Factorization centers, Cremona groups and the Grothendieck ring of varieties

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based on joint work with
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Motivation 1/3: associated curves to rational threefolds

- If $X \to Y$ is a birational map, we call centers that are blown up and blown **factorization centers**
- If X is a rational threefold, then there is a birational map $\mathbb{P}^3 \to X$ blowing up a curve C (or several curves)
- Example: $X = Q \cap Q' \subset \mathbb{P}^5$ smooth intersection of two quadrics, then it's rational if X has a line, and the curve C is a genus two curve in $\mathbb{P}^3$
- In general, is C determined by X?
  - Yes, if $k = \overline{k}$ (Clemens-Griffiths: intermediate Jacobian)
  - No, in general over perfect fields!
- Same question for rational surfaces over perfect fields, or in higher dimension over $\mathbb{C}$.
- E.g. associated K3 surfaces to rational cubic fourfolds?

Motivation 2/3: structure of the Cremona groups

The n-th Cremona group is $\mathrm{Cr}_n(k) = \mathrm{Bir}(\mathbb{P}^n_k)$.

Theorem (Noether)

Cremona group $\mathrm{Cr}_2(\mathbb{C})$ *is generated by* $\mathrm{PGL}_3(\mathbb{C})$ *and the Cremona involution* $[X : Y : Z] \mapsto [YZ : XZ : XY]$.

In general Cremona groups are big and complicated. Using MMP and Birkar's boundedness, Blanc-Lamy-Zimmermann recently constructed infinitely many homomorphisms $\mathrm{Cr}_n(k) \to \mathbb{Z}/2$ ($n \geq 3$).

Dolgachev's question

- Which Cremona groups are generated by involutions?
- Considered by Déserti and Blanc-Lamy-Zimmermann

This is not known even for $n = 2$ and most nonclosed fields, in spite of the fully understood links, generators and relations in $\mathrm{Cr}_2(k)$.

For $\mathrm{Cr}_n(\mathbb{C})$, $n \geq 3$ no set of generators is explicitly presented.

Motivation 3/3: structure of the Grothendieck ring

k field.

$K_0(\mathrm{Var}/k)$, the Grothendieck ring of varieties:

- generators: $[X]$ for X/k
- relations: $[X] = [Z] + [X \setminus Z]$ for all closed $Z \subset X$
- product $[X] \cdot [Y] = [X \times_k Y]$
- $\mathbb{L} = [\mathbb{A}^1]$

Some open questions:

1. X, Y smooth projective, $[X] = [Y] \overset{?}{\Longrightarrow} X, Y$ birational?
   (If $\mathrm{char}(k) = 0$, Larsent-Lunts: X, Y stably birational.)
   If X, Y smooth nonprojective, this is false (Borisov).
2. Describe $\mathrm{Ann}(\mathbb{L}^n)$ and L-equivalence $\mathbb{L}^n([X] - [Y]) = 0$. (Exists for non-stably birational Calabi-Yau, K3 surfaces, genus one curves; no general description known.)
3. Does $K_0(\mathrm{Var}/k)$ contain torsion?

Plan of this talk

1. Definition of the invariant $c(\phi)$
2. Factorization centers for surfaces, threefolds and fourfolds, with applications to the Grothendieck ring
3. Applications to Cremona groups

1. Definition of the invariant

Groupoid of birational types

Let Bir/k be the groupoid of birational types, that is:

- Objects = (smooth) projective varieties
- Morphisms = birational isomorphisms

Two different classical ways to generate all morphisms in Bir/k:

1. Weak Factorization in char. 0 [Abramovich-Karu-Matsuki-Wlodarczyk]: blow ups with smooth centers and their inverses
2. Minimal Model Program [Iskovskikh, Sarkisov, Reid, Corti, Hacon-McKernan]: Sarkisov links of type I, II, III, IV

The invariant $c(\phi)$ defined

- Consider $\phi : X \dashrightarrow Y$ birational isomorphism of smooth projective n-dimensional varieties
- Let $\mathrm{Ex}(\phi)$ be the set of exceptional divisors of ϕ
- Define

$$c(\phi) = \sum_{E' \in \mathrm{Ex}(\phi^{-1})} [E'] \; - \sum_{E \in \mathrm{Ex}(\phi)} [E] \in \mathbb{Z}[\mathrm{Bir}_{n-1}],$$

  that is c counts the birational types of divisors created by ϕ minus those contracted by ϕ. (Note: $c(\phi^{-1}) = -c(\phi)$.)
- For example, if $\phi : \widetilde{X} \to X$ is a smooth blow up with connected center Z of codimension c, then

$$c(\phi) = -[\mathbb{P}^{c-1} \times Z], \;\; c(\phi^{-1}) = [\mathbb{P}^{c-1} \times Z]$$

- If ϕ is a biregular isomorphism, or at least an isomorphism in codimension one, then $c(\phi) = 0$

Key Lemma

For any birational isomorphisms $\phi : X \dashrightarrow X'$, $\psi : X' \dashrightarrow X''$ we have

$$c(\psi \circ \phi) = c(\psi) + c(\phi).$$

In particular we have a group homomorphism

$$c : \mathrm{Bir}(X) \to \mathbb{Z}[\mathrm{Bir}_{n-1}].$$

Proof: Express $\mathrm{Ex}(\psi \circ \phi)$ and $\mathrm{Ex}((\psi \circ \phi)^{-1})$ in terms of $\mathrm{Ex}(\phi)$, $\mathrm{Ex}(\psi)$, $\mathrm{Ex}(\phi^{-1})$, $\mathrm{Ex}(\psi^{-1})$.

Remark

Exceptional divisors are ruled, so that the image of c belongs to $\mathrm{Bir}_{n-2} \stackrel{\times \mathbb{P}^1}{\to} \mathrm{Bir}_{n-1}$, however these maps are not injective (stable birationality $\neq$ birationality). To simplify the bookkeeping we sometimes consider the composition

$$c : \mathrm{Bir}(X, Y) \to \mathbb{Z}[\mathrm{Bir}_{n-1}] \to \mathbb{Z}[\mathrm{StBir}].$$

Simple examples

Example

Standard Cremona transformation

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, [X : Y : Z] \mapsto [YZ : XZ : XY]$$

can be decomposed as a blow up of 3 points followed by blow down of 3 lines so $c(\phi) = 3 - 3 = 0$. Here $1 \in \mathrm{StBir}$ is the stable birational class of a point.

More generally, let X, Y be smooth projective surfaces over **algebraically closed** field k. Then for any $\phi : X \dashrightarrow Y$ $c(\phi)$ is simply the number of points blown up minus the number of divisors contracted, so that

$$c(\phi) = \operatorname{rk} \mathrm{NS}(Y) - \operatorname{rk} \mathrm{NS}(X),$$

in particular c is zero for self-maps, and no ambiguity takes place.

2. Factorization centers for surfaces, threefolds and fourfolds, with applications to Grothendieck ring

Surfaces over perfect fields

For surfaces over a perfect fields, exceptional divisors are all $\mathbb{P}^1 \times Z$, where Z is a (uniquely determined) smooth connected zero-dimensional scheme, so we can think of

$$c : \mathrm{Bir}(X, Y) \to \mathbb{Z}[\mathrm{Var}^0/k],$$

where Var^0/k is the monoid of isomorphism classes of zero-dimensional smooth varieties.

Theorem

For any birational k-surfaces X, Y, the value $c(\phi)$ is independent of the choice of $\phi \in \mathrm{Bir}(X, Y)$.

Corollary

For $X = Y$ we get a zero map $c : \mathrm{Bir}(X) \to \mathbb{Z}[\mathrm{Var}^0/k]$. Explicitly this means that for any choice of factorization of $\phi \in \mathrm{Bir}(X)$ into blow ups and blow downs, centers of blow ups match the centers of blow downs up to isomorphism.

Proving the result for surfaces

k perfect field.

Proof

- We need to show that for any $\phi : X \to Y$, birational map of surfaces, $c(\phi)$ depends only on X and Y, not on ϕ
- Use MMP for surfaces over perfect field
- Easy cases: K_X nef, or X ruled over a curve of positive genus
- Interesting case: geometrically rational surfaces
- Manin-Iskovskikh: all such surfaces are del Pezzos and conic bundles
- Iskovskikh: all birational maps between such surfaces are compositions of Sarkisov links
- Make a claim and check all types of links
- Use some cohomological tricks (Grothendieck ring, étale cohomology, Gassmann triples) to reduce the number of links to check to just four, e.g. $dP_6 - dP_4 - dP_6$.

Consequences for the Grothendieck ring

k field, $d \geq 0$. Then $K_0(\mathrm{Var}^{\leq d}/k)$ defined by:

- generators: $[X]$ for X/k, $\dim(X) \leq d$
- relations: $[X] = [Z] + [X \setminus Z]$ for all closed $Z \subset X$
- not a ring, but has partially defined product

Example 0. $K_0(\mathrm{Var}^{\leq 0}/k) = \mathbb{Z}[\mathrm{Var}^0/k]$,
Example 1. $K_0(\mathrm{Var}^{\leq 1}/k) = \mathbb{Z}[\mathrm{Var}^0/k] \oplus \mathbb{Z}[\mathrm{SmProjCurves}/k]$.

We have (in general non-injective!) maps $K_0(\mathrm{Var}^{\leq d}/k) \to K_0(\mathrm{Var}^{\leq d+1}/k)$ and $K_0(\mathrm{Var}/k)$ is the colimit.

Corollary (equivalent to the result about $c(\phi)$ for surfaces)

For a perfect field k we have

$$0 \to K_0(\mathrm{Var}^{\leq 1}/k) \to K_0(\mathrm{Var}^{\leq 2}/k) \to \mathbb{Z}[\mathrm{Bir}_2/k] \to 0.$$

In particular, $K_0(\mathrm{Var}^{\leq 2}/k)$ is torsion-free, equality means birationality and L-equivalence is trivial; also $K_0(\mathrm{Var}^{\leq 2}/k) \to K_0(\mathrm{Var}/k)$ non-injective.

Factorization centers for complex threefolds

Proposition

If X is a smooth projective complex threefold, then $c : \mathrm{Bir}(X) \to \mathbb{Z}[\mathrm{Bir}_2/\mathbb{C}]$ is a zero map.

Proof

- Use intermediate Jacobian $J(X)$ (Clemens-Griffiths) and weak factorization: only need to consider blow ups of smooth connected curves; blowing up a smooth curve C replaces $J(X)$ by $J(X) \times J(C)$
- Decompositions of polarized abelian varieties into products of indecomposables are unique (Debarre); all $J(C)$ are indecomposable.
- Torelli theorem recovers C from $J(C)$; thus we have cancellation for curves of genus $g > 0$; plus a bit more work for points and $\mathbb{P}^1$'s blown up

Consequence: full control over $K_0(\mathrm{Var}^{\leq 3}/\mathbb{C})$.

Factorization centers for threefolds over perfect fields

What is the same as for $k = \mathbb{C}$

- Weak factorization works (in char. 0)
- Intermediate Jacobian $J(X)$ is defined [Deligne, Murre, Achter–Casalaina-Martin–Vial]
- Torelli Theorem holds for curves with $g \geq 2$ [Serre]

What is different

- Blown up curves of genus zero (conics) and genus one may contribute nontrivially to $c(\phi)$, $\phi \in \mathrm{Bir}(X)$
- No known examples with curves of genus zero
- **Only one known example** for curves of genus one (L-equivalent and D-equivalent elliptic quintics: next slide)
- No full classification of Sarkisov links known which played a key role in dimension two
- Structure of $\mathrm{K}_0(\mathrm{Var}^{\leq 3}/k)$ unclear; $\mathbb{L}$ is a zero-divisor

Elliptic quintics as factorization centers for threefolds

Let k be a nonclosed field which admits a normal degree 5 genus one curve $C \subset \mathbb{P}^4$ with no rational points, e.g. $k = \mathbb{Q}$.

Let $C' = \mathrm{Pic}^2(C)$, this is another elliptic quintic, and $C_{\overline{k}} \simeq C'_{\overline{k}}$, but over k, C' is **not isomorphic** to C (possibly except the case $j(C) = 1728$).

Proposition

For every C there exists $\phi \in Cr_3(k)$ such that $c(\phi) = [C] - [C']$ (and $[C] \neq [C']$ when C, C' are non-isomorphic)

Proof

- If $\gamma : Q^3 \dashrightarrow \mathbb{P}^3$ is the standard projection, then $c(\gamma) = 0$
- There is the Mori-Mukai link $\psi : Q^3 - 24 - \mathbb{P}^3$ with $c(\psi) = [C] - [C']$
- Thus for $\phi \in \mathrm{Cr}_3(k)$, $\phi = \gamma \circ \psi$, $c(\phi) = [C] - [C']$

Studying the image of c is a very interesting problem, but this concrete simple nonvanishing already has applications to Cremona groups.

More about elliptic quintics and L-equivalence

A related notion to absence of cancellation for factorization centers is L-equivalence. Smooth projective varieties X, Y of non-negative Kodaira dimension are called **L-equivalent** if

$$\mathbb{L}^n \cdot ([Y] - [X]) = 0$$

in the Grothendieck ring of varieties for some $n \geq 1$ (here $\mathbb{L} = [\mathbb{A}^1]$).

Known examples include some Calabi-Yaus [starting with Borisov] some K3 surfaces and elliptic quintics C, C'; they satisfy $\mathbb{L}^4([C] - [C']) = 0$ [Shinder–Zhang 2019]. All these pairs are also derived equivalent, but no general implication is currently known.

Construction on the previous slide is improving Shinder–Zhang:

$$[Q^3] + \mathbb{L}[C] = [\mathbb{P}^3] + \mathbb{L}[C'] \implies \mathbb{L}([C] - [C']) = 0,$$

(because $[\mathbb{P}^3] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 = [Q^3]$).

Factorization centers in dimension 4

Let X be a K3 surface of degree 12 and Picard rank one. Such an X exists over large enough number fields, and clearly over $k = \mathbb{C}$. Then X has a single derived equivalent partner, a K3 surface Y of the same type with $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

- Hassett–Lai, Ito-Miura-Okawa-Ueda: X and Y are L-equivalent. Construction of Hassett–Lai:

$$\begin{tikzcd} & & W \arrow[dl, "\mathrm{Bl}_X"'] \arrow[dr, "\mathrm{Bl}_Y"] & & \\ & \mathrm{Bl}_3(\mathbb{P}^4) \arrow[dl] & & \mathrm{Bl}_3(\mathbb{P}^4) \arrow[dr] & \\ \mathbb{P}^4 \arrow[rrrr, dashed, "\phi"] & & & & \mathbb{P}^4 \end{tikzcd}$$

- We see that $c(\phi) = [X] - [Y] \neq 0$. and $\mathbb{L}([X] - [Y]) = 0$ [Hassett-Lai]

L-equivalence of K3 surfaces and relation to nonvanishing of c, and to birational geometry of cubic fourfolds is very interesting.

Factorization centers in dimension ≥ 4

Let $\phi \in \mathrm{Cr}_4(k)$ be as in the Hassett-Lai diagram.

For each $j \geq 0$ we consider

$$\phi \times \mathrm{id}_{\mathbb{P}^j} \in \mathrm{Bir}(\mathbb{P}^4 \times \mathbb{P}^j) \simeq \mathrm{Cr}_{4+j}(k)$$

It follows from definitions that

$$c(\phi \times \mathrm{id}_{\mathbb{P}^j}) = [\mathbb{P}^j \times X] - [\mathbb{P}^j \times Y] \neq 0,$$

so $c : \mathrm{Cr}_n(k) \to \mathbb{Z}[\mathrm{StBir}]$ is nonzero for all $n \geq 4$.

There are also more interesting examples of higher nonvanishing, such as $\psi \in \mathrm{Cr}_5(k)$ with $c(\psi) = [T] - [T']$, a pair of derived equivalent and L-equivalent Calabi-Yau threefolds from the G_2 Grassmannian roof of Ito-Miura-Okawa-Ueda.

3. Applications to Cremona groups

Many homomorphisms $Cr_n(k) \to \mathbb{Z}$ ($n \geq 4$)

Let k be a large enough field as above to ensure existence of a K3 surface of degree 12 and Picard rank one. Recall that $\mathrm{Cr}_n(k) = \mathrm{Bir}(\mathbb{P}^n_k)$.

The Hasset-Lai construction depends on a family of parameters (K3 surface of degree 12 and 3 points on it), hence:

Corollary

For all $n \geq 4$ we get a surjective homomorphism

$$\mathrm{Cr}_n(k) \to \bigoplus_I \mathbb{Z},$$

where cardinality of I is $|k|$. As a consequence $\mathrm{Cr}_n(k)^{ab}$ admits $\bigoplus_I \mathbb{Z}$ as a direct summand.

$\mathrm{Cr}_2(\mathbb{C})^{ab} = 0$ (Noether), $\mathrm{Cr}_2(\mathbb{R})^{ab} = \oplus_I \mathbb{Z}/2$ (Zimmermann).
$\mathrm{Cr}_n(k)^{ab} \supset \oplus_I \mathbb{Z}/2$ for $n \geq 3$ (Blanc-Lamy-Zimmermann).

Regularizable birational automorphisms

Definition

An element $\phi \in \mathrm{Cr}_n(k)$ is regularizable if $\phi = \alpha^{-1}\gamma\alpha$ where

- $\alpha : \mathbb{P}^n \dashrightarrow X$ birational isomorphism
- $\gamma \in \mathrm{Aut}(X)$

All elements of finite order are regularizable, but there exist non regularizable elements.

Cheltsov's question (2003)

Which groups of birational automorphisms are generated by regularizable elements?

There are some positive results, mostly for surfaces and threefolds.

Negative answer to Dolgachev's and Cheltsov's questions

Corollary (of the nonvanishing results for c)

$Cr_n(k)$ is not generated by regularizable elements (in particular, by $\mathrm{PGL}_{n+1}(k)$ and any collection of elements of finite order, in particular, involutions) in each of the following cases:

1. *$n = 3$ and some nonclosed fields, e.g. $k = \mathbb{Q}$*
2. *$n \geq 4$, and most fields, e.g. $k = \mathbb{C}$*

(The $Cr_3(\mathbb{C})$ case is open.)

Proof

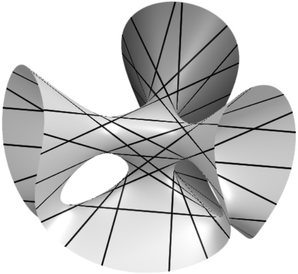
For every regularizable element

$$c(\phi) = c(\alpha^{-1}) + c(\gamma) + c(\alpha) = -c(\alpha) + 0 + c(\alpha) = 0,$$

so c is zero on the subgroup generated by regularizable elements.

Conclusion

- The invariant c records factorization centers, or exceptional divisors, for blow ups and blow downs for birational maps
- c is zero for surfaces over perfect fields and complex threefolds
- c is nonzero in more general contexts which has applications to our understanding of Cremona groups
- Studying the image of c is a worthwhile task, useful in rationality problems, understanding the Grothendieck ring, L-equivalence and applications to Cremona groups
- In this talk I concentrated on $c : \mathrm{Cr}_n(k) \to \mathbb{Z}[\mathrm{StBir}]$, however similar results will hold for $\mathrm{Bir}(X)$ for nonrational X ($X = \mathbb{P}^n$ is the most interesting and difficult case)
- This is work in progress; one dream would be to bound the image of c, using higher-dimensional MMP, Sarkisov links and possibly Birkar's boundedness



THE END / KONIEC

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