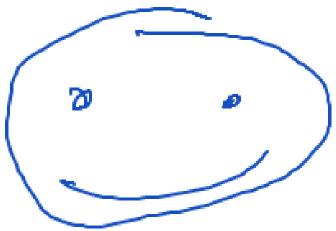

$\mathbb{P}^2$ $\qquad\qquad$ $\mathrm{Bir}\,\mathbb{P}^2$

Th. (Dolgachev, Iskovskikh, 2006)

Class. $G \underset{\text{fin.}}{\subset} \mathrm{Bir}\,\mathbb{P}^2$

Many groups, including complicated ones

e.g. $\mathfrak{A}_6$, $PSL_2(\mathbb{F}_7) \subset PGL_3(\mathbb{C})$

$(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2 \not\subset PGL_3$

$\subset \mathrm{Aut}\,\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathrm{Bir}\,\mathbb{P}^2$

$S$ Severi-Brauer surface $/K$, $\operatorname{char} K = 0$

S.B. variety $X$ of $\dim = n-1$ $\Leftrightarrow X_{\bar{K}} \cong \mathbb{P}^{n-1}_{\bar{K}}$

non-trivial $\Leftrightarrow X \not\cong \mathbb{P}^{n-1}$

S.B. varieties of $\dim = n-1$ $\overset{1:1}{\longleftrightarrow}$ central simple algebras of $\dim = n^2 / K$

Th.: $X(K) \neq \emptyset \Rightarrow X \cong \mathbb{P}^{n-1}$

Th. (Châtelet, 1942) $\operatorname{Aut} X = A^* / K^*$

$X$ SB var. $\longleftrightarrow$ c.s.a. $A$

Example: $\operatorname{Aut} \mathbb{P}^{n-1} \cong \operatorname{Mat}_{n\times n}{}^* / K^* = GL_n / K^*$

Main th. (—, 2020) $S$ non tr. S.B. surface $/\mathbb{K}$, char $= 0$

1. $G \underset{\text{fin.}}{\subset} \underline{\mathrm{Aut}\, S} \Rightarrow G \simeq \mathbb{Z}/n\mathbb{Z}$ for some $n = \prod p_i^{r_i}$, $p_i \equiv 1 \bmod 3$

$\mathbb{Z}/3n\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/3$,
(some)

$\mathbb{Z}/3n\mathbb{Z} \rtimes \mathbb{Z}/3$
(including $\mathbb{Z}/3 \rtimes \mathbb{Z}/3$)

2. $G \underset{\text{fin}}{\subset} \underline{\mathrm{Bir}\, S}$, $G$ not as above $\Rightarrow G \simeq \underline{(\mathbb{Z}/3\mathbb{Z})^3}$

3. For any group like this $\exists K$ $\exists S$ s.t. $G \subset \mathrm{Aut}\, S \subset \mathrm{Bir}\, S$

Corollary. $G \underset{\text{fin.}}{\subset} \operatorname{Bir} S \Rightarrow$ $G$ abelian, or has a normal abelian subgr. of index 3

Cf. Theorem of Serre – Yasinsky: $G \underset{\text{fin.}}{\subset} \operatorname{Bir} \mathbb{P}^2$
$\Rightarrow G$ contains a ~~fin.~~ normal ab. subgroup of index $\leq 7200$. $\left( (A_5 \times A_5) \rtimes \mathbb{Z}/2 \right)$

Th. $S/\mathbb{Q}$ non-tr. S.B. surface, $G \underset{\text{fin.}}{\subset} \operatorname{Bir} S \Rightarrow$

$$\Rightarrow \quad G \simeq (\mathbb{Z}/3)^r, \; r \leq 3.$$

Th. (Trepalin, 2020)

$S$ non-tr. S.B. surface $/K$, $G \underset{\text{fin.}}{\subset} \operatorname{Aut} S$

$$S/G \sim \mathbb{P}^2 \iff S/G(K) \neq \emptyset \iff |G| \,\vdots\, 3$$

$$S/G \sim S \iff S/G(K) = \emptyset \iff |G| \not\vdots\, 3$$

NB. $\mathbb{P}^2/G \sim \mathbb{P}^2$

Plan: 1. birational models of S.B. surfaces

2. restrictions on subgr. of Aut S and Bir S

3. examples

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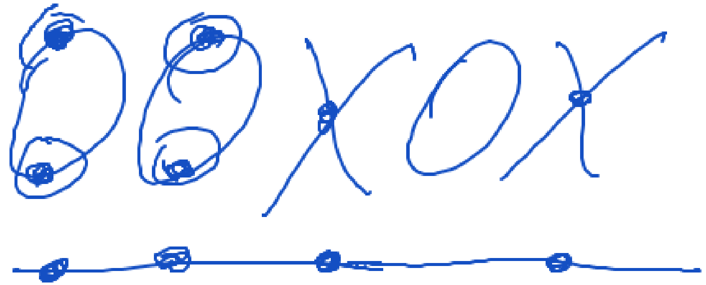
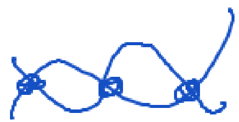
1. Th. $X$ S.B. variety, $X$ has a 0-cycle of degree coprime to $\dim X+1 \Rightarrow$
$\Rightarrow X \simeq \mathbb{P}^{\dim X}$

Th. (Nishimura–Lang) $X \sim Y$ smooth proper varieties, then $X(K) \neq \emptyset \Leftrightarrow Y(K) \neq \emptyset$

Corollary. A non-tr. S.B. surface is not bir. to a conic bundle, a dP surface of deg $\neq$ 3, 6, 9.

$S \dashrightarrow$ 

del Pezzo surface of degree $d$ has a 0-cycle of degree $d$



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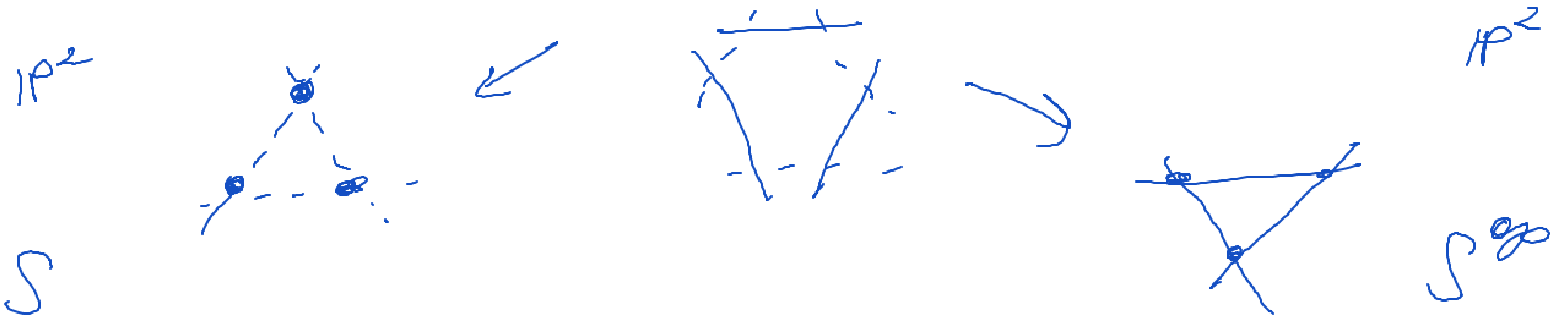
Th. (Weinstein, 1974–2019). $S$ nontr. S.B. surface$/k$ char $=0$, $S'$ is a del Pezzo surface with $\rho(S') = 1$, $S' \sim S$

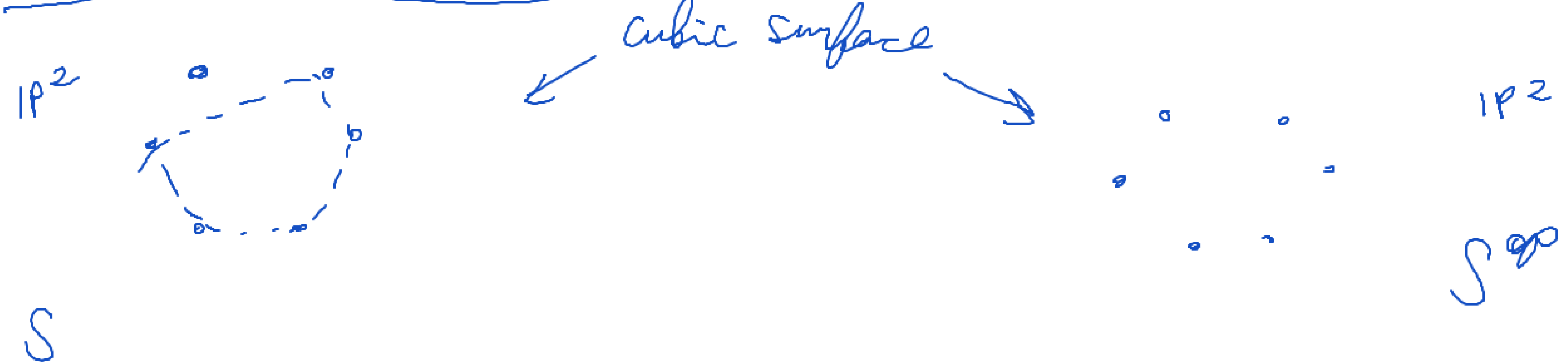
$\Rightarrow$ $S' \cong S$ or $S' \cong S^{op}$

$S \longleftrightarrow A$

$S^{op} \longleftrightarrow A^{op}$

Also, generators for $\mathrm{Bir}\, S$



G-MMP philosophy

$$\begin{tikzcd} & \tilde{S} \circlearrowleft G \arrow[dl] \arrow[dr, "G\text{-MMP}"] & \\ S \circlearrowleft G & & S' \circlearrowleft G \end{tikzcd}$$

$G \subset \mathrm{Bir}\, S$ fin.

$S'$ ~~either a conic bundle~~ or a dP surf. with $\rho(S')^G = 1$

We are left with dP surfaces of deg $= 3, 6, 9$

$\rho^G(S') = 1$, $S'$ has no points

$S$ or $S^{op}$

$G \circlearrowright S' \underset{\text{s.b.}}{\sim} S$ is a cubic surface

$\operatorname{Aut} S' \subset \underline{W(E_7)}$

Prime orders of elements: $\underline{2}$, $\underline{3}$, $\underline{5}$.

It appears that for $g$ of order 2 or 5

$\operatorname{Fix}_{S'_{\bar{k}}}(g)$ either contains a unique isolated point,

or a unique $\underline{\text{line}}$.

$\Rightarrow \quad |G| = 3^r \qquad \Rightarrow r \leq 4 \qquad G \subset \mathcal{H}_3 \rtimes \mathbb{Z}/3\mathbb{Z}$

$\ldots \Rightarrow G \subset (\mathbb{Z}/3)^3.$

Rem. $\exists K$, $\underset{\text{s.o.}}{S}$, $\underset{\text{cubic}}{\overline{S'}} \sim S$, $\underline{(\mathbb{Z}/3)^3} \subset \operatorname{Aut} S'$

$K = \mathbb{C}(\lambda, \mu)$

$$\lambda x^3 + \lambda^2 y^3 + \mu z^3 + \mu^2 z^3 = 0$$

$\omega$

$\omega$

$\omega$

$S(K) = \emptyset$

$\rho(S') \underline{= 3}$

Restrictions for subgroups of Aut S.

$G \underset{\text{fin.}}{\subset} \operatorname{Aut} S \Rightarrow |G|$ odd

nontr. S.B.

Pf.: supp. that $\exists g \in G$ $g^2 = 1$

$\operatorname{Fix}_{S_{\overline{K}}}(g) = \quad \subset \mathbb{P}^2_{\overline{K}}$

$\Rightarrow \exists$ pt. $\in S$

$S_{\overline{K}} \cong \mathbb{P}^2_{\overline{K}}$ $\quad S(K) \neq \emptyset$. $\square$.

$K = \mathbb{R}$

$x^2+y^2+z^2=0$

$\operatorname{Aut} \cong SO_3$

$\cup$

$\mathbb{Z}/n$, $D_{2n}$, $A_4$, $S_4$, $A_5$

Observation. For S.B. curves, groups of aut's are more complicated than for surfaces.

Th (Prokhorov, Shramov, 2017)

$X$ geom. rational 3-fold $/K$ char $=0$
(RC)

$G \underset{\text{fin}}{\subset} \text{Bir}\, X \implies$ $G$ contains a normal abelian subgroup of index $< 10^9$

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$K = \mathbb{C}(z)$ $\qquad \mathbb{Z}/\mathbb{C}$

$(K \supset \sqrt{1}$ char $=0)$ $\implies$ Aut $X$ has bounded fin. subgroups

Moreover, $G \underset{\text{fin}}{\subset} \text{Aut}\, X \implies$ $G$ abelian, $|G| \mid n^2$

$X$ S.B. variety, dim $= n-1$

~~sev~~ corr. to a ~~simp~~ central $\boxed{\text{Division}}$ alg.