DEFORMATION ALONG SUBSHEAVES, II

CLEMENS JÖRDER AND STEFAN KEBEKUS

ABSTRACT. Let $f : Y \to X$ be the inclusion map of a compact reduced subspace of a complex manifold, and let $\mathscr{F} \subseteq T_X$ be a subsheaf of the tangent bundle which is closed under the Lie bracket, but not necessarily a sheaf of $\mathscr{O}_X$-algebras. This paper discusses criteria to guarantee that infinitesimal deformations of f which are induced by $\mathscr{F}$ lift to positive-dimensional deformations of f, where f is deformed "along the sheaf $\mathscr{F}$".

In case where X is complex-symplectic and $\mathscr{F}$ the sheaf of locally Hamiltonian vector fields, this partially reproduces known results on unobstructedness of deformations of Lagrangian submanifolds. The proof is rather elementary and geometric, constructing higher-order liftings of a given infinitesimal deformation using flow maps of carefully crafted time-dependent vector fields.

CONTENTS

1. INTRODUCTION AND MAIN RESULT

1.1. **Introduction.** Let $f : Y \to X$ be the inclusion map of a compact reduced subspace of a complex manifold. We aim to deform the morphism f, keeping the complex spaces X and Y fixed. For this purpose let us fix a first-order infinitesimal deformation of f, say $\sigma \in H^0\big(Y,\, f^*T_X\big)$ — we refer to the earlier paper [KKL10, Sect. 1] for a discussion of infinitesimal deformations, and for other notions used here. We ask for conditions to guarantee that σ is effective. In other words, we ask for conditions that guarantee the existence of a disk $\Delta \subset \mathbb{C}$, centered about 0, and a family of morphisms,

$$\begin{array}{rccc} F: & \Delta \times Y & \to & X \\ & (t, y) & \mapsto & F_t(y) \end{array}$$

such that $F_0 = f$ and such that the infinitesimal deformation induced by the family, $\sigma_{F,0} := \frac{d}{dt}\big|_{t=0} F_t \in H^0\big(Y,\, f^*T_X\big)$, agrees with σ. In case when Y is a manifold, the most general result in this direction is due to Horikawa.

Theorem 1.1 (Horikawa's criterion, [Hor73]). *If $H^1\big(Y,\, f^*T_X\big)$ vanishes, then any first-order infinitesimal deformation of f is effective.* □

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Vanishing of the obstruction space $H^1\big(Y,\, f^*T_X\big)$ is a sufficient, but not a necessary condition for the existence of liftings. In settings where the geometry of the target manifold X is well-understood, it is often possible to prove existence of liftings even if $H^1\big(Y,\, f^*T_X\big)$ is large.

Earlier results. One situation where effectivity of infinitesimal deformations can sometimes be shown has been studied in [KKL10]. The authors of [KKL10] considered a coherent subsheaf $\mathscr{F} \subseteq T_X$ of $\mathscr{O}_X$-modules, closed under Lie-bracket, and an infinitesimal deformation induced by $\mathscr{F}$,

$$\sigma \in H^0\left(Y,\, \operatorname{Image} f^*\mathscr{F} \to f^*T_X\right).$$

It was shown in [KKL10, Thm. 1.5] that σ lifts to a family of morphisms if the space $H^1\big(Y,\, \operatorname{Image} f^*\mathscr{F} \to f^*T_X\big)$ vanishes. Examples of sheaves $\mathscr{F}$ that appear this way include (singular) foliations, logarithmic tangent sheaves, or the tensor product of T_X with the ideal sheaf of a subvariety.

Result of this paper. This paper is concerned with the case where the infinitesimal deformation σ is induced by a sheaf $\mathscr{F} \subseteq T_X$ which is closed under Lie-bracket, but is not necessarily a sheaf of $\mathscr{O}_X$-modules. Examples are given by sheaves of Hamiltonian vector fields on complex-symplectic manifolds, or more generally sheaves of vector fields whose flows stabilize a given tensor. The main result, formulated in Theorems 1.7 and 1.19, generalizes and improves on [KKL10, Thm. 1.5]. In case where X is complex-symplectic, and $Y \subset X$ is a Lagrangian submanifold, this reproduces results of Ran, Voisin, Kawamata, and others.

Aim and scope. Written for the IMPANGA Lecture Notes series, this paper aims at simplicity and clarity of argument. It does not strive to present the shortest proofs or most general results available. While everything said here can also be deduced from the abstract machinery of deformation theory, we argue in a rather elementary and geometric manner, constructing higher-order liftings of a given infinitesimal deformation using flow maps of carefully crafted time-dependent vector fields.

1.2. **Main result.** In order to formulate our result we start off with the necessary notation.

Notation 1.2. If X is any complex manifold, denote the sheaf of locally constant functions on X by $\mathbb{C}_X \subseteq \mathscr{O}_X$.

Throughout the present paper, we will frequently consider subsheaves $\mathscr{F} \subseteq T_X$, such as the sheaf of Hamiltonian vector fields on a complex-symplectic space, which are invariant under multiplication with constants, but not necessarily under multiplication with arbitrary regular functions. We call such $\mathscr{F}$ a *sheaf of $\mathbb{C}_X$-modules*. The main results of this paper consider the following setup.

Setup 1.3. *Let X be a complex manifold, and $Y \subseteq X$ a reduced complex subspace of X, with inclusion map $f : Y \to X$. Further, let $\mathscr{F} \subseteq T_X$ be a subsheaf of $\mathbb{C}_X$-modules, closed under Lie bracket.*

We aim to deform the inclusion map "along the subsheaf $\mathscr{F}$". The following two notions help to make this precise.

Definition 1.4 (Infinitesimal deformations locally induced by $\mathscr{F}$)**.** *In Setup 1.3, set*

$$\mathscr{F}_Y := \operatorname{Image}\left(f^{-1}\mathscr{F} \to f^*T_X\right) \subseteq f^*T_X.$$

We call $\mathscr{F}_Y$ the sheaf of infinitesimal deformations locally induced by $\mathscr{F}$*. Sections $\sigma \in H^0\big(Y,\, \mathscr{F}_Y\big)$ are called* infinitesimal deformations of f which are locally induced by $\mathscr{F}$.

Definition 1.5 (Obstruction sheaves). *In Setup 1.3, we call a subsheaf $\mathscr{G} \subseteq \mathscr{F}_Y$ an* obstruction sheaf for $\mathscr{F}$ *if the Lie-bracket of any two vector fields in $\mathscr{F}$ which agree along the image $f(Y)$ induces a section of $\mathscr{G}$.*

More precisely, $\mathscr{G} \subseteq \mathscr{F}_Y$ is called an obstruction sheaf for $\mathscr{F}$ if for any open set $U \subseteq X$ and any two vector fields $\vec{A}_1, \vec{A}_2 \in \mathscr{F}(U)$ satisfying $f^*\vec{A}_1 = f^*\vec{A}_2$, the preimage of the Lie-bracket is contained in $\mathscr{G}$,

$$f^* [\vec{A}_1, \vec{A}_2] \in \mathscr{G}\big(f^{-1}(U)\big).$$

Remark 1.6. Obstruction sheaves are generally not unique. Example 1.10 discusses a situation where there is more than one sheaf satisfying the requirements of Definition 1.5.

The main result of our paper essentially says that Setup 1.3, any infinitesimal deformation of f which is locally induced by $\mathscr{F}$ is effective if there exists an obstruction sheaf whose first cohomology group vanishes.

Theorem 1.7 (Existence of deformations). *In Setup 1.3, assume that Y is compact and assume that there exists an obstruction sheaf $\mathscr{G} \subseteq \mathscr{F}_Y$ for $\mathscr{F}$ such that $H^1\big(Y, \mathscr{G}\big) = 0$. Then any infinitesimal deformation $\sigma \in H^0\big(Y, \mathscr{F}_Y\big)$ locally induced by $\mathscr{F}$ is effective. In other words, there exists an open neighbourhood Δ of $0 \in \mathbb{C}$, and a family of morphisms*

$$\begin{array}{cccc} F: & \Delta \times Y & \to & X \\ & (t, y) & \mapsto & F_t(y) \end{array}$$

such that $F_0 = f$ and such that the infinitesimal deformation induced by the family, $\sigma_{F,0} := \frac{d}{dt}\big|_{t=0} F_t \in H^0\big(Y, f^*T_X\big)$, agrees with σ.

Understanding that the formulation of Theorem 1.7 may sound a little technical, we end the present subsection with two examples. In Section 1.3 we will then see that the family of morphisms whose existence is guaranteed by Theorem 1.7 can often be chosen in a way that geometrically relates to the sheaf $\mathscr{F}$.

Example 1.8 (Complex-symplectic manifolds). Let (X, ω) be a complex-symplectic manifold and $\mathscr{F} \subset T_X$ the sheaf of Hamiltonian vector fields. Let $Y \subseteq X$ be any compact complex submanifold, with inclusion map $f : Y \to X$. We assume that Y is Kähler and consider the sheaf $T_Y^{\perp} \subseteq f^*T_X$ of vector fields which are perpendicular to Y, with respect to the symplectic form ω. In other words, $T_Y^{\perp}$ is the sheaf associated to the presheaf

$$U \mapsto \left\{ \vec{A} \in (f^*T_X)(U) \;\middle|\; (f^*\omega)(\vec{A}, df(\vec{B})) = 0 \text{ for all } \vec{B} \in T_Y(U) \right\},$$

where $U \subset Y$ runs over the open subsets of Y. We will show in Section 6 that $H^0\big(Y, \mathscr{F}_Y\big) = H^0\big(Y, f^*T_X\big)$, and that $T_Y^{\perp}$ is an obstruction sheaf in this setting. Theorem 1.7 thus asserts that any infinitesimal deformation $\sigma \in H^0\big(X, f^*T_X\big)$ always lifts to a holomorphic family F of morphisms if the cohomology group $H^1\big(Y, T_Y^{\perp}\big)$ vanishes.

Remark 1.9 (Deformations of Lagrangian submanifolds). In the setting of Example 1.8, if $Y \subset X$ is a Lagrangian submanifold, then $T_Y^{\perp} = T_Y$. We obtain that vanishing of $H^1\big(Y, T_Y\big)$, the tangent space to the Kuranishi-family of deformations of Y, is the only obstruction to lifting a given infinitesimal deformation. This partially reproduces results of Ran, Kawamata and Voisin on the unobstructedness of deformations of Lagrangian subvarieties, cf. [Ran92, Voi92, Kaw92, Kaw97] and the references there.

Example 1.10 (Deformation along a foliation). Let X be a complex manifold and $\mathscr{F} \subseteq T_X$ a (regular) foliation, i.e., a sub-vectorbundle of T_X which is closed under

Lie bracket. Again, let $Y \subseteq X$ be any compact submanifold of X. Let $T \subset Y$ be the set of points where the foliation is tangent to Y,

$$T := \{y \in Y \mid \mathscr{F}|_y \subseteq T_Y|_y\}.$$

It is clear that $\mathscr{F}_Y = \mathscr{F}|_Y$. We will show in Section 7 that any of the two sheaves

$$\mathscr{G}_1 := \mathscr{F}_Y \quad \text{and} \quad \mathscr{G}_2 := \mathscr{F}_Y \otimes \mathscr{J}_T$$

are obstruction sheaves in this setting. Thus, if either $H^1(Y, \mathscr{F}_Y)$ or $H^1(Y, \mathscr{F}_Y \otimes \mathscr{J}_T)$ vanishes, then any infinitesimal deformation $\sigma \in H^0(Y, \mathscr{F}|_Y)$ lifts to a holomorphic family of morphisms.

1.3. **Deformations along $\mathscr{F}$.** Although the sheaf $\mathscr{F}$ appears in the assumptions of Theorem 1.7, its conclusion seems to disregard $\mathscr{F}$ entirely, as the family of morphisms obtained in Theorem 1.7 need not be related to $\mathscr{F}$ in any way. However, in all the examples we have in mind, there is a way to construct a family of morphisms $F : \Delta \times Y \to X$ that relates to $\mathscr{F}$ geometrically.

1.3.1. *Notation concerning higher-order infinitesimal deformations.* For a precise formulation of this result, we need to discuss locally closed subspaces of the Douady-space of all holomorphic maps $Y \to X$ which parametrize deformations *along* $\mathscr{F}$. The following notions concerning *higher-order* infinitesimal deformations of f will be used in the definition.

Definition 1.11 (Higher-order infinitesimal deformations, cf. [KKL10, Def. 2.12])**.** *Let $f : Y \to X$ be a morphism from a complex space Y to a complex manifold X. An n-*th order infinitesimal deformation *of f is a morphism*

$$f_n : \operatorname{Spec} \mathbb{C}[t]/t^{n+1} \times Y \to X$$

whose restriction to $Y \cong \operatorname{Spec} \mathbb{C} \times Y$ agrees with f.

The universal property of the Douady-space immediately yields another, equivalent, definition of higher-order infinitesimal deformations.

Notation 1.12 (Douady-space of morphisms, cf. [CP94, Sect. 2]). Let X, Y be complex spaces and assume that Y is compact. We denote the Douady-space of morphisms from Y to X by $\operatorname{Hom}(Y, X)$.

Fact 1.13 (Higher-order deformations as morphisms to the Douady-space)**.** *In the setting of Definition 1.11, assume additionally that Y is compact. To give an n-th order infinitesimal deformation of f, it is then equivalent to give a morphism*

$$f_n : \operatorname{Spec} \mathbb{C}[t]/t^{n+1} \to \operatorname{Hom}(Y, X)$$

whose closed point maps to the point $[f] \in \operatorname{Hom}(Y, X)$ representing the morphism f. □

Alternatively, any n-th order deformation of f can also be seen as a section in the pull-back of the n-th order jet-bundle of X. Jet bundles and their fundamental properties are reviewed in Section 2.1 below.

Fact 1.14 (Higher-order deformations as sections in Jet^n, [KKL10, Prop. 2.13])**.** *In the setting of Definition 1.11, to give an n-th order infinitesimal deformation of f, it is equivalent to give a section $f_n : Y \to f^* \mathrm{Jet}^n(X)$.* □

1.3.2. *Spaces of deformations along $\mathscr{F}$.* Sections in $\mathrm{Jet}^n(X)$, which appear in the description of higher-order deformations given in Fact 1.14, can be constructed using flows of (time-dependent) vector fields on X. We will recall in Section 2.3 that if $U \subseteq X$ is any open set, if $\vec{A}$ is any time-dependent vector field on U, and n any number, then the flow of $\vec{A}$ induces a section $\tau^n_{\vec{A}} : U \to \mathrm{Jet}^n(U)$. We call higher-order infinitesimal deformations of f to be *induced by time-dependent vector fields in $\mathscr{F}$* if they locally arise in this way.

Definition 1.15 (Deformations induced by time-dependent vector fields in $\mathscr{F}$)**.** *In Setup 1.3, an n-th order deformation $f_n : Y \to f^*\,\mathrm{Jet}^n(X)$ of f is said to be* locally induced by time-dependent vector fields in $\mathscr{F}$*, if there exists a cover of Y by open subsets of X, say $Y \subseteq \cup_{\alpha \in A} U_\alpha$, and for any $a \in A$ time-dependent vector fields of the following form,*

$$\vec{A}_a = \sum_{i=0}^{n} t^i \vec{A}_{a,i} \quad \textit{where} \quad \vec{A}_{a,i} \in \mathscr{F}(U_\alpha)$$

such that $f_n|_{U_a \cap Y} = \tau^n_{\vec{A}_a}|_{U_a \cap Y}$, for all $a \in A$.

Definition 1.16 (Spaces of deformations along $\mathscr{F}$)**.** *In Setup 1.3, assume additionally that Y is compact. A locally closed analytic subspace $H \subset \mathrm{Hom}(Y, X)$ containing $[f]$ is called* space of deformations along $\mathscr{F}$*, if for any infinitesimal deformation f_n of f that is locally induced by time-dependent vector fields in $\mathscr{F}$, the corresponding morphism* $\mathrm{Spec}\,\mathbb{C}[t]/t^{n+1} \to \mathrm{Hom}(Y, X)$ *factors through H.*

Example 1.17 (Complex-symplectic manifolds). In the setting of Example 1.8, where (X, ω) is a complex-symplectic manifold, set

$$H := \{ g \in \mathrm{Hom}(Y, X) \,|\, g^*\omega = f^*\omega \} \subseteq \mathrm{Hom}(Y, X).$$

Since (time-dependent) Hamiltonian vector fields preserve the symplectic form ω, it is clear that H is a space of deformations along $\mathscr{F}$.

Example 1.18 (Foliated manifolds). In the setting of Example 1.10, where $\mathscr{F} \subseteq T_X$ is a foliation, the existence of a space of deformations along $\mathscr{F}$ with very good properties has been shown in [KKL10, Cor. 5.6].

1.3.3. *Existence of deformations along $\mathscr{F}$.* In cases where a space of deformations along $\mathscr{F}$ exists, the deformation family constructed in Theorem 1.7 can be chosen to factor via that space. This complements and strengthens Theorem 1.7 in our special situation.

Theorem 1.19 (Existence of deformations along $\mathscr{F}$, strenghtening of Theorem 1.7)**.** *In the setting of Theorem 1.7, assume in addition that there exists a space $H \subseteq \mathrm{Hom}(Y, X)$ of deformations along $\mathscr{F}$. Then there exists a family $F : \Delta \times Y \to X$ such that F satisfies all properties stated in Theorem 1.7, and such that the associated map $F : \Delta \to \mathrm{Hom}(Y, X)$ factors through H.*

1.4. Outline of this paper, acknowledgements.

1.4.1. *Outline.* Section 2 summarizes fundamental facts concerning jet bundles associated with a complex manifold that will be needed in the sequel. Subsections 2.1 and 2.2 review the notions of jet bundles and time-dependent vector fields, respectively. The sections in the jet bundles arising from flow maps associated with time-dependent vector fields are discussed in the subsequent subsection 2.3.

Section 3 is the technical core of the paper. In Setup 1.3 the choice of an obstruction sheaf leads to the notion of *admissible* time-dependent vector fields on X. The rather technical definition of these time-dependent vector fields is justified by the properties of the jets induced by them, as it is formulated in Subsection 3.1 and proven in the remainder of Section 3.

The local definition and properties of admissible time-dependent vector fields and the induced jets are globalized in Section 4, yielding the notion of admissible higher-order infinitesimal deformations of the inclusion $f : Y \to X$. This leads us to a lifting criterion for infinitesimal higher-order deformations of f based on the cohomology vanishing assumption that appears in the statement of Theorem 1.7.

Once the result in Section 4 is established, the actual proof of Theorem 1.7 and Theorem 1.19 is a short argument that is outlined in Section 5.

The paper concludes with a detailed review of the two examples mentioned so far, namely the deformation on complex-symplectic manifolds and on foliated manifolds in Sections 6 and 7, respectively.

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2. Jet theory

The proof of Theorem 1.7 uses the convenient language of jet bundles. Sections 2.1 and 2.2 summarize basic facts and definitions about jet bundles and time-dependent vector fields, respectively. Section 2.3 contains an important formula concerning the jets induced by flow maps of time-dependent vector fields. A more detailed introduction to jets is found in [KKL10, Sect. 2] and the references quoted there.

2.1. Jet bundles.

Jet bundles generalize the notion of tangent bundles. If X is any complex manifold, an n-th order jet on X is an equivalence class of curve germs, where two germs are considered equivalent if they agree to n-th order. A precise definition is given as follows.

Definition 2.1 (Jets on complex manifolds)**.** *Let X be any complex manifold and $x \in X$ a point. An n-*th order jet *at $x \in X$ is a morphism*

$$\sigma : \mathrm{Spec}\ \mathbb{C}[t]/t^{n+1} \to X$$

of complex spaces which maps the closed point of $\mathrm{Spec}\ \mathbb{C}[t]/t^{n+1}$ *to x.*

Definition 2.2 (Jet bundle)**.** *If X is a complex manifold, define the n-*th order jet bundle of *X as*

$$\mathrm{Jet}^n(X) := \mathrm{Hom}\big(\mathrm{Spec}\ \mathbb{C}[t]/t^{n+1},\ X\big).$$

As a set, $\mathrm{Jet}^n(X)$ *equals the set of n-th order jets on X.*

Recall from Fact 1.14 that higher-order infinitesimal deformations f_n : $\mathrm{Spec}\ \mathbb{C}[t]/t^{n+1} \times Y \to X$ of f can be considered as X-morphisms $f_n : Y \to \mathrm{Jet}^n(X)$. A detailed description of the structure of jet bundles is therefore important.

Fact 2.3 (Affine bundle structure, cf. [KKL10, Fact 2.8.3])**.** *Let X be a manifold. Then the following facts hold true for any natural number $n \geq 0$:*

(1) *The complex space* $\mathrm{Jet}^n(X)$ *is a manifold. The obvious forgetful morphism $\pi_{n,m}$:* $\mathrm{Jet}^n(X) \to \mathrm{Jet}^m(X)$ *is holomorphic for all $n \geq m \geq 0$. There are natural isomorphisms* $\mathrm{Jet}^0(X) \cong X$ *and* $\mathrm{Jet}^1(X) \cong T_X$.

(2) *The morphism* $\pi_{n+1,n} : \operatorname{Jet}^{n+1}(X) \to \operatorname{Jet}^n(X)$ *has the structure of an affine bundle. The associated vector bundle of translations on* $\operatorname{Jet}^n(X)$ *is the pullback of the tangent bundle.* □

Notation 2.4. In the setting of Fact 2.3, if $\sigma, \tau \in \operatorname{Jet}^{n+1}(X)$ are two jets whose n-th order parts agree, $\pi_{n+1,n}(\sigma) = \pi_{n+1,n}(\tau)$, the affine bundle structure mentioned in Item 2 allows to express the difference between σ and τ as a tangent vector $\vec{v} \in T_X|_x$. In this context, we write $\vec{v} = \sigma - \tau$.

The following descriptions of jets is immediate from the universal property of the Hom space.

Fact 2.5 (Jet bundles in deformation theory)**.** *Let* $f : Y \to X$ *be a holomorphic map between a compact complex space* Y *and a complex manifold* X*. To give an* n*-th order jet at* $[f] \in \operatorname{Hom}(Y, X)$*, it is equivalent to give a holomorphic section* $\tau^n : Y \to f^* \operatorname{Jet}^n(X) := \operatorname{Jet}^n(X) \times_X Y$*.* □

2.2. **Time-dependent vector fields.** We follow the standard approach familiar from the theory of ordinary differential equations and define a time-dependent vector field on a complex manifold X as a vector field on the product of X and a "time axis".

2.2.1. *Notation.* The following notation concerning time-dependent vector fields on X and on the Cartesian product $X \times \mathbb{C}$ will be used throughout this paper.

Notation 2.6 (Cartesian product). Let X be a complex manifold. Consider the product $X \times \mathbb{C}$. Let t be the standard coordinate on $\mathbb{C}$, with associated vector field $\frac{\partial}{\partial t} \in H^0\big(X \times \mathbb{C},\, T_{X \times \mathbb{C}}\big)$. The projection from $X \times \mathbb{C}$ to the first factor is denoted by $p_X : X \times \mathbb{C} \to X$. Finally, let $j_t : X \to X \times \mathbb{C}, x \mapsto (x, t)$ be the inclusion map.

Notation 2.7 (Time-dependent vector fields). Let X be a complex manifold. A *time-dependent vector field* $\vec{A}$ on X is a vector field $\vec{A}$ on the product $X \times \mathbb{C}$ contained in the subspace

$$H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big) \subset H^0\big(X \times \mathbb{C},\, T_{X \times \mathbb{C}}\big).$$

For any time $t \in \mathbb{C}$, the restriction of $\vec{A}$ to time t is denoted by

$$\vec{A}_t := j_t^*(\sigma) \in H^0(X,\, T_X)$$

Definition 2.8 (Vector field with constant flow in time)**.** *If* $\vec{A}$ *is any time-dependent vector field on a complex manifold* X*, set*

$$D(\vec{A}) := \tfrac{\partial}{\partial t} + \vec{A} \in H^0\big(X \times \mathbb{C},\, T_{X \times \mathbb{C}}\big).$$

We call $D(\vec{A})$ *the* vector field with constant flow in time associated with $\vec{A}$*.*

2.2.2. *Calculus involving time-dependent vector fields.* For later reference, we state without proof a formula involving time-dependent vector fields and their Lie brackets. The formula is easily checked by a direct computation in local coordinates.

Lemma 2.9 (Calculus of time-dependent vector fields)**.** *Let* $\vec{A}, \vec{B} \in p_X^*(T_X)$ *be any two time-dependent vector fields on* X*, and let* $n \in \mathbb{N}$ *be any number. Using Notation 2.6, the following equations hold.*

$$[\vec{A}, t^n \cdot \vec{B}] = [t^n \cdot \vec{A}, \vec{B}] = t^n \cdot [\vec{A}, \vec{B}] \tag{2.9.1}$$

$$\mathrm{L}_{D(\vec{A})}\big(\tfrac{t^n}{n!} \cdot \vec{B}\big) = \tfrac{t^n}{n!} \cdot [\vec{A}, \vec{B}] + \tfrac{t^{n-1}}{(n-1)!} \cdot \vec{B} + \tfrac{t^n}{n!} \cdot \tfrac{\partial}{\partial t} \vec{B} \tag{2.9.2}$$

□

2.3. **Jets induced by time-dependent vector fields.** If X is a complex manifold, $x \in X$ a point and $\vec{A} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ a time-dependent vector field on X, then the local flow of $\vec{A}$ through x induces a curve germ γ_x at x, uniquely determined by the properties $\gamma_x(0) = x$ and $\gamma_x'(t) = \vec{A}_t(\gamma_x(t))$ for all t. We denote the associated jets as follows.

Definition 2.10 (Jets induced by time-dependent vector fields). *Let $\vec{A} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ be a time-dependent vector field on a complex manifold X. We denote by*

$$\tau^n_{\vec{A}} : X \to \mathrm{Jet}^n(X)$$

the holomorphic section which assigns to each point $x \in X$ the n-th order jet at $x \in X$ associated with the $\vec{A}$-integral curve through x.

The following is the key observation of this paper and of the previous paper [KKL10]. In its simplest form, Fact 2.11 considers two time-dependent vector fields $\vec{A}$, $\vec{B}$ on X whose associated n-th order jets $\tau^n_{\vec{A}}$ and $\tau^n_{\vec{B}}$ agree at a point $x \in X$. It gives a formula for the difference between $(n+1)$-st order jets $\tau^{n+1}_{\vec{A}}(x)$ and $\tau^{n+1}_{\vec{B}}(x)$ which, using the affine bundle structure of $\mathrm{Jet}^{n+1}(X) \to \mathrm{Jet}^n(X)$, can be identified with a tangent vector $\tau^{n+1}_{\vec{A}}(x) - \tau^{n+1}_{\vec{B}}(x) \in T_X|_x$. This formula allows for explicit computations of obstruction cocycles relevant when trying to lift infinitesimal deformations from n-th to $(n+1)$-st order. We refer to [KKL10, Sect. 3] for a more detailed discussion.

Fact 2.11 (Difference formula, cf. [Jö10, Cor. 2.5.3], [KKL10, Thm. 4.3]). *Let $\vec{A}_1, \dots, \vec{A}_n, \vec{B} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ be time-dependent vector fields on a complex manifold X. Let $x \in X$ be any point such that the induced n-th order jets agree at x,*

$$\tau^n_{\vec{A}_1}(x) = \tau^n_{\vec{A}_2}(x) = \cdots = \tau^n_{\vec{B}}(x).$$

Using Fact 2.3 and Notation 2.4 to identify the difference between the $(n+1)$-st order jets $\tau^{n+1}_{\vec{A}_n}(x)$ and $\tau^{n+1}_{\vec{B}}(x)$ with a tangent vector in $T_X|_x$, the difference is given as

$$\tau^{n+1}_{\vec{B}}(x) - \tau^{n+1}_{\vec{A}_n}(x) = \big(\mathrm{L}_{D(\vec{A}_1)} \circ \cdots \circ \mathrm{L}_{D(\vec{A}_n)}\, D(\vec{B})\big)_0(x) \in T_X|_x,$$

where $L_{D(\vec{A}_i)}$ denotes Lie-derivative with respect to the vector field $D(\vec{A}_i)$ with constant flow in time. □

3. Admissible vector fields

We consider the following setup throughout the present section.

Setup 3.1. *Let X be a complex manifold, equipped with a Lie-closed subsheaf $\mathscr{F} \subseteq T_X$ of $\mathbb{C}_X$-modules. Let $Y \subseteq X$ be a reduced complex subspace of X with inclusion map $f : Y \to X$, and let further $\mathscr{G} \subseteq \mathscr{F}_Y$ be an obstruction sheaf for $\mathscr{F}$, in the sense of Definition 1.5.*

The aim of this section is to define and discuss *admissible* vector fields. These are time-dependent vector fields whose induced jets are particularly well-behaved when used to deform the inclusion map $f : Y \to X$. Since the deformation of f will be defined *locally* on Y, we do not assume compactness of Y in this section.

To begin, we fix the notion a time-dependent vector field in $\mathscr{F}$, see Definition 1.15.

Notation 3.2 (Time-dependent vector fields in $\mathscr{F}$). A time-dependent vector field $\vec{A} \in H^0(X \times \mathbb{C}, p_X^*(T_X))$ is said to be a *time-dependent vector field in $\mathscr{F}$*, if it can be expressed as a finite sum

$$\vec{A} = \sum_{i=1}^{n} t^i \vec{A}_i$$

where $\vec{A}_i \in H^0(X, \mathscr{F})$ is a time-independent vector field in $\mathscr{F}$, and t is the standard coordinate on $\mathbb{C}$.

Definition 3.3 (Admissible vector field)**.** *In Setup 3.1, let $\vec{A} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ be a time-dependent vector field in $\mathscr{F}$, and let $n \geq 1$ be any natural number. The vector field $\vec{A}$ is said to be n-*admissible for the obstruction sheaf *$\mathscr{G}$, if for any natural number $1 \leq m \leq n$, the restriction to Y of $\mathrm{L}^m_{D(\vec{A})} \vec{A}$ at time $t = 0$ is a section of the obstruction sheaf. In other words, if*

$$\big(\mathrm{L}^m_{D(\vec{A})} \vec{A}\big)_0\big|_Y \in H^0\big(Y, \mathscr{G}\big).$$

Remark 3.4 (Admissible fields and derivative in time direction). If $\vec{A} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ is any time-dependent vector field on X, then

$$\mathrm{L}_{D(\vec{A})} \vec{A} = \mathrm{L}_{\vec{A}+\frac{\partial}{\partial t}} \vec{A} = \mathrm{L}_{\frac{\partial}{\partial t}} \vec{A} = -\,\mathrm{L}_{\vec{A}} \tfrac{\partial}{\partial t} = -\,\mathrm{L}_{D(\vec{A})} \tfrac{\partial}{\partial t}.$$

More generally, we have $\mathrm{L}^m_{D(\vec{A})} \vec{A} = -\,\mathrm{L}^m_{D(\vec{A})} \frac{\partial}{\partial t}$. If $\vec{A}$ is n-admissible for $\mathscr{G}$, then $\big(\mathrm{L}^m_{D(\vec{A})} \frac{\partial}{\partial t}\big)_0\big|_Y \in H^0\big(Y, \mathscr{G}\big)$ for all $1 \leq m \leq n$.

Remark 3.5 (Time-independent vector fields in $\mathscr{F}$ are admissible). A time-independent section of $\mathscr{F}$ is n-admissible for arbitrary n and arbitrary obstruction sheaf, when considered as time-dependent vector field. In other words, any field

$$\vec{A} \in H^0\big(X, \mathscr{F}\big) \subset H^0\big(X \times \mathbb{C},\, p_X^{-1}(\mathscr{F})\big)$$

is n-admissible for any $\mathscr{G}$.

3.1. **Jets induced by admissible fields.** The geometric meaning of Definition 3.3 is perhaps not obvious. However, the usefulness of the concept will immediately become clear once we look at jets induced by admissible vector fields. The following two propositions, which form the technical core of this paper, summarize the main features. Proofs are given in Subsections 3.2–3.5 below.

Proposition 3.6 (Extension of jets from n-th to $(n+1)$-st order)**.** *In Setup 3.1, let $\vec{A} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ be an n-admissible vector field for $\mathscr{G}$ with $n \geq 1$. If $\vec{\Delta} \in H^0\big(X, \mathscr{F}\big) \subset H^0\big(X, T_X\big)$ is any time-independent vector field in $\mathscr{F}$ whose restriction to Y lies in the obstruction sheaf, $\vec{\Delta}|_Y \in H^0\big(Y, \mathscr{G}\big)$, then there exists a time-dependent vector field $\vec{B} \in H^0\big(X \times \mathbb{C},\, p_X^*(T_X)\big)$ such that the following holds true.*

(1) *The time-dependent vector field $\vec{B}$ is $(n+1)$-admissible for $\mathscr{G}$.*
(2) *The n-th order jets induced by $\vec{A}$ and $\vec{B}$ agree on Y,*

$$\tau^n_{\vec{A}}\big|_Y = \tau^n_{\vec{B}}\big|_Y.$$

(3) *The difference between the induced $(n+1)$-st order jets is given by $\vec{\Delta}$,*

$$\tau^n_{\vec{B}}\big|_Y - \tau^n_{\vec{A}}\big|_Y = \vec{\Delta}\big|_Y.$$

Proposition 3.7 (Differences of jets induced by admissible vector fields)**.** *In Setup 3.1, let $n \geq 1$ be any number and $\vec{A}, \vec{B} \in H^0\big(X \times \mathbb{C}, p_X^*(T_X)\big)$ be two time-dependent vector fields, both of them n-admissible for the obstruction sheaf $\mathscr{G}$. If the induced n-th order jets agree on Y,*

$$\tau^n_{\vec{A}}\big|_Y = \tau^n_{\vec{B}}\big|_Y,$$

then the difference between the $(n+1)$-st order deformations of f lies in the obstruction sheaf $\mathscr{G}$. In other words,

$$\tau^{n+1}_{\vec{B}}\big|_Y - \tau^{n+1}_{\vec{A}}\big|_Y \in H^0\big(Y, \mathscr{G}\big).$$

The proofs of Propositions 3.6 and 3.7 are quite elementary, but somewhat lengthy and tedious. The reader interested in gaining an overview of the argumentation is advised to skip Sections 3.2–3.5 on first reading and continue with Section 4 on page 14, where Propositions 3.6 and 3.7 are used to lift first-order infinitesimal deformations of f to arbitrary order.

3.2. **Preparation for the proof of Proposition 3.6.** The proof of Proposition 3.6 relies on the following computational lemma.

Lemma 3.8. *In the setup of Proposition 3.6, let $\vec{E}$ be any time-independent vector field in $\mathscr{F}$ and consider the time-dependent vector field in $\mathscr{F}$*

$$\vec{B} := \vec{A} + \frac{t^n}{n!}\vec{\Delta} + \frac{t^{n+1}}{(n+1)!}\vec{E}. \tag{3.8.1}$$

Then the following equalities hold up to higher-order terms of t, for all $1 \leq m \leq n$,

$$\mathrm{L}^m_{D(\vec{B})}(\vec{B}) \equiv \mathrm{L}^m_{D(\vec{A})}(\vec{A}) + \tfrac{t^{n-m}}{(n-m)!}\vec{\Delta} \qquad \mathrm{mod}\ (t^{n-m+1}) \tag{3.8.2}$$

$$\mathrm{L}^{n+1}_{D(\vec{B})}\,\vec{B} \equiv \mathrm{L}^{n+1}_{D(\vec{A})}(\vec{A}) + \vec{E} + n[\vec{A}, \vec{\Delta}] \qquad \mathrm{mod}\ (t). \tag{3.8.3}$$

Lemma 3.8 easily follows from a direct but rather tedious computation. Details are found in the preprint version of this paper, available on the arXiv.

3.3. **Proof of Proposition 3.6.** Consider the time-dependent vector field

$$\vec{B} := \vec{A} + \frac{t^n}{n!}\vec{\Delta} + \frac{t^{n+1}}{(n+1)!}\vec{E}, \quad \text{where} \quad \vec{E} := -n[\vec{A}, \vec{\Delta}] - \big(\mathrm{L}^{n+1}_{D(\vec{A})}\,\vec{A}\big)_0. \tag{3.8.4}$$

Since $\vec{\Delta}|_Y$ is a section of the obstruction sheaf $\mathscr{G}$, Equation (3.8.2) immediately implies that $\vec{B}$ is n-admissible. By choice of $\vec{E}$, Equation (3.8.3) shows that $\vec{B}$ is in fact $(n+1)$-admissible. This already shows Property (1) claimed in Proposition 3.6.

Proof of Proposition 3.6, Property (2). To show Property (2), we will prove by induction that

$$\tau^m_{\vec{B}}\big|_Y = \tau^m_{\vec{A}}\big|_Y, \quad \text{for all } 1 \leq m \leq n. \tag{3.8.5}$$

We start the induction with the case where $m = 1$. In this case Equation (3.8.5) simply asserts that the vector fields $D(\vec{A})$ and $D(\vec{B})$ agree along Y at time $t = 0$. That, however, is clear by choice of $\vec{B}$.

For the inductive step, assume that Equation (3.8.5) was shown for a certain number $1 \leq m < n$. It will then follow from Fact 2.11 that the difference between the $(m+1)$-st order jets is given as

$$\tau^{m+1}_{\vec{B}}\big|_Y - \tau^{m+1}_{\vec{A}}\big|_Y = \big(\mathrm{L}^m_{D(\vec{A})}\, D(\vec{B})\big)_0\Big|_Y. \tag{3.8.6}$$

But since

$$\begin{aligned} \mathrm{L}^m_{D(\vec{A})} D(\vec{B}) &= \mathrm{L}^m_{D(\vec{A})} \left(D(\vec{A}) + \tfrac{t^n}{n!}\vec{\Delta} + \tfrac{t^{n+1}}{(n+1)!}\vec{E} \right) && \text{by (3.8.4)} \\ &= \mathrm{L}^m_{D(\vec{A})} \left(\tfrac{t^n}{n!}\vec{\Delta} + \tfrac{t^{n+1}}{(n+1)!}\vec{E} \right) \\ &\equiv \tfrac{t^{n-m}}{(n-m)!}\vec{\Delta} \mod (t^{n-m+1}) && \text{by (2.9.2),} \end{aligned}$$

it is clear that the difference (3.8.6) vanishes as required. This finishes the proof of Property (2) claimed in Proposition 3.6.

Proof of Proposition 3.6, Property (3). Fact 2.11 and Property (2) together imply that the difference between the $(n+1)$-st order jets is given by

$$\tau^{n+1}_{\vec{B}}\big|_Y - \tau^{n+1}_{\vec{A}}\big|_Y = \big(\mathrm{L}^n_{D(\vec{A})} D(\vec{B})\big)_0\Big|_Y.$$

As in the proof of Property (2) above we obtain that

$$\mathrm{L}^n_{D(\vec{A})} D(\vec{B}) \equiv \vec{\Delta} \mod (t),$$

finishing the proof of Proposition 3.6. □

3.4. **Preparation for the proof of Proposition 3.7.** The proof of Proposition 3.7 makes use of two computational lemmas concerning Lie derivatives that are formulated and proved in the current Section 3.4. The actual proof of Proposition 3.7 is given in Section 3.5. To start, recall the classical Jacobi identity, formulated in terms of Lie derivatives.

Remark 3.9 (Jacobi identity). Let $\vec{X}, \vec{Y}, \vec{Z} \in H^0(X, T_M)$ be vector fields on a complex manifold M. Written in terms of Lie derivatives rather than Lie brackets, the Jacobi identity asserts that

$$\mathrm{L}_{[\vec{X},\vec{Y}]} \vec{Z} = \mathrm{L}_{\vec{X}} \circ \mathrm{L}_{\vec{Y}} \vec{Z} - \mathrm{L}_{\vec{Y}} \circ \mathrm{L}_{\vec{X}} \vec{Z}. \tag{3.9.1}$$

Lemma 3.10. *In the setup of Proposition 3.7, define*

$$\begin{aligned} \vec{R}^2 &:= [D(\vec{A}), D(\vec{B})] = \mathrm{L}_{D(\vec{A})} D(\vec{B}) && \textit{and inductively} \\ \vec{R}^m &:= [D(\vec{A}), \vec{R}^{m-1}] = \mathrm{L}^{m-1}_{D(\vec{A})} D(\vec{B}) && \textit{for } m > 2. \end{aligned}$$

If $\vec{Z} \in H^0\big(X \times \mathbb{C}, T_{X\times\mathbb{C}}\big)$ is any vector field on the product $X \times \mathbb{C}$ and $m \geq 2$ any number, then $\mathrm{L}_{\vec{R}^m} \vec{Z}$ can be expressed as a linear combination of terms $\vec{T}$ of the form

$$\vec{T} = \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_m}(\vec{Z}),$$

where all $\vec{F}_i$ are either equal to $D(\vec{A})$ or equal to $D(\vec{B})$.

Proof. We prove Lemma 3.10 by induction on m. If $m = 2$, then the Jacobi Identity (3.9.1) asserts that

$$\mathrm{L}_{\vec{R}^2} \vec{Z} = \mathrm{L}_{[D(\vec{A}),D(\vec{B})]} \vec{Z} = \mathrm{L}_{D(\vec{A})} \circ \mathrm{L}_{D(\vec{B})} \vec{Z} - \mathrm{L}_{D(\vec{B})} \circ \mathrm{L}_{D(\vec{A})} \vec{Z},$$

showing the claim in case where $m = 2$. For $m > 2$, the Jacobi identity gives

$$\mathrm{L}_{\vec{R}^m} \vec{Z} = \mathrm{L}_{[D(\vec{A}),\vec{R}^{m-1}]} \vec{Z} = \mathrm{L}_{D(\vec{A})} \circ \mathrm{L}_{\vec{R}^{m-1}} \vec{Z} - \mathrm{L}_{\vec{R}^{m-1}} \circ \mathrm{L}_{D(\vec{A})} \vec{Z},$$

showing the claim inductively. □

Lemma 3.11. *In the setup of Proposition 3.7, if $\vec{F}_1, \ldots, \vec{F}_n$ is a sequence of time-dependent vector fields such that all $\vec{F}_i$ are either equal to $D(\vec{A})$ or equal to $D(\vec{B})$, then*

$$\Big(\mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\big(\tfrac{\partial}{\partial t}\big)\Big)_0\Big|_Y \in H^0(Y, \mathscr{G}). \tag{3.11.1}$$

Proof. If all $\vec{F}_i$ are equal to $D(\vec{A})$ or all $\vec{F}_i$ are equal to $D(\vec{B})$, then the statement follows from Remark 3.4. We can thus assume without loss of generality that $\vec{F}_n = D(\vec{B})$, and that at least one of the $\vec{F}_i$, for $i < n$, is equal to $D(\vec{A})$.

As a first step in the proof of Lemma 3.11, we show the following claim.

Claim 3.11.2. Let $1 \leq i \leq n-2$ be any number, and consider the vector field[1]

$$\begin{aligned}\vec{T} &:= \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_{i-1}} \circ \mathrm{L}_{[\vec{F}_i, \vec{F}_{i+1}]} \circ \mathrm{L}_{\vec{F}_{i+2}} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\left(\tfrac{\partial}{\partial t}\right) \\ &= \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_{i-1}} \circ \mathrm{L}_{\vec{F}_i} \circ \mathrm{L}_{\vec{F}_{i+1}} \circ \mathrm{L}_{\vec{F}_{i+2}} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\left(\tfrac{\partial}{\partial t}\right) - \\ &\quad \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_{i-1}} \circ \mathrm{L}_{\vec{F}_{i+1}} \circ \mathrm{L}_{\vec{F}_i} \circ \mathrm{L}_{\vec{F}_{i+2}} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\left(\tfrac{\partial}{\partial t}\right).\end{aligned} \tag{3.11.3}$$

Then $\vec{T}_0\big|_Y \in H^0\big(Y, \mathscr{G}\big)$.

Using a somewhat tedious inductive argument, which we leave to the reader, one verifies that the vector field $\vec{T}$ can be expressed as follows,

$$\vec{T} := \sum_{\substack{\alpha \text{ a subsequence} \\ \text{of } (1,2,\ldots,i-1)}} [\vec{P}^\alpha, \vec{T}^\alpha]$$

where the subsequence α is written as, $\alpha = \big(\alpha(1), \ldots, \alpha(k)\big)$, where $\overline{\alpha} = \big(\overline{\alpha}(1), \ldots, \overline{\alpha}(i-1-k)\big)$ denotes the complementary subsequence, and where

$$\begin{aligned}\vec{P}^\alpha &:= \mathrm{L}_{\vec{F}_{\alpha(1)}} \circ \cdots \circ \mathrm{L}_{\vec{F}_{\alpha(k)}} \circ \mathrm{L}_{\vec{F}_i}\left(\vec{F}_{i+1}\right) && \text{and} \\ \vec{T}^\alpha &:= \mathrm{L}_{\vec{F}_{\overline{\alpha}(1)}} \circ \cdots \circ \mathrm{L}_{\vec{F}_{\overline{\alpha}(l)}} \circ \mathrm{L}_{\vec{F}_{i+2}} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\left(\tfrac{\partial}{\partial t}\right).\end{aligned}$$

There are two things to note in this setting.

(1) Since $\mathscr{F}$ is closed under Lie-bracket, the restriction to time $t = 0$ of the vector field $\vec{T}^\alpha$ is a section of $\mathscr{F}$, that is, $\big(\vec{T}^\alpha\big)_0 \in H^0\big(X, \mathscr{F}\big)$, for all subsequences α.
(2) The vector field $(\vec{P}^\alpha)_0 \in H^0\big(X, \mathscr{F}\big)$ is the restriction to time $t = 0$ of an iterated Lie bracket of at most $(n-1)$ time-dependent vector fields in $\mathscr{F}$, all inducing the same $(n-1)$-st order jets on $Y \subseteq X$. Fact 2.11 therefore implies that $(\vec{P}^\alpha)_0$ vanishes on Y, again for all subsequences α.

Consequently, the restriction of $\vec{T}_0$ to Y satisfies

$$\vec{T}_0|_Y = \sum_\alpha\, [\vec{P}^\alpha_0, \vec{T}^\alpha_0] = \sum_\alpha\, [\vec{P}^\alpha_0 + \vec{T}^\alpha_0, \vec{T}^\alpha_0] \in H^0\big(Y, \mathscr{G}\big),$$

by Definition 1.5. This ends the proof of Claim 3.11.2.

Claim 3.11.2 asserts that Equation (3.11.1) holds if and only if it holds after permuting the operators $\mathrm{L}_{\vec{F}_i}$ and $\mathrm{L}_{\vec{F}_{i+1}}$. In other words, Equation (3.11.1) holds if and only if

$$\Big(\mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_{i+1}} \circ \mathrm{L}_{\vec{F}_i} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\left(\tfrac{\partial}{\partial t}\right)\Big)_0\Big|_Y \in H^0\big(Y, \mathscr{G}\big).$$

Using the classical "bubble-sort" algorithm, we can therefore sort the $\vec{F}_i$ and assume without loss of generality that there exists a number k such that $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_k$ are all equal to $D(\vec{A})$, whereas $\vec{F}_{k+1}, \ldots, \vec{F}_n$ are all equal to $D(\vec{B})$. In this situation an argument similar to, but easier than the proof of Claim 3.11.4 then shows the following.

[1] The equality in (3.11.3) is again Jacobi's identity in the form of Remark 3.9.

Claim 3.11.4. If $\vec{S}$ denotes the following vector field,

$$\vec{S} := \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\big(\tfrac{\partial}{\partial t}\big) - \mathrm{L}_{D(\vec{B})} \circ \mathrm{L}_{\vec{F}_2} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\big(\tfrac{\partial}{\partial t}\big),$$

then $\vec{S}_0|_Y \in H^0\big(Y, \mathscr{G}\big)$.

The proof of Claim 3.11.4 is left to the reader. Claim 3.11.4 allows to replace $\vec{F}_1 = D(\vec{A})$ by $D(\vec{B})$. Applying the 'Sorting Claim 3.11.2' and the 'Replacement Claim 3.11.4' exactly k times, this reduces the Lemma 3.11 to the case where $\vec{F}_1 = \cdots = \vec{F}_n = D(\vec{B})$, where the lemma is known to be true, thus finishing the proof of Lemma 3.11. □

3.5. **Proof of Proposition 3.7.** By Fact 2.11, the time-independent vector field

$$\vec{\Delta} := \big(\mathrm{L}^n_{D(\vec{A})}\, D(\vec{B})\big)_0 \in H^0\big(X, T_X\big)$$

at time $t = 0$ describes the difference between the $(n+1)$-st order jets $\tau^{n+1}_{\vec{A}}$ and $\tau^{n+1}_{\vec{B}}$ along $Y \subseteq X$. We need to show that the restriction $\vec{\Delta}|_Y$ is a section of the obstruction sheaf $\mathscr{G}$. For simplicity of argument, we discuss the cases $n = 1$ and $n > 1$ separately in the next two subsections.

Proof in case $n = 1$. If $n = 1$, expand the definition of $\vec{\Delta}$ and of $D(\vec{A})$ and $D(\vec{B})$ to obtain

$$\vec{\Delta}\big|_Y = \big(\mathrm{L}_{D(\vec{A})}\, D(\vec{B})\big)_0\big|_Y = \big(\mathrm{L}_{\vec{A}}\, \vec{B}\big)_0\big|_Y + \underbrace{\big(\mathrm{L}_{\frac{\partial}{\partial t}}\, \vec{B}\big)_0\big|_Y}_{\in \mathscr{G} \text{ by Rem. 3.4}} + \underbrace{\big(\mathrm{L}_{\vec{A}}\, \tfrac{\partial}{\partial t}\big)_0\big|_Y}_{\in \mathscr{G} \text{ by Rem. 3.4}}.$$

To conclude, it thus suffices to show that

$$\big(\mathrm{L}_{\vec{A}}\, \vec{B}\big)_0\big|_Y = \big(\mathrm{L}_{\vec{A}_0}\, \vec{B}_0\big)\big|_Y \in H^0\big(Y, \mathscr{G}\big).$$

This, however, follows from the assumption that $\vec{A}$ and $\vec{B}$ be admissible vector fields, so that $\vec{A}_0, \vec{B}_0 \in H^0\big(X, \mathscr{F}\big)$, and from Definition 1.5 of *obstruction sheaf*. This proves Proposition 3.7 in case where $n = 1$. □

Proof in case $n > 1$. Consider the time-dependent vector field

$$\vec{R} := \mathrm{L}^{n-1}_{D(\vec{A})}\, D(\vec{B}).$$

Fact 2.11 and the assumed equality of n-th order jets imply that $\vec{R}_0|_Y = 0$. As one consequence, we obtain that

$$\big(L_{\vec{R}_0}\vec{A}_0\big)\big|_Y = \big(L_{\vec{R}_0+\vec{A}_0}\vec{A}_0\big)\big|_Y \in H^0\big(Y, \mathscr{G}\big),$$

according to Definition 1.5, where obstruction sheaves were introduced. The equation

$$\vec{\Delta}\big|_Y = \big(\mathrm{L}_{D(\vec{A})}\, \vec{R}\big)_0\big|_Y = -\big(\mathrm{L}_{\vec{R}}\, D(\vec{A})\big)_0\big|_Y = \underbrace{-\,\mathrm{L}_{\vec{R}_0}\big(\vec{A}_0\big)\big|_Y}_{\in \mathscr{G}} - \big(\mathrm{L}_{\vec{R}}\, \tfrac{\partial}{\partial t}\big)_0\big|_Y$$

thus asserts that to prove Proposition 3.7, it suffices to show that

$$\big(\mathrm{L}_{\vec{R}}\, \tfrac{\partial}{\partial t}\big)_0\big|_Y \in H^0\big(Y, \mathscr{G}\big). \tag{3.11.5}$$

To this end, recall from Lemma 3.10 that $\mathrm{L}_{\vec{R}}\, \frac{\partial}{\partial t}$ can be expressed as a linear combination of terms $\vec{T}$ of the form

$$\vec{T} = \mathrm{L}_{\vec{F}_1} \circ \cdots \circ \mathrm{L}_{\vec{F}_n}\big(\tfrac{\partial}{\partial t}\big)$$

with $\vec{F}_i \in \{D(\vec{A}), D(\vec{B})\}$ for $1 \leq i \leq n$. Lemma 3.11 then asserts that every one of these terms is contained in $H^0\big(Y, \mathscr{G}\big)$ when restricted at time $t = 0$ to Y, thus finishing the proof of Proposition 3.7 in case where $n > 1$. $\square$

4. ADMISSIBLE HIGHER-ORDER INFINITESIMAL DEFORMATIONS

In this section, we generalize the notion of admissibility to jets of arbitrary order. We employ Proposition 3.6 and 3.7 to show that - under a suitable cohomology vanishing assumption - first-order infinitesimal deformations that are locally induced by admissible vector fields can always be lifted to *admissible* infinitesimal deformations of arbitrary order. We maintain the assumptions spelled out in Setup 3.1 on page 8.

Definition 4.1 (Admissible higher-order infinitesimal deformations). *In Setup 3.1, let $n \geq 1$. A section $\tau^n : Y \to \mathrm{Jet}^n(X)$ over f is said to be* admissible for the obstruction sheaf $\mathscr{G}$*, if there exists a cover of $Y \subset X$ by open subsets, say $Y \subseteq \bigcup_{j \in J} X_j^\circ$, and time-dependent vector fields $\vec{A}_j$ on X_j° for any $j \in J$ such that the following two conditions hold true for every $j \in J$.*

1. *The vector field $\vec{A}_j$ is n-admissible for the obstruction sheaf $\mathscr{G}|_{Y_i^\circ}$, where $Y_j^\circ := Y \cap X_j^\circ$.*
2. *The restriction of τ^n to Y_j° is induced by $\vec{A}_j$. In other words, $\tau^n\big|_{Y_j^\circ} = \tau^n_{\vec{A}_j}\big|_{Y_j^\circ}$.*

Remark 4.2. Recall from Remark 3.5 that time-independent vector fields in $\mathscr{F}$ are always admissible. In Setup 3.1, any infinitesimal deformation $\sigma \in H^0\big(Y, \mathscr{F}_Y\big)$ locally induced by $\mathscr{F}$ is therefore admissible for the obstruction sheaf $\mathscr{G}$ in the sense of Definition 4.1 above.

Proposition 4.3 (Lifting admissible sections to arbitrary order). *In Setup 3.1, suppose that $H^1\big(Y, \mathscr{G}\big) = 0$. Then any section $\tau^n : Y \to \mathrm{Jet}^n(X)$ over f that is admissible for $\mathscr{G}$ admits a lift to a section $\tau^{n+1} : Y \to \mathrm{Jet}^{n+1}(X)$ that is likewise admissible for $\mathscr{G}$.*

Proof. Fix an open cover $Y \subseteq \bigcup_{j \in J} X_j^\circ$ and n-admissible vector fields $\vec{A}_j$ as in Definition 4.1. We write $Y_j^\circ := X_j^\circ \cap Y$, $X_{jk}^\circ := X_j^\circ \cap X_k^\circ$ and $Y_{jk}^\circ := Y_j^\circ \cap Y_k^\circ$ for $j, k \in J$.

Since $\tau^n_{\vec{A}_j}\big|_{Y_{jk}^\circ} = \tau^n_{\vec{A}_k}\big|_{Y_{jk}^\circ}$ for $j, k \in J$ by Item (2) in Assumptions 4.1, we may calculate the difference between $(n+1)$-st order jets. By Assumption (1) and by Proposition 3.7, this difference is described by a section in the obstruction sheaf $\mathscr{G}$, that is,

$$\vec{C}_{jk} := \tau^{n+1}_{\vec{A}_j}\big|_{Y_{jk}^\circ} - \tau^{n+1}_{\vec{A}_k}\big|_{Y_{jk}^\circ} \in \mathscr{G}(Y_{jk}^\circ) \tag{4.3.1}$$

The general properties of affine bundles imply that the family $(\vec{C}_{jk})_{jk}$ is a Čech 1-cocycle of the sheaf $\mathscr{G}$ with respect to the open cover $Y = \bigcup_j Y_j^\circ$. The cohomology vanishing assumption ensures that $(\vec{C}_{jk})_{jk}$ is a Čech 1-coboundary, that is, that there exist sections $\vec{C}_j \in \mathscr{G}(Y_j^\circ)$ such that

$$\vec{C}_{jk} = \vec{C}_k\big|_{Y_{jk}^\circ} - \vec{C}_j\big|_{Y_{jk}^\circ} \in \mathscr{G}(Y_{jk}^\circ). \tag{4.3.2}$$

Taking (4.3.1) and (4.3.2) together, the sections

$$\tau^{n+1}_{\vec{A}_j}\big|_{Y_j^\circ} + \vec{C}_j : Y_i^\circ \to \mathrm{Jet}^{n+1}(X), \tag{4.3.3}$$

with j running over the index set J, glue, giving a section $\tau^{n+1} : Y \to \mathrm{Jet}^{n+1}(X)$ over τ^n.

Refining the open cover, Equation (4.3.3) and Proposition 3.6 allow to assume that $\vec{C}_j$ is induced by a vector field on X_j° which is $(n+1)$-admissible for $\mathscr{G}|_{Y_j^\circ}$, as required. □

5. Proof of Theorems 1.7 and 1.19

We maintain assumptions and notation of Theorems 1.7 and 1.19, where X is a complex manifold, $\mathscr{F} \subset T_X$ a Lie-closed subsheaf of $\mathbb{C}_X$-modules, where $Y \subset X$ is a reduced, compact complex subspace with inclusion map $f : Y \to X$, and $\mathscr{G}$ an obstruction sheaf for $\mathscr{F}$. Let $\sigma \in H^0\big(Y,\, \mathscr{F}_Y\big)$ be an infinitesimal deformation of f that is locally induced by $\mathscr{F}$, and assume that $H^1\big(Y,\, \mathscr{G}\big) = 0$.

5.1. **Proof of the Theorem 1.7.** By Proposition 4.3, the section $\tau^1 : Y \to \operatorname{Jet}^1(X)$ corresponding to the first order deformation $\sigma \in H^0\big(Y,\, \mathscr{F}_Y\big)$ of f inductively admits lifts to sections

$$\tau^n : Y \to \operatorname{Jet}^n(X)$$

for any natural number n. The family $(\tau^n)_{n\in\mathbb{N}}$ corresponds to a formal curve

$$\operatorname{Spec} \mathbb{C}[[t]] \to \operatorname{Hom}(Y, X) \tag{5.0.4}$$

at $[f] \in \operatorname{Hom}(Y, X)$ whose associated Zariski tangent vector equals σ. The existence of a holomorphic curve at $[f] \in \operatorname{Hom}(Y, X)$ with derivative σ then follows from a classical result of Michael Artin, [Art68, Thm. 1.2]. □

5.2. **Proof of Theorem 1.19.** If $H \subseteq \operatorname{Hom}(Y, X)$ is a subspace of deformations along $\mathscr{F}$, then the formal curve (5.0.4) factors through H,

$$\operatorname{Spec} \mathbb{C}[[t]] \to H \subseteq \operatorname{Hom}(Y, X).$$

In particular, the holomorphic curve at $[f] \in \operatorname{Hom}(Y, X)$ given by [Art68] can be required to lie in H. □

6. Example: Embeddings into complex-symplectic manifolds

In this section, we apply Theorem 1.7 to embedding morphisms into complex-symplectic manifolds. The following assumptions will be maintained throughout the present section.

Setup 6.1. *Let (X, ω) be a complex-symplectic manifold, let $\mathscr{F} \subset T_X$ be the subsheaf of Hamiltonian vector fields, and $f : Y \to X$ the inclusion of a compact submanifold.*

6.1. **Infinitesimal deformations locally induced by Hamiltonian vector fields.** We start off with a discussion of the sheaf of infinitesimal deformations locally induced by $\mathscr{F}$. Perhaps somewhat surprisingly, it will turn out that *all* infinitesimal deformations of f are locally induced by $\mathscr{F}$, as long as Y is either a curve, surface, or a Kähler manifold.

Notation 6.2. If $U \subseteq Y$ is any open set, and $\vec{A} \in \big(f^*T_X\big)(U)$, consider the associated section $\eta_{\vec{A}} := \big(f^*\omega\big)(\vec{A}, \cdot) \in f^*\Omega^1_X$ and the associated form $\xi_{\vec{A}} := (df)(\eta_{\vec{A}}) \in \Omega^1_Y$.

Proposition 6.3 (Infinitesimal deformations locally induced by Hamiltonians)**.** *In Setup 6.1, let $\vec{A} \in H^0\big(Y,\, f^*T_X\big)$ be an infinitesimal deformation of f. If $\xi_{\vec{A}}$ is closed, then $\vec{A}$ is locally induced by $\mathscr{F}$.*

Proof. The question being local on Y, we can assume that there are coordinates $x_1, \dots, x_n, y_1, \dots, y_m$ on X such that the submanifold Y is given as $Y = \{x_1 = \dots = x_n = 0\}$. If $\xi_{\vec{A}}$ is closed, it can locally be written as $\xi_{\vec{A}} = dg$, where $g \in \mathscr{O}_Y$ is a suitable holomorphic function. The section $\eta_{\vec{A}}$ is then written as

$$\eta_{\vec{A}} = dg + \sum_{i=1}^{n} g_i(y_1, \dots, y_m) \cdot dx_i$$

where again $g_i \in \mathscr{O}_Y$ are suitable holomorphic functions. Consider the function

$$G := g + \sum_{i=1}^{n} g_i(y_1, \dots, y_m) x_i,$$

defined on a neighborhood of Y in X. To finish the proof, observe that the image of its associated Hamiltonian vector field Hamilton(G) in f^*T_X agrees with $\vec{A}$. □

Corollary 6.4 (All infinitesimal deformations are locally induced by Hamiltonians)**.** *If Y is Kähler or if $\dim Y < 3$, then any infinitesimal deformation of f is locally induced by $\mathscr{F}$. In other words,*

$$H^0(Y, \text{Image}\, \mathscr{F}_Y \to f^*T_X) = H^0(Y, f^*T_X).$$

Proof. The proof follows immediately from Proposition 6.3 and from the fact that any holomorphic 1-form on a compact curve, surface or Kähler manifolds is automatically closed, cf. [BHPVdV04, Chapt. IV.2] and [Voi07, Thm. 8.28]. □

6.2. **Obstruction sheaf in the symplectic setup.** Next, we show that in the symplectic setting, there exists an obstruction sheaf that can be understood geometrically. We recall the notation of vector fields that are perpendicular to Y, already introduced in Example 1.8.

Notation 6.5. Let $T_Y^\perp \subseteq f^*T_X$ be the subsheaf of sections which are perpendicular to Y, i.e., the sheaf associated to the presheaf

$$U \mapsto \{\vec{A} \in f^*(T_X)(U) \mid (f^*\omega)(\vec{A}, \vec{B}) = 0 \text{ for all } \vec{B} \in T_Y(U)\}$$

where $U \subset Y$ runs over the open subsets. The sheaf $T_Y^\perp$ is a subbundle of f^*T_X. Its rank equals $\operatorname{codim}_X Y$.

We aim to show that $T_Y^\perp$ is an obstruction sheaf, in the sense of Definition 1.5. To start, we need to show that $T_Y^\perp \subseteq \mathscr{F}_Y$. In order to prove this, apply the bundle isomorphism $T_X \cong \Omega_X^1$ induced by ω to both sides of the inclusion. To prove $T_Y^\perp \subseteq \mathscr{F}_Y$ we thus need to show that sections of the kernel of $df : f^*\Omega_X^1 \to \Omega_Y^1$ can locally be extended to closed forms, defined on a neighborhood of Y. In complete analogy to the proof of Proposition 6.3, this follows from a short calculation in local coordinates which we leave to the reader.

The following proposition then shows the remaining property required for $T_Y^\perp$ to be an obstruction sheaf.

Proposition 6.6 (Lie brackets of Hamiltonian vector fields)**.** *Let $U \subseteq X$ be any open subset, $Y^\circ := Y \cap U$, and let $\vec{F}, \vec{G} \in \mathscr{F}(U)$ be two Hamiltonian vector fields on U that agree along Y°. Then the Lie bracket $[\vec{F}, \vec{G}]$ is perpendicular to Y. In other words, $f^*[\vec{F}, \vec{G}] \in T_Y^\perp(Y^\circ)$.*

Proof. The statement of Proposition 6.6 being local on Y, it suffices to show that

$$\omega\left(\vec{V}, [\vec{F}, \vec{G}]\right)|_{Y^\circ} = 0 \tag{6.6.1}$$

for any vector field $\vec{V} \in H^0\big(U,\, T_X\big)$ which is tangent to Y°, i.e., which satisfies

$$\vec{V}|_y \in T_Y|_y \subseteq T_X|_y \quad \text{for all } y \in Y^\circ.$$

Given any such $\vec{V}$, the equality $(\vec{F} - \vec{G})|_{Y^\circ} = 0$ implies that

$$\mathrm{L}_{\vec{F}-\vec{G}}(\vec{V})|_{Y^\circ} = 0. \tag{6.6.2}$$

Equality (6.6.1) then follows with

$$\begin{aligned} 0 &= \mathrm{L}_{\vec{F}-\vec{G}}\big(\omega(\vec{V}, \vec{G})\big)|_{Y^\circ} && \text{since } (\vec{F} - \vec{G})|_{Y^\circ} = 0 \\ &= \omega(\mathrm{L}_{\vec{F}-\vec{G}}\, \vec{V}, \vec{G})|_{Y^\circ} + \omega(\vec{V}, \mathrm{L}_{\vec{F}-\vec{G}}\, \vec{G})|_{Y^\circ} && \text{since } \vec{F} \text{ and } \vec{G} \text{ preserve } \omega \\ &= \omega(\vec{V}, [\vec{F}, \vec{G}])|_{Y^\circ} && \text{by (6.6.2)} \end{aligned}$$

This finishes the proof of Proposition 6.6. □

Corollary 6.7 (Obstruction sheaf in the symplectic setup)**.** *The sheaf $T_Y^\perp \subset \mathscr{F}_Y$ is an obstruction sheaf for the sheaf $\mathscr{F} \subset T_X$ of Hamiltonian vector fields.* □

6.3. **A space of deformations along $\mathscr{F}$.** As a last step in the discussion of symplectic spaces, we aim to identify a space of deformations along $\mathscr{F}$. Since Hamiltonian vector fields preserve the symplectic form by definition, the following space is a candidate.

Definition 6.8 (Space of morphisms with prescribed pull-back of ω)**.** *Consider subspace*

$$\mathrm{Hom}_\omega(Y, X) := \{g \in \mathrm{Hom}(Y, X) \,|\, g^*(\omega) = f^*(\omega)\} \subseteq \mathrm{Hom}(Y, X)$$

with its natural structure as a (not necessarily reduced) complex space.

The following is now an elementary consequence of the fact that Hamiltonian vector fields preserve ω.

Fact 6.9 ($\mathrm{Hom}_\omega(Y, X)$ is a space of deformations along $\mathscr{F}$, cf. [Jö10, Prop. 3.1.10])**.** *If n is any number and f_n any n-th order infinitesimal deformation of f that is locally induced by time-dependent Hamiltonian vector fields, then the corresponding morphism*

$$f_n : \mathrm{Spec}\ \mathbb{C}[t]/t^{n+1} \to \mathrm{Hom}(Y, X)$$

factors via $\mathrm{Hom}_\omega(Y, X)$. In particular, $\mathrm{Hom}_\omega(Y, X) \subseteq \mathrm{Hom}(Y, X)$ is a space of deformations along $\mathscr{F}$. □

6.4. **Summary of results in the symplectic case.** Putting Corollaries 6.4, 6.7, Fact 6.9 and the main results of this paper, Theorems 1.7 and 1.19 together, the following corollary summarizes our results in the symplectic setting.

Corollary 6.10. *Let $f : Y \to X$ be the embedding of a compact curve, surface of Kähler submanifold into a complex-symplectic manifold (X, ω). Assume that $H^1\big(Y,\, T_Y^\perp\big) = 0$. Then any infinitesimal deformation $\sigma \in H^0\big(Y,\, f^*T_X\big)$ is effective.*

More is true: There exists a family $F : \Delta \times Y \to X$ of morphisms such that the infinitesimal deformation induced by F agrees with σ, i.e., $\sigma_{F,0} = \sigma$, and such that $F_t^*(\omega) = f^*(\omega)$ for any $t \in \Delta$. □

6.5. **Generalisations.** Many of the results contained in this section carry over to the case where X is not necessarily symplectic, but carries a two-tensor $\omega \in H^0\big(X,\, \Omega^1_X \otimes \Omega^1_X\big)$, which need not be alternating, symmetric, or non-degenerate. Details are found in [Jö10].

7. Example: Deformation along a foliation

This subsection is concerned with the case when X carries a regular foliation, that is, a subbundle $\mathscr{F} \subseteq T_X$ closed under Lie bracket. We fix the following setup for the present section.

Setup 7.1. *Let $\mathscr{F} \subseteq T_X$ be a regular foliation on a complex manifold X, and let $f : Y \to X$ be the inclusion of a compact submanifold $Y \subseteq X$.*

7.1. **Infinitesimal deformations locally induced by the foliation.** Since $\mathscr{F}$ is a subbundle of T_X, it is clear that in the context of Setup 7.1, the sheaf of infinitesimal deformations locally induced by $\mathscr{F}$ is $\mathscr{F}_Y = f^*\mathscr{F}$. The space of infinitesimal deformations locally induced by $\mathscr{F}$ is then $H^0\big(Y,\, f^*\mathscr{F}\big)$.

7.2. **Obstruction sheaf in the foliated setup.** Of course, the restricted subbundle $\mathscr{F}_Y \subset f^*(T_X)$ is an obstruction sheaf for the foliation $\mathscr{F}$. If the leaves intersect the submanifold $f(Y)$ transversely, then $\mathscr{F}_Y$ is in fact the only possible obstruction sheaf. However, if the set

$$T := \{y \in Y \mid \mathscr{F}|_y \subseteq T_Y|_y\} \tag{7.1.1}$$

of points where the foliation is tangent to Y is non-empty, then the following proposition asserts that the proper subsheaf $\mathscr{G} := \mathscr{J}_T \otimes \mathscr{F}_Y \subset \mathscr{F}_Y$ is an obstruction sheaf as well.

Proposition 7.2. *In Setup 7.1, let $U \subset X$ be any open subset, set $Y^\circ := Y \cap U$ and $T^\circ := T \cap U$. If $\vec{A}, \vec{B} \in \mathscr{F}(U)$ are two vector fields in $\mathscr{F}$ that agree along Y°, then the Lie bracket vanishes along T°,*

$$f^*\big([\vec{A}, \vec{B}]\big)|_{T^\circ} = 0 \in \mathscr{F}_Y(T^\circ).$$

Proof. We consider vector fields as derivations acting on the structure sheaf $\mathscr{O}_X$. From this point of view, we need to show $\big([\vec{A}, \vec{B}].g\big)(x) = 0$ for any point $x \in T^\circ$ and any germ of function $g \in \mathscr{O}_{X,x}$. We know that

$$[\vec{A}, \vec{B}].g = [\vec{A}, \vec{B} - \vec{A}].g = \vec{A}.\big(\underbrace{(\vec{B} - \vec{A}).g}_{=:a}\big) - \underbrace{(\vec{B} - \vec{A}).(\vec{A}.g)}_{=:b},$$

The terms a and b vanish along Y° because $\vec{A}|_{Y^\circ} = \vec{B}|_{Y^\circ}$. Since $\vec{A} \in \mathscr{F}$ is tangent to Y at all points $x \in T^\circ$, the assertion follows. □

Corollary 7.3 (Obstruction sheaf in the case of a foliation)**.** *In Setup 7.1, any sheaf $\mathscr{G}$ satisfying*

$$\mathscr{J}_T \otimes \mathscr{F}|_Y \subseteq \mathscr{G} \subseteq \mathscr{F}|_Y$$

is an obstruction sheaf for the foliation $\mathscr{F}$. □

7.3. **A space of deformations along $\mathscr{F}$.** A space of deformations along $\mathscr{F}$ has been constructed in [KKL10]. The following notation is useful in the description of its main property.

Notation 7.4 (Velocity vector field for families of morphisms, cf. [KKL10, Sect. 1.B]). Let $F : \Delta \times Y \to X$ be a family of morphisms such that $F_0 = f$. Given a point $y \in Y$, we can consider the curve

$$F_y : \Delta \to X, \quad t \mapsto F(t, y).$$

Given $t_0 \in \Delta$ and taking derivatives in t for all y at time $t = t_0$, this gives a section

$$\sigma_{F,t_0} \in H^0\big(Y,\, (F_{t_0})^* T_X\big),$$

called *velocity vector field at time t_0*.

Fact 7.5 (Space of deformations along a foliation, cf. proof of [KKL10, Cor. 5.6]). *In Setup 7.1, there exists a space of deformations along $\mathscr{F}$, denoted*

$$\operatorname{Hom}_{\mathscr{F}}(Y, X) \subseteq \operatorname{Hom}(Y, X),$$

with the following additional property. If $F : \Delta \to \operatorname{Hom}_{\mathscr{F}}(Y, X)$ is any holomorphic curve germ, with associated family of morphisms $F : \Delta \times Y \to X$, then the velocity vector fields are in the pull-back of the foliation $\mathscr{F}$, for all times $t \in \Delta$. In other words, $\sigma_{F,t} \in H^0\big(Y,\, F_t^*\mathscr{F}\big)$ for all $t \in \Delta$. □

The actual statement of [KKL10, Cor. 5.6] only implies that infinitesimal deformations induced by *time-independent* vector fields in $\mathscr{F}$ factor through $Hom_{\mathscr{F}}(Y, X)$. However, the proof in *loc. cit.* can be generalized with minimal changes to the case of *time-dependent* vector fields, as required in Definition 1.16.

7.4. **Summary of results in the case of a foliated manifold.** Using Corollary 7.3 and Fact 7.5, the following corollary summarizes our results in the case of a foliated manifolds.

Corollary 7.6. *In Setup 7.1, let $\mathscr{G}$ be any subsheaf satisfying*

$$\mathscr{J}_T \otimes \mathscr{F}_Y \subseteq \mathscr{G} \subseteq \mathscr{F}|_Y,$$

where $T \subseteq Y$ is the space defined in (7.1.1) *above. Assume that $H^1\big(Y, \mathscr{G}\big) = 0$.*

Then any infinitesimal deformation $\sigma \in H^0\big(Y, \mathscr{F}|_Y\big)$ is effective. More is true: there exists a family $F : \Delta \times Y \to X$ of morphisms such that $F_0 = f$, $\sigma_{F,0} = \sigma$ and, such that the velocity vector fields $\sigma_{F,t}$ are contained in $H^0\big(Y,\, F_t^*\mathscr{F}\big)$, for all $t \in \Delta$. □

Remark 7.7. It might be worth noting that to obtain the conclusions of Corollary 7.6, it suffices to prove vanishing $H^1\big(Y, \mathscr{G}\big) = 0$ for a single obstruction sheaf $\mathscr{G}$. Since the obstruction sheaf is often not uniquely defined, this gives extra leeway which might be useful in applications.

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CLEMENS JÖRDER, MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITT FREIBURG, ECKERSTRASSE 1, 79104 FREIBURG IM BREISGAU, GERMANY
E-mail address: `clemens.joerder@math.uni-freiburg.de`

STEFAN KEBEKUS, MATHEMATISCHES INSTITUT, ALBERT-LUDWIGS-UNIVERSITT FREIBURG, ECKERSTRASSE 1, 79104 FREIBURG IM BREISGAU, GERMANY
E-mail address: `stefan.kebekus@math.uni-freiburg.de`
URL: `http://home.mathematik.uni-freiburg.de/kebekus`