A Gysin formula for Hall-Littlewood polynomials

Piotr Pragacz*
Institute of Mathematics, Polish Academy of Sciences
Sniadeckich 8, 00-956 Warszawa, Poland
P.Pragacz@impan.pl

To Bill Fulton on his 75th birthday

Abstract

We give a formula for pushing forward the classes of Hall-Littlewood polynomials in Grassmann bundles, generalizing Gysin formulas for Schur $S$- and $P$-functions.

Let $E \to X$ be a vector bundle of rank $n$ over a nonsingular variety $X$ over an algebraically closed field of any characteristic. Let $\pi : G^qE \to X$ be the Grassmann bundle parametrizing rank $q$ quotients of $E$. A goal of this note is to give a formula for the image via $\pi_*$ of Hall-Littlewood classes from the Grassmann bundle. This formula generalizes some Gysin formulas for Schur $S$- and $P$-functions. In particular, it generalizes the formula in [5, Prop. 1.3(ii)], and provides an explanation of its intriguing coefficient. We refer to [1] for all unexplained here notions and notation from algebraic geometry.

Let $\tau_E : Fl(E) \to X$ be the flag bundle parametrizing flags of quotients of $E$ of ranks $n, n-1, \ldots, 1$. Suppose that $x_1, \ldots, x_n$ is a sequence of the Chern roots of $E$. Let $t$ be an indeterminate. The main formula will be located in $A(X)[t]$ or in $H^*(X, \mathbb{Z})[t]$ for a complex variety $X$.

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we define

$$R_\lambda(E; t) = (\tau_E)_*(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j}(x_i - tx_j)). \tag{1}$$

The Grassmann bundle $\pi : G^q(E) \to X$ is endowed with a tautological sequence of vector bundles

$$0 \to S \to \pi^*E \to Q \to 0,$$

where $\text{rank}(Q) = q$. Let $r = n - q$ be the rank of $S$. Suppose that $x_1, \ldots, x_q$ are the Chern roots of $Q$ and $x_{q+1}, \ldots, x_n$ are the ones of $S$.

2010 Mathematics Subject Classification. 14C17, 14M15, 05E05.

Keywords. push-forward of a cycle, Grassmann bundle, flag bundle, Hall-Littlewood polynomial, Schur $P$-function.

*Research is supported by a MNiSzW grant.
Proposition 1. For sequences $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$ of non-negative integers, we have

$$
\pi_*(R_\lambda(Q;t)R_\mu(S;t) \prod_{i \leq q < j} (x_i - tx_j)) = R_{\lambda \mu}(E;t),
$$

where $\lambda \mu = (\lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_r)$ is the juxtaposition of $\lambda$ and $\mu$.

Proof. Consider a commutative diagram

$$
\begin{array}{ccc}
F(S) \times G_\mu(E) & \xrightarrow{\pi} & F(E) \\
\tau_Q \times \tau_S & \downarrow & \downarrow \tau = \tau_E \\
G_\mu(E) & \xrightarrow{\pi} & X
\end{array}
$$

It follows that

$$
\pi_*(\tau_Q \times \tau_S)_* = \tau_*.
$$

(2)

Using Eq.(1) for $Q$ and $S$ and Eq.(2), we obtain

$$
\pi_*(R_\lambda(Q;t)R_\mu(S;t) \prod_{i \leq q < j} (x_i - tx_j)) = \pi_*(\tau_Q)_* (x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \leq q} (x_i - tx_j)) \cdot (\tau_S)_* (x_n^{\mu_n} \prod_{q < i < j} (x_i - tx_j)) \prod_{i \leq q < j} (x_i - tx_j).
$$

$$
= \pi_*(\tau_Q \times \tau_S)_* (x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \leq q} (x_i - tx_j)) x_{q+1}^{\mu_{q+1}} \cdots x_n^{\mu_n} \prod_{q < i < j} (x_i - tx_j) \prod_{i \leq q < j} (x_i - tx_j).
$$

$$
= \tau_*(x_1^{\lambda_1} \cdots x_q^{\lambda_q} x_{q+1}^{\mu_{q+1}} \cdots x_n^{\mu_n} \prod_{i < j} (x_i - tx_j)).
$$

$$
= R_{\lambda \mu}(E;t).
$$

In the argument above, we have used the following equality:

$$
\prod_{i < j \leq q} (x_i - tx_j) \prod_{q < i < j} (x_i - tx_j) \prod_{i \leq q < j} (x_i - tx_j) = \prod_{i < j} (x_i - tx_j).
$$

(3)

We now set

$$
v_m(t) = \prod_{i=1}^m \frac{1 - t^i}{1 - t} = (1 + t)(1 + t + t^2) \cdots (1 + t + \cdots + t^{m-1}).
$$

(3)

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Consider the maximal intervals $I_1, \ldots, I_d$ in $\{1, \ldots, n\}$, where the sequence $\lambda$ is constant. For $i \in \{1, \ldots, n\}$, we write $n(i)$ for the number of the interval containing $i$. Let $m_1, \ldots, m_d$ be the lengths of the intervals $I_1, \ldots, I_d$. So we have $m_1 + \cdots + m_d = n$. We set

$$
v_\lambda(t) = \prod_{i=1}^d v_{m_i}(t).
$$

(4)
Let $S_n$ be the symmetric group of permutations of $\{1, \ldots, n\}$. We define a subgroup $S_\lambda^n$ of $S_n$ as the stabilizer of $\lambda$. Of course, $S_\lambda^n = \prod_{i=1}^{d} S_{m_i}$.

Finally, we associate to a sequence $\lambda$ a $(d-1)$-step flag bundle (with steps of lengths $m_i$) $\eta_\lambda : Fl_\lambda(E) \to X$, parametrizing flags of quotients of $E$ of ranks $n - m_1, n - m_1 - m_2, \ldots, n - m_1 - m_2 - \cdots - m_2$.

**Example 2.** Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition (see [4, I,1,Ex.9]) with $k \leq n$. Let $\lambda = \nu_0 n - k$ be the sequence $\nu$ with $n - k$ zeros added at the end. Then $d = k + 1$, $(m_1, \ldots, m_d) = (1^k, n - k)$, $v_\lambda(t) = v_{n-k}(t)$, $S_\lambda^n = S_1^k \times S_{n-k}$, and $\eta_\lambda : Fl_\lambda(E) \to X$ is the flag bundle, often denoted by $r_{E}^k$, parametrizing quotients of $E$ of ranks $k, k - 1, \ldots, 1$.

We extend the definition of $P_\lambda(E; t)$ (loc.cit.) from partitions to sequences of nonnegative integers $\lambda$; we set $P_\lambda(E; t) = \frac{1}{v_\lambda(t)} R_\lambda(E; t)$.

Let us record the following particular case.
Corollary 5. Let $\nu$ be a strict partition with length $k \leq n$. Let $\lambda = \nu^{n-k}$. We have

$$P_\lambda(E; t) = (\tau_E)_{\nu^k} \left( x_1^{\nu_1} \cdots x_k^{\nu_k} \prod_{i<j, i \leq k} (x_i + tx_j) \right).$$

As a consequence of Propositions 1 and 3 and Eq.(5), we obtain the following result.

Theorem 6. Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be sequences of non-negative integers. Then we have

$$\pi_* \left( \prod_{i \leq q} (x_i - tx_j)P_\lambda(Q; t)P_\mu(S; t) \right) = \frac{v_{\lambda\mu}(t)}{v_{\lambda}(t)v_{\mu}(t)} P_{\lambda\mu}(E; t).$$

We first consider the specialization $t = 0$.

Example 7. We recall Schur $S$-functions. Let $s_i(E)$ denotes the $i$th complete symmetric function in the roots $x_1, \ldots, x_n$, given by

$$\sum_{i \geq 0} s_i(E) = \prod_{j=1}^n \frac{1}{1-x_j}.$$

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, we define

$$s_\lambda(E) = \left| s_{\lambda_i-i+j}(E) \right|_{1 \leq i, j \leq n}.$$

(See also [4, I.3].) Equivalently,

$$s_\lambda(E) = (\tau_E)_{\nu^\lambda}(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}).$$

We see that $P_\lambda(E; t) = s_\lambda(E)$ for $t = 0$. Under this specialization, the theorem becomes

$$\pi_* \left( (x_1 \cdots x_q)^r s_\lambda(Q) \cdot s_\mu(S) \right) = \pi_* \left( s_{\lambda_i+r, \ldots, \lambda_i+r}(Q) \cdot s_\mu(R) \right) = s_{\lambda\mu}(E),$$

a result obtained originally in [3, Prop. p. 196] and [2, Prop. 1].

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition $\mu$. One can rearrange $\lambda$ by a sequence of operations $(\ldots, i, j, \ldots) \rightarrow (\ldots, j-1, i+1, \ldots)$ applied to pairs of successive integers. Either one arrives at a sequence of the form $(\ldots, i, i+1, \ldots)$, in which case $s_\lambda(E) = 0$, or one arrives in $d$ steps at a partition $\mu$, and then $s_\lambda(E) = (-1)^d s_\mu(E)$.

Corollary 8. Let $\nu$ and $\sigma$ be strict partitions of lengths $k \leq q$ and $h \leq r$. It follows from Eq.(3) that

$$\frac{v_{\nu\nu^{q-k}\sigma\sigma^{r-h}}(t)}{v_{\nu^{q-k}}(t)v_{\sigma^{r-h}}(t)} = \begin{bmatrix} n-k-h \\ q-h \end{bmatrix}(t),$$

the Gaussian polynomial. Thus the theorem applied to the sequences $\lambda = \nu^{q-k}$ and $\mu = \sigma^{r-h}$ yields the following equation:

$$\pi_* \left( \prod_{i \leq q} (x_i - tx_j)P_\nu(Q; t)P_\sigma(S; t) \right) = \begin{bmatrix} n-k-h \\ q-h \end{bmatrix}(t) \cdot P_{\nu\sigma}(E; t). \quad (6)$$
We need the following property of Gaussian polynomials, which should be known but we know no precise reference.

**Lemma 9.** The Gaussian polynomial

\[
\left[ \begin{array}{c} a + b \\ a \end{array} \right](t)
\]

specialized at \( t = -1 \) is zero if \( a \cdot b \) is odd, and is equal to the binomial coefficient

\[
\left( \left[ \frac{(a + b)/2}{a/2} \right] \right)
\]

otherwise.

**Proof.** We have

\[
\left[ \begin{array}{c} a + b \\ a \end{array} \right](t) = \frac{(1 - t)(1 - t^2) \cdots (1 - t^{a + b})}{(1 - t)(1 - t^a)(1 - t) \cdots (1 - t^b)}.
\]

Since \( t = -1 \) is a zero with multiplicity 1 of the factor \( (1 - t^d) \) for even \( d \), and a zero with multiplicity 0 for odd \( d \), the order of the rational function \( \left[ \begin{array}{c} a + b \\ a \end{array} \right](t) \) at \( t = -1 \) is equal to

\[
\left( \left[ \frac{(a + b)/2}{a/2} \right] \right) - \left( \left[ \frac{a/2}{2} \right] \right) - \left( \left[ \frac{b/2}{2} \right] \right).
\]

The order (7) is equal to 1 when \( a \) and \( b \) are odd, and 0 otherwise. In the former case, we get the claimed vanishing, and in the latter one, the product of the factors with even exponents is equal to

\[
\left[ \begin{array}{c} \left[ \frac{(a + b)/2}{a/2} \right] \\ a/2 \end{array} \right](t^2).
\]

The value of this function at \( t = -1 \) is equal to \( \left[ \begin{array}{c} \left[ \frac{(a + b)/2}{a/2} \right] \\ a/2 \end{array} \right](1) \) which is the binomial coefficient

\[
\left( \left[ \frac{(a + b)/2}{a/2} \right] \right).
\]

This is the requested value since the remaining factors with an odd exponent give 2 in the numerator and the same number in the denominator.

The assertions of the lemma follow. \( \Box \)

We now consider the specialization \( t = -1 \).

**Example 10.** Consider Schur \( P \)-functions \( P_\lambda(E) = P_\lambda \) defined as follows. For a strict partition \( \lambda = (\lambda_1 > \ldots > \lambda_k) \) with odd \( k \),

\[
P_\lambda = P_{\lambda_1} \cdot P_{\lambda_2} \cdots \lambda_k - P_{\lambda_2} \cdot P_{\lambda_1, \lambda_3} \cdots \lambda_k + \cdots + P_{\lambda_k} \cdot P_{\lambda_1, \ldots, \lambda_{k-1}},
\]

and with even \( k \),

\[
P_\lambda = P_{\lambda_1, \lambda_2} \cdot P_{\lambda_3, \ldots, \lambda_k} - P_{\lambda_1, \lambda_3} \cdot P_{\lambda_2, \lambda_4, \ldots, \lambda_k} + \cdots + P_{\lambda_1, \lambda_k} \cdot P_{\lambda_2, \ldots, \lambda_{k-1}}.
\]

Here, \( P_i = \sum s_\mu \), the sum over all hook partitions \( \mu \) of \( i \), and for positive \( i > j \) we set

\[
P_{i,j} = P_i \cdot P_j + 2 \sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j}.
\]
Equivalently, for a strict partition $\lambda$ of length $k$, we have

$$P_{\lambda}(E) = (r_{E}^{k})_{*}\left(x_{1}^{\lambda_{1}} \cdots x_{k}^{\lambda_{k}} \prod_{i<j, i \leq k} (x_{i} + x_{j})\right).$$

By Corollary 5, we see that $P_{\lambda}(E) = P_{\lambda}(E; t)$ for $t = -1$.

We now use the notation from Corollary 8. Specializing $t = -1$ in Eq.(6), we get by Lemma 9

$$\pi_{*}(c_{qr}(Q \otimes S) \cdot P_{\nu}(Q) \cdot P_{\sigma}(S)) = d_{\nu, \sigma} \cdot P_{\nu \sigma}(E),$$

where $d_{\nu, \sigma} = 0$ if $(q - k)(r - h)$ is odd and

$$d_{\nu, \sigma} = \left(\frac{[(n - k - h)/2]}{[(q - k)/2]}\right)$$

otherwise. This result was obtained originally in [5, Prop. 3.1(ii)] in a different way. The present approach gives an explanation of the intriguing coefficient $d_{\nu, \sigma}$.

Suppose that $\lambda = (\lambda_{1}, \ldots, \lambda_{k})$ is not a strict partition. If there are repetitions of elements in $\lambda$, then the right-hand side is zero; if not then $P_{\lambda} = (-1)^{l}P_{\mu}$, where $l$ is the length of the permutation which rearranges $(\lambda_{1}, \ldots, \lambda_{k})$ into the corresponding strict partition $\mu$.

We thank Witold Kraśkiewicz and Anders Thorup for helpful discussions.

References


