Order of tangency between manifolds

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Abstract

We study the order of tangency between two manifolds of same dimension and give that notion three quite different geometric interpretations. Related aspects of the order of tangency are also discussed.

Keywords and Phrases. Order of tangency. Order of contact. Jet prolongations. Tower of Grassmannians. Intersection number.

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1 Introduction

In the present paper we discuss the order of tangency (or that of contact) between manifolds and their relations to enumerative geometry started with classical Schubert calculus.

Two plane curves, both nonsingular at a point x^0, are said to have a contact of order at least k at x^0 if, in properly chosen regular parametrizations, those two curves have identical Taylor polynomials of degree k about x^0.

Formulas enumerating contacts were widely investigated. For example in [2] the authors derive a formula for the number of contacts of order n between members of specified $(n-1)$-parameter family of plane curves and a generic plane curve of a sufficiently large degree.

Contact problems of this sort have been of both classical and modern interest, particularly in light of Hilbert's 15th problem to make rigorous the classical calculations of enumerative geometry, especially those undertaken by Schubert [12]. The situation regarding simple (i.e., second–order) contacts between families of varieties is now well understood thanks in large measure to the contact formula of Fulton, Kleiman and MacPherson [5]. The above mentioned formula in [2] generalises that given by Schubert in [13] for the number of triple contacts between a fixed plane curve and a specified 2-parameter family of curves. Schubert made his computations through the use of what have come to be known as "Schubert triangles". This theory has been made completely rigorous by Roberts and Speiser, see e.g. [11], and independently by Collino and Fulton [1].

Apart from contact formulas, an important role is played by the "order of tangency". Let us discuss this notion for Thom polynomials. Among important properties of Thom polynomials we record their positivity closely related to

Schubert calculus (see, e.g., [9] and [10] for a survey). Namely, the order of tangency allows one to define the jets of Lagrangian submanifolds. The space of these jets fibers nicely over the Lagrangian Grassmannian and leads to a positive decomposition of a Lagrangian Thom polynomial in the basis of Lagrangian Schubert cycles.

In this paper, we give three approaches to the order of tangency. The first one (in section 2) is by the Taylor polynomial. The second one (a mini-max procedure in section 3) makes use of curves. The third approach (in section 4) is by Grassmann bundles. We show that these three approaches are equivalent. We work here with manifolds over reals, but everything carries over for the complex manifolds.

In the three last sections, we discuss some issues related to the "closeness" of pairs of geometric objects: branches of algebraic sets, relations with contact geometry and Lagrangian tangency order.

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2 By Taylor

One situation that is frequently encountered at the crossroads of geometry and analysis deals with pairs of manifolds which are the graphs of functions of same number of variables. Such graphs can intersect, or touch each other, at a beforehand prescribed point, with various degrees of proximity.

Our departing point is a definition of such proximity going precisely in the spirit of a benchmark reference book [8], p. 18, although not formulated *expressis verbis* there. Two manifolds M and $\widetilde{M}$ in $\mathbb{R}^m$, both of the same dimension p, intersecting at $x^0 \in M \cap \widetilde{M}$ *have at* x^0 *the order of tangency at least* k when there exist neighbourhoods U and $\widetilde{U}$ in $\mathbb{R}^p$, parametrisations $q\colon (U, u^0) \to (M, x^0)$, $\tilde{q}\colon (\widetilde{U}, \tilde{u}^0) \to (\widetilde{M}, x^0)$, and a diffeomorphism $\Phi\colon U \to \widetilde{U}$ such that

$$\Big(\tilde{q} \circ \Phi - q\Big)(u) \;=\; o\Big(\big|u - u^0\big|^k\Big) \tag{1}$$

when $U \ni u \to u^0$. (Then, clearly, $\tilde{q}(\tilde{u}^0) = x^0$ and $\Phi(u^0) = \tilde{u}^0$.) It is straightforward that this definition does not depend on the choice of local parametrisations q and $\tilde{q}$. As a matter of record, basically the same definition is evoked in Proposition on page 4 in [7] (a diffeomorphism, named F, explicitly appears there in the one before last line, if only in the C^ω context). In [7] there is also proposed the following reformulation of (1).

Proposition 1 *We have*

$$T^k_{u^0} q \;=\; T^k_{u^0}\big(\tilde{q} \circ \Phi\big)\,, \tag{2}$$

where $T^k_{u^0}(\cdot)$ *means the Taylor polynomial about* u^0 *of order* k.

Indeed, we have

$$\begin{aligned}\tilde q\circ\Phi(u)-q(u)\ &=\big(\tilde q\circ\Phi(u)-T^k_{u^0}\big(\tilde q\circ\Phi\big)(u-u^0)\big)\\ &+\ \big(T^k_{u^0}\big(\tilde q\circ\Phi\big)(u-u^0)-T^k_{u^0}\,q(u-u^0)\big)+\big(T^k_{u^0}\,q(u-u^0)-q(u)\big)\,,\end{aligned}$$

where the first and last summands are $o\Big(\big|u-u^0\big|^k\Big)$ by Taylor. Under (1), so is the middle summand

$$T^k_{u^0}\big(\tilde q\circ\Phi\big)(u-u^0)-T^k_{u^0}\,q(u-u^0)\ =\ o\Big(\big|u-u^0\big|^k\Big)$$

and (2) follows from the following general result.

Lemma 1 *Let $w\in\mathbb{R}[u_1,\,u_2,\ldots,\,u_p]\,,\ \deg w\le k,\ w(u)=o\big(|u|^k\big)$ when $u\to 0$ in $\mathbb{R}^p$. Then w is identically zero.*

Proof goes by induction on $k\ge 0$, with an obvious start for $k=0$. Then, assuming this for the polynomials of degrees smaller than $k\ge 1$ and taking a polynomial w of degree k as in the wording of the lemma, we can assume without loss of generality that w is *homogeneous* of degree k (the terms of lesser degrees vanish altogether by the inductive assumption.) Let $\mathbf{u}\in\mathbb{R}^p$, $|\mathbf{u}|=1$, be otherwise arbitrary. Then

$$t^k w(\mathbf{u})=w(t\mathbf{u})=o\big(|t\mathbf{u}|^k\big)=o\big(|t|^k\big)\quad\text{when } t\to 0\,.$$

Hence $w(\mathbf{u})=0$ and the vanishing of w follows.

Remark. Our having parametrisations q and $\tilde q$ with *different* domains U and $\widetilde U$, and, in consequence, the recourses to diffeomorphism Φ, are all redundant (such is the line of exposition in [7]). Below in section 4 we dispense with such generality, have parametrisations with same domain and of very specific type – they are going to be just *graphs* of smooth mappings going from p dimensions to $m-p$ dimensions. This, clearly, is not going to restrict the generality of the discussion.

3 By curves

In the present section we entirely stick to the notation introduced in section 2. Our second approach uses *pairs of curves* lying, respectively, in M and $\widetilde M$. We naturally assume that $T_{x^0}M=T_{x^0}\widetilde M$. Our momentary objective is to show that

Theorem 1 *Manifolds M and $\widetilde M$ have at x^0 the order of tangency at least k ($k\ge 1$) iff*

$$\min_v\Big(\max_{\gamma,\,\tilde\gamma}\big(\max\big\{l\colon\ |\gamma(t)-\tilde\gamma(t)|=o\big(|t|^l\big)\ \text{when } t\to 0\,\big\}\big)\Big)\ \ge\ k\,. \tag{3}$$

The **minimum** *is taken over all* $0 \neq v \in T_{x^0}M = T_{x^0}\widetilde{M}$*. The* **outer maximum** *is taken over all pairs of curves* $\gamma \subset M$*,* $\tilde\gamma \subset \widetilde{M}$ *such that* $\gamma(0) = x^0 = \tilde\gamma(0)$*, and – both non-zero! – velocities* $\dot\gamma(0)$*,* $\dot{\tilde\gamma}(0)$ *are both parallel to* v*. The* **inner maximum** *is taken over admissible positive integers only.*

To proceed with proof, suppose that the order of tangency at point x^0 in question (cf. section 2) is exactly k. Then it is quick to show that the integer on the left hand side of inequality (3) is at least k. Indeed, for every fixed vector v as above, $v = dq(u^0)\mathbf{u}$ (without loss of generality, $\mathbf{u}$ is like in the proof of Lemma 1), one can take $\delta(t) = q(u^0 + t\mathbf{u})$ and $\tilde\delta(t) = \tilde q\big(\Phi(u^0 + t\mathbf{u})\big)$. Then

$$|\delta(t) - \tilde\delta(t)| \;=\; o\big(|t\mathbf{u}|^k\big) \;=\; o\big(|t|^k\big)$$

and so, in that inequality,

$$\max_{\gamma,\,\tilde\gamma}\Big(\max\big\{\, l\colon |\gamma(t) - \tilde\gamma(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\Big) \;\geq\; k\,.$$

In view of the arbitrariness in our choice of v, the same remains true after taking the minimum over all admissible v's on inequality's left hand side.

The opposite non-sharp inequality is more involved. In order to justify it we suppose again that the order of tangency under consideration is exactly k — and watch the two manifolds in the vicinity of x^0 via a handy local C^∞ diffeomorphism after which

$$(M,\, x^0) = \big\{x_{p+1} = x_{p+2} = \cdots = x_m = 0\big\}$$

and

$$(\widetilde{M},\, x^0) = \big\{x_j = F^j(x_1,\, x_2, \ldots,\, x_p)\,,\; j = p+1,\, p+2, \ldots,\, m\big\}$$

for some functions F^j. Having the manifolds so neatly (graph-like) positioned, we take simple parametrisations q and $\tilde q$ with same domain and the 'transfer' diffeomorphism $\Phi = \mathrm{id}$:

$$q(u_1,\, u_2, \ldots,\, u_p) = (u_1,\, u_2, \ldots,\, u_p,\, 0,\, 0, \ldots,\, 0)\,,$$

$$\tilde q\big(u_1,\, u_2, \ldots,\, u_p\big) = \Big(u_1,\, u_2, \ldots,\, u_p,\, F\big(u_1,\, u_2, \ldots,\, u_p\big)\Big)\,,$$

where $F = \big(F^{p+1},\, F^{p+2}, \ldots,\, F^m\big)$. The order of tangency being k, there hold

$$T^k_{u^0}(q) = T^k_{u^0}\big(\tilde q\big) \qquad \text{and} \qquad T^{k+1}_{u^0}(q) \neq T^{k+1}_{u^0}\big(\tilde q\big)\,,$$

that is

$$T^k_{u^0}(F) = 0 \qquad \text{and} \qquad T^{k+1}_{u^0}(F) \neq 0\,.$$

It follows that there exist an integer $j \in \{p+1,\, p+2, \ldots,\, m\}$ and a vector $\mathbf{w} \in \mathbb{R}^p$ such that

$$T^k_{u^0}(F^j)(\mathbf{w}) \;=\; 0 \qquad \text{and} \qquad T^{k+1}_{u^0}\big(F^j\big)(\mathbf{w}) \neq 0\,. \tag{4}$$

Let now u and $\tilde{u}$ be two smooth curves in $\mathbb{R}^p$ passing at $t = 0$ by u^0 and such that the vectors $\dot{u}(0)$ and $\dot{\tilde{u}}(0)$ are both non-zero and parallel to $\mathbf{w}$.

These curves *in parameters* give rise to curves $\delta(t) = q(u(t))$ and $\tilde{\delta}(t) = \tilde{q}(\tilde{u}(t))$ *in the manifolds*, both having at $t = 0$ non-zero speeds parallel to the vector $\mathbf{v} \colon = dq(u^0)\mathbf{w} = d\tilde{q}(u^0)\mathbf{w}$. We will now estimate from above the left hand side of the inequality (3) using, no wonder, $\mathbf{v}$, δ, and $\tilde{\delta}$:

$$\begin{aligned} |\delta(t) - \tilde{\delta}(t)| &= \sqrt{|u(t) - \tilde{u}(t)|^2 + |F\big(\tilde{u}(t)\big)|^2} \\ &\geq |F\big(\tilde{u}(t)\big)| \geq |F^j\big(\tilde{u}(t)\big)| \neq o\big(|t|^{k+1}\big)\,, \end{aligned} \tag{5}$$

where the last inequality necessitates an explanation. In fact, by (4) and for every $c \neq 0$

$$T^{k+1}_{u^0}\big(F^j\big)(tc\mathbf{w}) = (ct)^{k+1} T^{k+1}_{u^0}\big(F^j\big)(\mathbf{w}) \neq o\big(|t|^{k+1}\big) \quad \text{when} \quad t \to 0\,.$$

But $\tilde{u}(t) - \tilde{u}(0) = ct\mathbf{w} + o\big(|t|\big)$ for some non-zero c, hence

$$T^{k+1}_{u^0}\big(F^j\big)\big(\tilde{u}(t) - \tilde{u}(0)\big) \neq o\big(|t|^{k+1}\big) \quad \text{when} \quad t \to 0$$

as well. Also, and generally, $F^j(u) = T^{k+1}_{u^0}\big(F^j\big)(u - u^0) + O\Big(\big|u - u^0\big|^{k+2}\Big)$ when $u \to u^0$ in $\mathbb{R}^p$. Therefore, $F^j\big(\tilde{u}(t)\big) \neq o\big(|t|^{k+1}\big)$ as written in (5).

Now it is important to note that the produced couple of curves δ and $\tilde{\delta}$ is completely general for that chosen vector $\mathbf{v}$. Hence it follows that – for this precise vector $\mathbf{v}\,!$ – the quantity

$$\max_{\gamma,\,\tilde{\gamma}}\Big(\max\big\{\, l \colon |\gamma(t) - \tilde{\gamma}(t)| = o\big(|t|^l\big) \text{ when } t \to 0\big\}\Big)$$

does not exceed k. Understandingly, so does the minimum of such quantities over all v's in $T_{x^0}M = T_{x^0}\widetilde{M}$. Theorem 1 is now proved.

4 By Grassmannians

Our third approach is based on the introductory pages of [7] where a natural tower of consecutive Grassmannians is being attached to every given local smooth parametrisation q as used in the preceding sections. However, to allow for a recursive definition of tower's members, a more general framework is needed.

Namely, to every *immersion* $h \colon N \to N'$, N – an n-dimensional manifold, N' – an n'-dimensional manifold, the author of [7] firstly attaches the image map $\mathcal{G}h \colon N \to G_n(N')$ of its tangent map $d\,h$, where $G_n(N')$ is the Grassmann bundle, over N', of all n-planes tangent to N':

$$\mathcal{G}h(s) \;=\; dh(s)(T_sN)\,.$$

We stick to the notation from section 2, use as previously the pair of parametrisations q and $\tilde{q}$, but from now on dispense with a local diffeomorphism Φ – cf. Remark in section 2. So we are now given the mappings

$$\mathcal{G}\,q:\,U\longrightarrow G_p(M)\,,\qquad \mathcal{G}\,\tilde{q}:\,U\longrightarrow G_p\Big(\widetilde{M}\Big)\,.$$

Upon putting $m_0 = m$, $M^{(0)} = M$, $\widetilde{M}^{(0)} = \widetilde{M}$, $\mathcal{G}^{(1)} = \mathcal{G}$, there emerge a couple of sequences of recursively defined mappings

$$\mathcal{G}^{(l)}q:\,U\longrightarrow G_p\big(M^{(l-1)}\big)\,,\quad l\geq 2$$

and

$$\mathcal{G}^{(l)}\tilde{q}:\,U\longrightarrow G_p\Big(\widetilde{M}^{(l-1)}\Big)\,,\quad l\geq 2\,,$$

where $M^{(l)} = G_p\big(M^{(l-1)}\big)$, $\widetilde{M}^{(l)} = G_p\Big(\widetilde{M}^{(l-1)}\Big)$, and $m_l = \dim M^{(l)} = \dim \widetilde{M}^{(l)}$ for $l = 1, 2, 3, \ldots$, as on page 2 in [7]. Now our objective is to show

Theorem 2 *Manifolds M and $\widetilde{M}$ have at x^0 the order of tangency at least k ($k \geq 1$) iff for every parametrisations q and $\tilde{q}$ of the vicinities of x^0 in, respectively, M and $\widetilde{M}$, there holds*

$$\mathcal{G}^{(k)}q(u^0)\;=\;\mathcal{G}^{(k)}\tilde{q}(u^0)\,. \tag{6}$$

(Observe that $q(u^0) = \tilde{q}(u^0) = x^0$.)
In the way of proving we will assume without loss of generality that both M and $\widetilde{M}$ are, in the vicinities of x^0, just graphs of smooth mappings, and the parametrisations q and $\tilde{q}$ themselves are the graphs of those mappings.[1] That is, in what follows $q(u) = (u, f(u))$, $f\colon U \to \mathbb{R}^{m-p}$ and similarly $\tilde{q}(u) = (u, \tilde{f}(u))$, $\tilde{f}\colon U \to \mathbb{R}^{m-p}$, $x^0 = \big(u^0, f(u^0)\big) = \big(u^0, \tilde{f}(u^0)\big)$. Now, assuming (2) we will prove by induction on $l \leq k$ that

$$\mathcal{G}^{(l)}q(u^0)\;=\;\mathcal{G}^{(l)}\tilde{q}(u^0)\,, \tag{7}$$

and for $l = k$ get (6).

An added value of this derivation will be the control over the sets of natural local coordinates in the Grassmannians in question. With this information at hand the implication (6) $\Rightarrow$ (2) will follow in no time.

So take $l = 1$ and note that, by definition,

$$\mathcal{G}^{(1)}q(u)\;=\;\Big(u,\,f(u),\,\mathrm{span}\{\partial_j + f_j(u)\colon j = 1,\,2,\ldots,\,p\}\Big) \tag{8}$$

and the same holds true with $\tilde{f}$ instead of f. (f_j means the partial derivative of the vector function f with respect to u_j.) Under (2) we know that the first

[1] This technical simplification possibly requires one extra local diffeomorphism in the ambient space $(\mathbb{R}^m,\, x^0)$.

derivatives of q and $\tilde{q}$ coincide at u^0. In our graph setup this means that all the first partial derivatives of f and $\tilde{f}$ coincide at u^0. The equality (7) is got for $l = 1$.

Before doing the induction step observe that the description (8) of $\mathcal{G}^{(1)}q(u)$ is not handy. Yet there are charts in each Grassmannian! In the case of $G_p(M)$ which is locally $\mathbb{R}^m \times G_p$, the chart in the fibre G_p good for (8) consists of all the entries in the bottommost rows (indexed by numbers $p+1$, $p+2$, $\dots$, m) in the $m \times p$ matrices

$$\left[\ v_1 \ \middle|\ v_2 \ \middle|\ \dots \ \middle|\ v_p \ \right]$$

with non-zero upper $p \times p$ minor, **after** multiplying the matrix on the right by the inverse of that upper $p \times p$ submatrix.

That is to say, taking as the local coordinates all the entries in the rows No $p+1, \dots, m$ of the following matrix

$$\begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \\ \multicolumn{4}{|c|}{\mathbf{v_j^i},\quad i = p+1, \dots, m \quad j = 1, \dots, p} \end{bmatrix} \begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \end{bmatrix}^{-1}.$$

Or, more explicitly still, these coordinates are the entries of

$$\boxed{\mathbf{v_j^i},\quad i = p+1, \dots, m \quad j = 1, \dots, p} \begin{bmatrix} v_1^1 & v_2^1 & \cdots & v_p^1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^p & v_2^p & \cdots & v_p^p \end{bmatrix}^{-1}.$$

In such a chart (8) is stenographed to simply

$$\mathcal{G}^{(1)}q(u) \;=\; \Big(u,\ f(u),\ f_{\mathrm{I}}(u)\Big)\,, \tag{9}$$

where $f_{\mathrm{I}}(u)$ is our shorthand notation subsuming all the first partials of the components of f which are in the number $p \times (m-p)$. The beginning of induction is now encoded under the handy form (9). Similar Grassmannian charts and stenography are instrumental in the induction step.

$l \Rightarrow l+1$. We suppose for certain $1 \le l < k$ that $\mathcal{G}^{(l)}q\colon U \to M^{(l)}$ is already written down in appropriate Grassmann coordinates as

$$\mathcal{G}^{(l)}q(u) \;=\; \Big(u,\ f(u),\ f_{\mathrm{I}}(u),\ f_{\mathrm{II}}(u), \dots,\ f_{\mathrm{L}}(u)\Big)\,, \tag{10}$$

where f_ν means the aggregate of all partials of f of order ν, going in the number $(m-p)p^\nu$, with all Schwarz symmetries just **ignored** (or: not known). This visualisation is central in the proof. We are going to reproduce it one order up.

The punch line comes – first to compute the next mapping

$$\mathcal{G}\big(\mathcal{G}^{(l)}\big)q\colon U \to G_p(M^{(l)})\,,$$

and second to visualise it in appropriately chosen new ampler coordinates. One goes into the bundle of tangent p-planes to $M^{(l)}$, whose base is $M^{(l)}$. Therefore, a running point in that base is to be noted and written in the first place. By the definition of the operation $\mathcal{G}$, that point is to be followed by the image of the differential of $\mathcal{G}^{(l)}q$ at that point. As to the running point, it is visualised by the RHS of the equation (10). As to the differential, it is the linear span of p vectors – the partial derivatives at u with respect to u_j, $j = 1, 2, \ldots, p$ of the RHS of (10).

These partials, when assembled together vertically as the columns in an $m_l \times p$ matrix, feature the uppermost $p \times p$ identity sub-matrix, much like on the RHS in (8):

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline \multicolumn{4}{|c|}{(i,j) - \text{entries}\,, \quad i = p+1, ..., m_l} \\ \multicolumn{4}{|c|}{\qquad\qquad\qquad j = 1, ..., p} \end{bmatrix}$$

Therefore, all the entries of the bottommost $(m_l - p) \times p$ submatrix are the evaluations of the pertinent local coordinates in the Grassmannian $G_p(M^{(l)})$ at the image of the differential of $\mathcal{G}^{(l)}q$. (The dimension of $G_p(M^{(l)})$ is $m_{l+1} = m_l + p(m_l - p)$.) These entries consist of [having already previously appeared] derivatives $f_{\mathrm{I}}(u)$, $f_{\mathrm{II}}(u), \ldots, f_{\mathrm{L}}(u)$ and of newly computed partial derivatives of order $l + 1$, stenographed as $f_{\mathrm{L}+1}(u)$. Having the similar visualisation of $\mathcal{G}^{(l+1)}\tilde{q}(u)$ and knowing that the equality (2) – now without a redundant Φ – holds true at $u = u^0$, one gets (7) for $l + 1$.

As for the inverse implication (6) $\Rightarrow$ (2), it is now clear.

Theorem 2 is now proved.

5 An algebraic geometry example

Our work does not address an important class of objects – branches of algebraic sets which often happen to be tangent one to another with various degrees of closeness. Here is an example of such situation excerpted from [3]:

$C = \{(x, y)\colon (y - x^2)^2 = x^5\}$, cf. Figure 2 on page 37 there. The two branches of C issuing from the point $(0, 0)$, $C_- = \{y = x^2 - x^{5/2}\,,\ x \geq 0\}$ and $C_+ = \{y = x^2 + x^{5/2}\,,\ x \geq 0\}$ could be naturally extended to one-dimensional manifolds D_- and D_+, both of class C^2 – the graphs of functions $y_-(x) = x^2 - |x|^{5/2}$

and $y_+(x) = x^2 + |x|^{5/2}$, respectively. The Taylor polynomials of order 2 about $x = 0$ of y_- and y_+ coincide. Hence D_- and D_+ have at $(0, 0)$ the order of tangency at least 2 (cf. section 2).

This example suggests that, in the real algebraic geometry category, it would be pertinent to use non-integer measures of closeness. For instance

$$\sup\{\alpha > 0 : \; y_+(x) - y_-(x) = o(|x|^\alpha) \text{ when } x \to 0\}\,.$$

This *generalised* order of tangency would be 5/2 in the Colley-Kennedy example.

6 Relation with contact geometry

Unsurprisingly, the notion of order of contact proves useful not only in algebraic geometry (cf. the Introduction), but also in geometry *tout court.* One not so obvious application in the *real* category deals with the real contact structures in three dimensions. Our summarising it here follows closely Section 1.6 in [6]. The author considers a couple $\Sigma \subset M$, M – a *contact* 3-dimensional manifold and Σ – a fixed embedded surface in it. Contact means M being endowed with a contact structure, say ξ, in TM.

When one approaches a fixed point $p \in \Sigma$ by points q staying within Σ, a natural question is about the order of smallness of the angle $\angle\big(T_q\Sigma,\, \xi_q\big)$. If that angle is an O of the distance of q to p to power k (the distance measured in any chosen, and hence every, set of smooth local coordinates about p), then it is said that ξ has the order of contact at least k with Σ at p. (Therefore, what is discussed in this section differs a little from the notion of closeness of a pair of manifolds investigated in the preceding sections. Yet the added value is substantial.)

That is, to say that that the *new* order of contact is at least 1 at a given point p tantamounts to saying that $\xi_p = T_p\Sigma$. And it is exactly 0 at p whenever $\xi_p \neq T_p\Sigma$.

So it comes as a not small surprise that this elementary notion allows one to characterise the contact structures as such! Namely, a theorem proved in [6] asserts that a rank-2 tangent distribution ξ on a 3-dimensional M is contact iff ξ has the new order of contact **at most** 1 with every surface Σ embedded in M, and this at every point of Σ.

The next natural question in this direction is whether it is possible to similarly characterise contact structures on $(2n+1)$-dimensional manifolds, $n \geq 2$. The author of [6] says nothing in this respect.

7 Lagrangian tangency order

In this section, following [4], we discuss some discrete symplectic invariants. Let us fix a symplectic space. If $H_1 = \cdots = H_n$ define a smooth submanifold L in the symplectic space, then the tangency order of a curve $f : \mathbb{R} \to M$ to L

is the minimum of orders of vanishing at 0 of the functions $H_1 \circ f$ ... $H_n \circ f$. We denote the tangency order of f to L by $t(f, L)$. The Lagrangian tangency order $L(t, f)$ of a curve f is the maximum of $t(f, L)$ over all smooth Lagrangian submanifolds L of the symplectic space. It can be shown (*loc.cit.* Proposition 2.5) that if $f : \mathbb{R} \to M$ and $g : \mathbb{R} \to M$ are good analytic parametrisations of the same curve, then $Lt(f) = Lt(g)$.

Let N be a subset of the symplectic space. We define the tangency order of the germ of N to the germ of a submanifold L, $t[N, L]$ to be the minimum of $t(f, L)$ over all parametrised curve germs f such that $Im(f) \subset N$.

The Lagrangian tangency order of N, $Lt(N)$, is defined to be the maximum of $t[N, L]$ over all smooth Lagrangian submanifold-germs L in the symplectic space.

There is another symplectic invariant – *the index of isotropy of* N, which is defined to be the maximal order of tangency between a smooth submanifold containing N and an isotropic submanifold of the same dimension as M.

For comparison of the Lagrangian tangency order and the index of isotropy, with applications to singularities, we refer the reader to [4].

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