{C})$ is a bijection, and morphisms $\mathcal{M}_d \to T$ with T a finite type $\mathbb{C}$-scheme factor uniquely over M_d.
- M_d is an irreducible quasi-projective variety of dimension 19 with only finite quotient singularities.
- The locus of (S, L) with $\rho(S) \geq k$ is a countably infinite union of codim. $k-1$ subvarieties of M_d.

Twisted K3 surfaces

Definition

The (cohomological) *Brauer group* of a K3 surface S is

$$\mathrm{Br}(S) := \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m) \cong \mathrm{H}^2(S, \mathcal{O}_S^*)_{\text{tors}}.$$

A *twisted K3 surface* is a pair (S, α) with S K3 and $\alpha \in \mathrm{Br}(S)$.

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{NS}(S) \longrightarrow & \mathrm{H}^2(S,\mathbb{Z}) & \longrightarrow & \mathrm{H}^2(S,\mathcal{O}_S) & \xrightarrow{\exp} \mathrm{H}^2(S,\mathcal{O}_S^*) \longrightarrow 0 \\ & \cap & & \wr\| & \\ & \mathrm{H}^2(S,\mathbb{C}) & \xrightarrow{=} & \mathrm{H}^{0,2}(S) \oplus \mathrm{H}^{1,1}(S) \oplus \mathrm{H}^{2,0}(S) & \end{array}$$

Any $\alpha \in \mathrm{Br}(S)$ equals $\exp(B^{0,2})$ for some $B \in \mathrm{H}^2(S, \mathbb{Q})$, unique up to $\mathrm{H}^2(S, \mathbb{Z})$ and $\mathrm{NS}(S) \otimes \mathbb{Q}$. This is called a *B-field lift* of α.

Let $T(S) := \mathrm{NS}(S)^\perp \subset \mathrm{H}^2(S, \mathbb{Z})$. There is an isomorphism

$$\begin{aligned} \mathrm{Br}(S) &\cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}), \;\; \alpha \mapsto (B, -) \\ &\cong (\mathbb{Q}/\mathbb{Z})^{\oplus 22-\rho(S)} \end{aligned}$$

Twisted Hodge structure

Hodge structure $\widetilde{\mathrm{H}}(S, B, \mathbb{Z})$ of K3 type on $\mathrm{H}^*(S, \mathbb{Z})$:

$$\widetilde{\mathrm{H}}^{2,0}(S, B) := \mathbb{C}[\exp(B)\sigma] \subset \mathrm{H}^*(S, \mathbb{C}),$$

σ nowhere degenerate holomorphic 2-form; $\exp(B)\sigma := \sigma + B \wedge \sigma$.
Independent of lift $B \Rightarrow$ define $\widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}) := \widetilde{\mathrm{H}}(S, B, \mathbb{Z})$.

The *Picard group* of (S, α) is

$$\mathrm{Pic}(S, \alpha) := \widetilde{\mathrm{H}}^{1,1}(S, \alpha) \cap \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z})$$

and its *transcendental lattice*:

$$T(S, \alpha) := \mathrm{Pic}(S, \alpha)^{\perp} \subset \widetilde{\mathrm{H}}(S, \alpha, \mathbb{Z}).$$

When α is trivial, we have $\mathrm{Pic}(S, \alpha) = (\mathrm{H}^0 \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^4)(S, \mathbb{Z})$ and $T(S, \alpha) = T(S)$. In general, there is an isomorphism of lattices

$$T(S, \alpha) \cong \ker(\alpha \colon T(S) \to \mathbb{Q}/\mathbb{Z}),$$

in fact $T(S, \alpha) = \exp(B) \ker(\alpha)$.

K3 surfaces and cubic fourfolds

Let $X \subset \mathbb{P}^5_{\mathbb{C}}$ be a smooth cubic fourfold. Let $h^2 \in \mathrm{H}^4(X,\mathbb{Z})$ be the square of the hyperplane class.

Definition

X is *special* if X contains a surface T whose cohomology class is not a multiple of h^2. Let $K_d := \operatorname{sat}\langle h^2, T\rangle$ (here $d = \operatorname{disc} K_d$).

Hassett (2000): Special cubics form an infinite union of irreducible divisors $\mathcal{C}_d$ in the 20-dimensional moduli space $\mathcal{C}$ of cubic fourfolds. Here $d \equiv 0, 2 \mod 6$ and $d > 6$.

Definition

A polarized K3 surface (S, L) is *associated* to $X \in \mathcal{C}_d$ if

$$\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}} := L^{\perp} \subset \mathrm{H}^2(S,\mathbb{Z})$$

is Hodge isometric to $K_d^{\perp} \subset \mathrm{H}^4(X,\mathbb{Z})$, up to a sign and a Tate twist. Then $\deg L = d$.

Theorem (Hassett)

A cubic X has an associated K3 surface if and only if $X \in \mathcal{C}_d$ with
$(**)$ *d is even and not divisible by 4, 9, or any odd prime $p \equiv 2(3)$.*

Conjecture X is rational $\Leftrightarrow$ X has an associated K3 surface.

Huybrechts: associated *twisted* K3 surfaces (based on Addington–Thomas). When $\rho(S) = 1$, so $\operatorname{Pic} S = \mathbb{Z}L$ and $T(S) = \mathrm{H}^2(S, \mathbb{Z})_{\mathrm{pr}}$:

(S, α) is associated to $X \in \mathcal{C}_{d'}$ if $T(S, \alpha)$ is Hodge isometric to $K_{d'}^{\perp} \subset \mathrm{H}^4(X, \mathbb{Z})$, up to a sign and a Tate twist. Then $d' = \deg(L) \cdot \operatorname{ord}(\alpha)^2$.

X has an associated twisted K3 if and only if $X \in \mathcal{C}_{d'}$ with
$(**')$ $d' = dr^2$ for some integers d and r, where d satisfies $(**)$.

K3s and cubics in families

Hassett: When $(**)$ holds, there is a surjective rational map $\varphi\colon \mathrm{M}_d \dashrightarrow \mathcal{C}_d$ so that (S, L) is associated to $\varphi(S, L)$. It has degree 1 when $d \equiv 2\,(6)$ and degree 2 when $d \equiv 0\,(6)$.

Goal: generalize this to twisted K3 surfaces.

First step: generalize M_d.

- A *polarized twisted K3 surface* of degree d and order r is a triple (S, L, α) with S a K3 surface, L a polarization on S of degree d and $\alpha \in \mathrm{Br}(S)$ of order r.
- (S, L, α) and (S', L', α') are isomorphic if there is an isomorphism $f\colon S \to S'$ such that $f^*L' = L$ and $f^*\alpha' = \alpha$.

Claim: There is no coarse moduli space for polarized twisted K3 surfaces of fixed degree and order which is a locally noetherian scheme.

Non-existence result for moduli spaces

Consider the moduli functor

$$\begin{aligned} \mathcal{N}_d[r] \colon (\mathsf{Sch}/\mathbb{C})^{\mathrm{op}} &\to \mathsf{Sets} \\ T &\mapsto \{(f \colon S \to T, L, \alpha)\}/\cong \end{aligned}$$

where $(f \colon S \to T, L)$ is a family of polarized K3s of degree d and $\alpha \in \mathrm{H}^0(T, R^2 f_* \mathbb{G}_{\mathrm{m}})$ s.t. $\alpha_t \in \mathrm{H}^2(S_t, \mathbb{G}_{\mathrm{m}})[r]$ for $t \in T(\mathbb{C})$.

Suppose there is a coarse moduli space $\mathcal{N}_d[r] \to \mathrm{N}_d[r]$. Consider

$$\begin{array}{ccc} (S \to T, L, \alpha) & \quad & \mathcal{N}_d[r] \longrightarrow \mathrm{N}_d[r] \\ \downarrow \xi(T) & & \xi \downarrow \qquad\qquad \downarrow \exists! \pi \\ (S \to T, L) & & \mathcal{M}_d \longrightarrow \mathrm{M}_d \end{array}$$

For $y \in \mathrm{N}_d[r](\mathbb{C})$ corresponding to (S, L, α), $x = \pi(y)$ corresponds to (S, L). Hence,

$$(\mathrm{N}_d[r])_x = \{(S, L, \alpha) \mid \alpha \in \mathrm{Br}(S)[r]\}/\operatorname{Aut}(S, L).$$

Non-existence result for moduli spaces

- If $d > 2$, let $U \subset \mathrm{M}_d$ be the open subset where $\mathrm{Aut}(S, L)$ is trivial. Then for $x \in U$,

$$\begin{aligned}(\mathrm{N}_d[r])_x &= \mathrm{Br}(S)[r] \\ &\cong \mathrm{Hom}(T(S), \tfrac{1}{r}\mathbb{Z}/\mathbb{Z}) \cong (\mathbb{Z}/r\mathbb{Z})^{\oplus 22-\rho(S)}.\end{aligned}$$

  Hence $\pi|_{\mathrm{N}_d[r]\times_{\mathrm{M}_d} U}$ is ramified over the locus where $\rho(S) > 1$. This is not closed, so $\mathrm{N}_d[r]$ is not locally noetherian.
- When $d = 2$, let $U \subset M_2$ be the open subset where $\mathrm{Aut}(S, L) = \mathbb{Z}/2\mathbb{Z}$. Then $\pi|_{\mathrm{N}_d[r]\times_{\mathrm{M}_d} U}$ is ramified over the locus where $\rho(S) > 2$, which is not closed.

By a similar argument, there is no locally noetherian coarse moduli space for polarized K3 surfaces of degree d and order exactly r.

Modification of Brauer group

- Recall: $\mathrm{Br}(S) \cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus 22-\rho(S)}$.
- Define: $\widetilde{\mathrm{Br}}(S) := \mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}, \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus 21}$.

Have a surjection $\pi_{\mathrm{Br}} \colon \widetilde{\mathrm{Br}}(S) \to \mathrm{Br}(S)$, isomorphism iff $\rho(S) = 1$.

Theorem 1 (B.)

There exists a coarse moduli space M^r_d for triples (S, L, α) with (S, L) a polarized K3 surface of deg. d and $\alpha \in \widetilde{\mathrm{Br}}(S)$ of order r, which is a finite disjoint union of 19-dimensional irreducible quasi-projective varieties with at most finite quotient singularities.

The Kummer sequence induces a map $q \colon \mathrm{H}^1(S, \mathbb{G}_{\mathrm{m}}) \to \mathrm{H}^2(S, \mu_r)$. One can show that $\widetilde{\mathrm{Br}}(S)[r] \cong \mathrm{H}^2(S, \mu_r)/q(L)$.

Compare Bragg (2019): moduli stack $\mathcal{B}^r_d$ of triples (S, L, α) with S K3, L ample line bundle on S of deg. d and $\alpha \in \mathrm{H}^2(S, \mu_r)$. If B^r_d is its coarse moduli space, M^r_d is an open subspace of $\mathrm{B}^r_d \otimes \mathbb{C}$.

Relation with cubic fourfolds

Take α in

$$\widetilde{\mathrm{Br}}(S)[r] = \mathrm{Hom}(\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}, \tfrac{1}{r}\mathbb{Z}/\mathbb{Z}) \cong \tfrac{1}{r}\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}^{\vee} / \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}^{\vee}.$$

Let $w \in \frac{1}{r}\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{pr}}^{\vee} \subset \mathrm{H}^2(S,\mathbb{Q})$ be a representative of α; this is a B-field lift of $\pi_{\mathrm{Br}}(\alpha)$. We set $\widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z}) := \widetilde{\mathrm{H}}(S,w,\mathbb{Z})$.

Recall:

$(**')$ $d' = dr^2$ for some integers d and r, where d satisfies $(**)$.

Theorem 2 (B.)

Condition $(**')$ holds if and only if for all $X \in \mathcal{C}_{d'}$ and all d, r as in $(**')$, there is a triple (S, L, α) with (S, L) a polarized K3 of degree d and $\alpha \in \widetilde{\mathrm{Br}}(S)$ of order r, and a Hodge isometry

$$\mathrm{H}^4(X,\mathbb{Z}) \supset K_{d'}^{\perp} \cong \exp(w)\ker(\alpha) \subset \widetilde{\mathrm{H}}(S,\alpha,\mathbb{Z})$$

up to a sign and a Tate twist.

Construction of M^r_d

Recall the construction of M_d:

- For any polarized K3 (S, L) of degree d, $H^2(S, \mathbb{Z})_{pr}$ is isomorphic to a fixed lattice Λ_d. An isomorphism $\varphi\colon H^2(S, \mathbb{Z})_{pr} \to \Lambda_d$ is called a *marking*.
- There is a fine (analytic) moduli space M^{mar}_d of *marked polarized K3 surfaces* (S, L, φ) of degree d. It is an open submanifold of

$$\mathcal{D}(\Lambda_d) := \{x \in \mathbb{P}(\Lambda_d \otimes \mathbb{C}) \mid (x)^2 = 0,\ (x, \overline{x}) > 0\}.$$

- M_d is the quotient of M^{mar}_d by $\widetilde{O}(\Lambda_d) = \ker(O(\Lambda_d) \to \operatorname{Disc}\Lambda_d)$. It is a Zariski open subset of $\mathcal{D}(\Lambda_d)/\widetilde{O}(\Lambda_d)$, a quasi-projective variety (Bailey–Borel).

Note: a marking $\varphi\colon H^2(S, \mathbb{Z})_{pr} \to \Lambda_d$ induces an isomorphism $\widetilde{Br}(S) = \operatorname{Hom}(H^2(S, \mathbb{Z})_{pr}, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(\Lambda_d, \mathbb{Q}/\mathbb{Z})$.

Construction of M_d^r

Define the following spaces:

- $M_d^{mar}[r] := M_d^{mar} \times \mathrm{Hom}(\Lambda_d, \frac{1}{r}\mathbb{Z}/\mathbb{Z})$. This is a coarse moduli space for tuples (S, L, φ, α) with (S, L, φ) a marked polarized K3 of degree d and $\alpha \in \widetilde{\mathrm{Br}}(S)[r]$.
- $M_d[r] := M_d^{mar}[r]/\widetilde{\mathrm{O}}(\Lambda_d)$. This is a coarse moduli space for triples (S, L, α) with $(S, L) \in M_d$ and $\alpha \in \widetilde{\mathrm{Br}}(S)[r]$.

The space $M_d[r]$ is a finite disjoint union of components

$$M_v := \left(M_d^{mar} \times \{v\}\right) / \mathrm{Stab}(v)$$

where $v \in \mathrm{Hom}(\Lambda_d, \frac{1}{r}\mathbb{Z}/\mathbb{Z})$ and $\mathrm{Stab}(v) \subset \widetilde{\mathrm{O}}(\Lambda_d)$ is the stabilizer of v. Each M_v is a 19-dimensional irreducible quasi-projective variety with at most finite quotient singularities.

- Let M_d^r be the union of the components M_v where $v \in \mathrm{Hom}(\Lambda_d, \frac{1}{r}\mathbb{Z}/\mathbb{Z})$ has order r. □

Analogue of Hassett's map

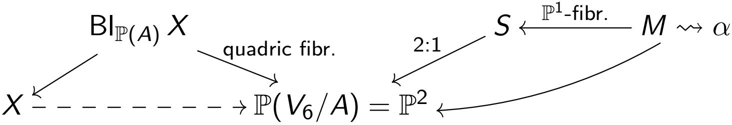
One can show that d' satisfies $(**')$ if and only if there exist

- $v \in \mathrm{Hom}(\Lambda_d, \frac{1}{r}\mathbb{Z}/\mathbb{Z})$
- a finite cover $\widetilde{\mathrm{M}}_v \xrightarrow{f} \mathrm{M}_v$
- a surjective rational map $\varphi\colon \widetilde{\mathrm{M}}_v \dashrightarrow \mathcal{C}_{d'}$ (degree 1 or 2)

such that $f(S, L, \widetilde{\alpha})$ is associated to $\varphi(S, L, \widetilde{\alpha})$.

Example: $d' = 8$, so $d = r = 2$. The space M_2^2 has 3 components. For one of them, M_{v_0}, there exist f and φ as above; f is an isomorphism and φ is birational. So get $\varphi\colon \mathrm{M}_{v_0} \overset{1:1}{\dashrightarrow} \mathcal{C}_8$.

If $X \in \mathcal{C}_8$, then $X \subset \mathbb{P}(V_6)$ contains a plane $\mathbb{P}(A)$. For X generic:



Then (S, α) is associated to X (Kuznetsov).