A Gysin formula for Hall-Littlewood polynomials

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To Bill Fulton on his 75th birthday

Abstract

We give a formula for pushing forward the classes of Hall-Littlewood polynomials in Grassmann bundles, generalizing Gysin formulas for Schur S- and P-functions.

Let $E \to X$ be a vector bundle of rank n over a nonsingular variety X over an algebraically closed field. Denote by $\pi : G^q(E) \to X$ the Grassmann bundle parametrizing rank q quotients of E. Let $\pi_* : A(G^q(E)) \to A(X)$ be the homomorphism of the Chow groups of algebraic cycles modulo rational equivalence, induced by pushing-forward cycles (see [3, Chap. 1]). There exists an analogous map of cohomology groups. A goal of this note is to give a formula (see Theorem 7) for the image via π_* of Hall-Littlewood classes from the Grassmann bundle.

Hall-Littlewood polynomials appeared implicitly in Hall's study [5] of the combinatorial lattice structure of finite abelian p-groups, and explicitly in the work of Littlewood on some problems of representation theory [8]. A detailed account of the theory of Hall-Littlewood functions is given in [9].

The formula in Theorem 7 generalizes some Gysin formulas for Schur S- and P-functions. In particular, it generalizes the formula in [11, Prop. 1.3(ii)], and provides an explanation of its intriguing coefficient. We refer to [4] for general information about the appearance of Schur S- and Q-functions in cohomological studies of algebraic varieties.

Let t be an indeterminate. The main formula will be located in $A(X)[t]$, or in the extension $H^*(X, \mathbf{Z})[t]$ of the cohomology ring for a complex variety X. Let $\tau_E : Fl(E) \to X$ be the flag bundle parametrizing flags of quotients of E of ranks $n, n-1, \ldots, 1$. Suppose that $x_1, \ldots, x_n$ is a sequence of the Chern roots of E. For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we define

$$R_\lambda(E;t) = (\tau_E)_*\big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j}(x_i - t x_j)\big)\,, \tag{1}$$

2010 *Mathematics Subject Classification.* Primary 14C17, 14M15, 05E05.

Keywords. push-forward of a cycle, Grassmann bundle, flag bundle, Hall-Littlewood polynomial, Schur P-function.

*This work was supported by National Science Center (NCN) grant No. 2014/13/B/ST1/00133.

where $(\tau_E)_*$ acts on each coefficient of the polynomial in t separately. (The same convention will be used for other flag bundles.)

The Grassmann bundle $\pi : G^q(E) \to X$ is endowed with the tautological exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0\,,$$

where $\mathrm{rank}(Q) = q$. Let $r = n - q$ be the rank of S. Suppose that $x_1 \ldots, x_q$ are the Chern roots of Q and $x_{q+1}, \ldots, x_n$ are the ones of S.

Proposition 1. *For sequences $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$ of non-negative integers, we have*

$$\pi_*\big(R_\lambda(Q;t) R_\mu(S;t) \prod_{i \le q < j} (x_i - t x_j)\big) = R_{\lambda\mu}(E;t),$$

where $\lambda\mu = (\lambda_1, \ldots, \lambda_q, \mu_1, \ldots, \mu_r)$ is the juxtaposition of λ and μ.

Proof. Consider a commutative diagram

$$\begin{tikzcd} Fl(Q) \times_{G^q(E)} Fl(S) \arrow[r, "\cong"] \arrow[d, "\tau_Q \times \tau_S"'] & Fl(E) \arrow[d, "\tau = \tau_E"] \\ G^q(E) \arrow[r, "\pi"'] & X \end{tikzcd}$$

It follows that

$$\pi_* (\tau_Q \times \tau_S)_* = \tau_*\,. \tag{2}$$

Using Eq.(1) for Q and S and Eq.(2), we obtain

$$\begin{aligned}
&\pi_*\big(R_\lambda(Q;t) R_\mu(S;t) \prod_{i \le q < j} (x_i - t x_j)\big) \\
&= \pi_*\big((\tau_Q)_*\big(x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \le q} (x_i - t x_j)\big) \cdot (\tau_S)_*\big(x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{q < i < j} (x_i - t x_j)\big) \prod_{i \le q < j} (x_i - t x_j)\big) \\
&= \pi_* (\tau_Q \times \tau_S)_* \Big(x_1^{\lambda_1} \cdots x_q^{\lambda_q} \prod_{i < j \le q} (x_i - t x_j) x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{q < i < j} (x_i - t x_j) \prod_{i \le q < j} (x_i - t x_j)\Big) \\
&= \tau_* (x_1^{\lambda_1} \cdots x_q^{\lambda_q} x_{q+1}^{\mu_1} \cdots x_n^{\mu_r} \prod_{i<j} (x_i - t x_j)) \\
&= R_{\lambda\mu}(E;t)\,.
\end{aligned}$$

In the argument above, we have used the following equality:

$$\prod_{i < j \le q} (x_i - t x_j) \prod_{q < i < j} (x_i - t x_j) \prod_{i \le q < j} (x_i - t x_j) = \prod_{i<j} (x_i - t x_j)\,. \ \square$$

We now set

$$v_m(t) = \prod_{i=1}^{m} \frac{1 - t^i}{1 - t} = (1+t)(1+t+t^2) \cdots (1 + t + \cdots + t^{m-1}). \tag{3}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a sequence of nonnegative integers. Consider the maximal subsets I_1,...,I_d in $\{1, \ldots, n\}$, where the sequence λ is constant. Let $m_1, \ldots, m_d$ be the cardinalities of I_1,...,I_d. So we have $m_1 + \cdots + m_d = n$. We set

$$v_\lambda(t) = \prod_{i=1}^{d} v_{m_i}(t)\,. \tag{4}$$

Let S_n be the symmetric group of permutations of $\{1, \ldots, n\}$. We define a subgroup S_n^λ of S_n as the stabilizer of λ. Of course,

$$S_n^\lambda = \prod_{i=1}^{d} S_{m_i}\,.$$

Finally, we associate to a sequence λ a $(d-1)$-step flag bundle (with steps of lengths m_i)

$$\eta_\lambda : Fl_\lambda(E) \to X\,,$$

parametrizing flags of quotients of E of ranks

$$n - m_d, n - m_d - m_{d-1}, \ldots, n - m_d - m_{d-1} - \cdots - m_2\,. \tag{5}$$

Example 2. Let $\nu = (\nu_1 > \ldots > \nu_k > 0)$ be a strict partition (see [9, I,1,Ex.9]) with $k \le n$. Let $\lambda = \nu 0^{n-k}$ be the sequence ν with $n-k$ zeros added at the end. Then $d = k+1$, $(m_1, \ldots, m_d) = (1^k, n-k)$, $v_\lambda(t) = v_{n-k}(t)$, $S_n^\lambda = (S_1)^k \times S_{n-k}$, and $\eta_\lambda : Fl_\lambda(E) \to X$ is the flag bundle, often denoted by τ_E^k, parametrizing quotients of E of ranks $k, k-1, \ldots, 1$.

If $\lambda = (a^p b^{n-p})$, then $d = 2$, $(m_1, m_2) = (p, n-p)$, $v_\lambda(t) = v_p(t) v_{n-p}(t)$, $S_n^\lambda = S_p \times S_{n-p}$, and η_λ is here the Grassmann bundle $\pi : G^p(E) \to X$.

We shall now need some results from [9, III]. Let $y_1, \ldots, y_n$ and t be independent indeterminates. We record the following equation from [9, III, (1.4)]:

Lemma 3. *We have*

$$\sum_{w \in S_n} w\Big(\prod_{i<j} \frac{y_i - t y_j}{y_i - y_j}\Big) = v_n(t)\,.$$

For a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we define

$$R_\lambda(y_1, \ldots, y_n; t) = \sum_{w \in S_n} w\Big(y_1^{\lambda_1} \cdots y^{\lambda_n} \prod_{i<j} \frac{y_i - t y_j}{y_i - y_j}\Big)$$

Arguing as in [9, III (1.5)], we show with the help of Lemma 3 the following result.

Proposition 4. *The polynomial $v_\lambda(t)$ divides $R_\lambda(y_1, \ldots, y_n; t)$, and we have*

$$R_\lambda(y_1, \ldots, y_n; t) = v_\lambda(t) \sum_{w \in S_n / S_n^\lambda} w\Big(y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j, \lambda_i \neq \lambda_j} \frac{y_i - t y_j}{y_i - y_j}\Big)\,.$$

Let us invoke the following description of the Gysin map for the flag bundle $\eta_\lambda : Fl_\lambda(E) \to X$ with the help of a symmetrizing operator. Recall that $A(Fl_\lambda(E))$ as an $A(X)$-module is generated by S_n^λ-invariant polynomials in the Chern roots of E (see [1, Thm 5.5]). We define for an S_n^λ-invariant polynomial $f = f(y_1, \ldots, y_n)$,

$$\partial_\lambda(f) = \sum_{w \in S_n/S_n^\lambda} w\Big(\frac{f(y_1, \ldots, y_n)}{\prod_{i<j, \lambda_i \neq \lambda_j}(y_i - y_j)}\Big).$$

The following result is a particular case of [2, Prop. 2.1] (in the situation of Corollary 6, the result was shown already in [10, Sect. 2]).

Proposition 5. *With the above notation, we have*

$$(\eta_\lambda)_*\big(f(x_1, \ldots, x_n)\big) = \big((\partial_\lambda f)(y_1, \ldots, y_n)\big)(x_1, \ldots, x_n).$$

It follows from Propositions 4 and 5 that

$$R_\lambda(E;t) = v_\lambda(t)(\eta_\lambda)_*\Big(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{i<j, \lambda_i \neq \lambda_j}(x_i - tx_j)\Big),$$

where $x_1, \ldots, x_n$ are the Chern roots of E.

Let λ be a sequence of nonnegative integers. Extending [9, III, 2], we set

$$P_\lambda(E;t) = \frac{1}{v_\lambda(t)} R_\lambda(E;t). \tag{6}$$

It follows from Proposition 4 that $P_\lambda(E;t)$ is a polynomial in the Chern classes of E and t.

Let us record the following particular case.

Corollary 6. Let ν be a strict partition with length $k \leq n$. Set $\lambda = \nu 0^{n-k}$. We have

$$P_\lambda(E;t) = (\tau_E^k)_*\Big(x_1^{\nu_1} \cdots x_k^{\nu_k} \prod_{i<j, i \leq k}(x_i - tx_j)\Big).$$

As a consequence of Propositions 1 and 4, using Eq.(6), we obtain the following result.

Theorem 7. *Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be sequences of nonnegative integers. Then we have*

$$\pi_*\Big(\prod_{i \leq q < j}(x_i - tx_j) P_\lambda(Q;t) P_\mu(S;t)\Big) = \frac{v_{\lambda\mu}(t)}{v_\lambda(t) v_\mu(t)} P_{\lambda\mu}(E;t).$$

We first consider the specialization $t = 0$.

Example 8. We recall Schur S-functions. Let $s_i(E)$ denotes the ith complete symmetric function in the roots $x_1, \ldots, x_n$, given by

$$\sum_{i \geq 0} s_i(E) = \prod_{j=1}^n \frac{1}{1 - x_j}.$$

Given a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$, we set

$$s_\lambda(E) = \left| s_{\lambda_i - i + j}(E) \right|_{1 \leq i,j \leq n} .$$

(See also [9, I, 3].) Translating the Jacobi-Trudi formula (*loc.cit.*) to the Gysin map for $\tau_E : Fl(E) \to X$ (see, e.g. [11, Sect. 4]), we have

$$s_\lambda(E) = (\tau_E)_*(x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}).$$

We see that $P_\lambda(E;t) = s_\lambda(E)$ for $t = 0$. Under this specialization, the theorem becomes

$$\begin{aligned} \pi_*\big((x_1 \cdots x_q)^r s_\lambda(Q) s_\mu(S)\big) &= \pi_*\big(s_{\lambda_1+r,\ldots,\lambda_q+r}(Q) s_\mu(S)\big) \\ &= s_{\lambda\mu}(E) , \end{aligned}$$

a result obtained originally in [7, Prop. p. 196] and [6, Prop. 1].

If a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is not a partition, then $s_\lambda(E)$ is either 0 or $\pm s_\mu(E)$ for some partition μ. One can rearrange λ by a sequence of operations $(\ldots, i, j, \ldots) \mapsto (\ldots, j-1, i+1, \ldots)$ applied to pairs of successive integers. Either one arrives at a sequence of the form $(\ldots, i, i+1, \ldots)$, in which case $s_\lambda(E) = 0$, or one arrives in d steps at a partition μ, and then $s_\lambda(E) = (-1)^d s_\mu(E)$.

Corollary 9. Let ν and σ be strict partitions of lengths $k \leq q$ and $h \leq r$. It follows from Eq.(3) that

$$\frac{v_{\nu 0^{q-k} \sigma 0^{r-h}}(t)}{v_{\nu 0^{q-k}}(t) v_{\sigma 0^{r-h}}(t)} = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix} (t) \cdot (1+t)^e ,$$

the Gaussian polynomial times $(1+t)^e$ where e is the number of common parts of ν and σ. Thus the theorem applied to the sequences $\lambda = \nu 0^{q-k}$ and $\mu = \sigma 0^{r-h}$ yields the following equation:

$$\pi_*\Big(\prod_{i \leq q < j} (x_i - t x_j) P_\nu(Q;t) P_\sigma(S;t) \Big) = \begin{bmatrix} n-k-h \\ q-k \end{bmatrix} (t) \cdot (1+t)^e \cdot P_{\lambda\mu}(E;t) . \quad (7)$$

We need the following property of Gaussian polynomials, which should be known but we know no precise reference.

Lemma 10. *At $t = -1$, the Gaussian polynomial*

$$\begin{bmatrix} a+b \\ a \end{bmatrix} (t)$$

specializes to zero if ab is odd and to the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor}$$

otherwise.

Proof. We have

$$\begin{bmatrix} a+b \\ a \end{bmatrix} (t) = \frac{(1-t)(1-t^2) \cdots (1-t^{a+b})}{(1-t) \cdots (1-t^a)(1-t) \cdots (1-t^b)} .$$

Since $t=-1$ is a zero with multiplicity 1 of the factor $(1-t^d)$ for even d, and a zero with multiplicity 0 for odd d, the order of the rational function $\left[\begin{smallmatrix} a+b \\ a \end{smallmatrix}\right](t)$ at $t=-1$ is equal to

$$\lfloor (a+b)/2 \rfloor - \lfloor a/2 \rfloor - \lfloor b/2 \rfloor . \tag{8}$$

The order (8) is equal to 1 when a and b are odd, and 0 otherwise. In the former case, we get the claimed vanishing, and in the latter one, the product of the factors with even exponents is equal to

$$\begin{bmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{bmatrix} (t^2) .$$

The value of this function at $t=-1$ is equal to $\left[\begin{smallmatrix} \lfloor a+b/2 \rfloor \\ \lfloor a/2 \rfloor \end{smallmatrix}\right](1)$ which is the binomial coefficient

$$\binom{\lfloor (a+b)/2 \rfloor}{\lfloor a/2 \rfloor} .$$

This is the requested value since the remaining factors with an odd exponent give 2 in the numerator and the same number in the denominator.

The assertions of the lemma follow. $\square$

We now consider the specialization $t=-1$.

Example 11. Consider Schur P-functions $P_\lambda(E) = P_\lambda$ (or $P_\lambda(y_1,\dots,y_n) = P_\lambda$) defined as follows. For a strict partition $\lambda = (\lambda_1 > \ldots > \lambda_k > 0)$ with odd k,

$$P_\lambda = P_{\lambda_1} P_{\lambda_2,\dots,\lambda_k} - P_{\lambda_2} P_{\lambda_1,\lambda_3,\dots,\lambda_k} + \cdots + P_{\lambda_k} P_{\lambda_1,\dots,\lambda_{k-1}} ,$$

and with even k,

$$P_\lambda = P_{\lambda_1,\lambda_2} P_{\lambda_3,\dots,\lambda_k} - P_{\lambda_1,\lambda_3} P_{\lambda_2,\lambda_4,\dots,\lambda_k} + \cdots + P_{\lambda_1,\lambda_k} P_{\lambda_2,\dots,\lambda_{k-1}} .$$

Here, $P_i = \sum s_\mu$, the sum over all hook partitions μ of i, and for positive $i > j$ we set

$$P_{i,j} = P_i P_j + 2 \sum_{d=1}^{j-1} (-1)^d P_{i+d} P_{j-d} + (-1)^j P_{i+j} .$$

(See also [9, III, 8].) It was shown in [12, p. 225] that for a strict partition λ of length k,

$$P_\lambda(y_1,\dots,y_n) = \sum_{w \in S_n/(S_1)^k \times S_{n-k}} w\Big(y_1^{\lambda_1} \cdots y_n^{\lambda_n} \prod_{i<j, i \le k} \frac{y_i + y_j}{y_i - y_j} \Big)$$

(see also [9, III, 8]). This implies

$$P_\lambda(E) = (\tau_E^k)_* \Big(x_1^{\lambda_1} \cdots x_k^{\lambda_k} \prod_{i<j, i\le k} (x_i + x_j) \Big) .$$

By Corollary 6, using its notation, we thus get $P_\lambda(E;t) = P_\nu(E)$ for $t=-1$.

We now use the notation from Corollary 9. Specializing $t=-1$ in Eq.(7), we get by Lemma 10

$$\pi_*\big(c_{qr}(Q \otimes S) P_\nu(Q) P_\sigma(S) \big) = d_{\nu,\sigma} P_{\nu\sigma}(E) ,$$

where $d_{\nu,\sigma} = 0$ if $(q-k)(r-h)$ is odd and

$$d_{\nu,\sigma} = (-1)^{(q-k)h} \binom{\lfloor (n-k-h)/2 \rfloor}{\lfloor (q-k)/2 \rfloor}$$

otherwise. This result was obtained originally in [11, Prop. 1.3(ii)] in a different way. The present approach gives an explanation of the intriguing coefficient $d_{\nu,\sigma}$.

Suppose that $\lambda = (\lambda_1, \dots, \lambda_k)$ is not a strict partition. If there are repetitions of elements in λ, then P_λ is zero; if not then $P_\lambda = (-1)^l P_\mu$, where l is the length of the permutation which rearranges $(\lambda_1, \dots, \lambda_k)$ into the corresponding strict partition μ.

We thank Witold Kraśkiewicz, Itaru Terada and Anders Thorup for helpful discussions, and the referee for suggesting several improvements of the text.

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