Working notes:
On symplectic Kempf-Laksov flag bundles

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Abstract. — We introduce analogs of the Kempf–Laksov desingularizations of Schubert bundles in (non-necessary maximal) symplectic Grassman bundles. In this setting, these are smooth flag bundles that are only birational to Schubert bundles. This construction allows us to obtain some universal Gysin formulas for isotropic Schubert bundles.

Introduction

In Sect. 2, we define smooth flag bundles birational to isotropic Schubert bundles. These are analogous to the flag bundles of Kempf and Laksov in **[KL74]** which desingularize the Schubert bundles when working with the general linear groups. These are constructed as a chain of zero locus in projective bundles of lines. These generalize the flag bundles of Kempf and Laksov **[KL74]**. loci

In **[Kaz00]**, Kazarian has also constructed a "desingularization" (non-transversality assumptions) of Schubert bundles using zero locus and projective bundles of lines, in the Lagrangian case ($d = n$), working with the Grassmannian as a base. It seems however that our construction is more easily adapted to non-Lagrangian case, and it is our goal to work with base X. loci

1. Schubert bundles

Let $(E,\omega) \to X$ be a rank $2n$ symplectic vector bundle for the symplectic form $\omega\colon E \otimes E \to L$ with value in a line bundle $L \to X$, over a variety X. For $d \in \{1,\dots,n\}$, let $\mathbf{G}^{\omega}_d(E)$ be the Grassmann bundle of isotropic d-planes in the fibers of E. For a vector space $V \in E_x$ let denote V^{ω} its symplectic complement

$$V^{\omega} := \{w \in E_x \colon \omega(v,w) = 0, \text{ for all } v \in V\}.$$

Let

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E_n{}^{\omega} \subsetneq \cdots \subsetneq E_0{}^{\omega} = E$$

be a reference flag of isotropic subbundles and co-isotropic subbundles of E, where $\operatorname{rank}(E_i) = i$. For the sake of uniformity of notation, for $i = 0,1,\dots,n$, denote as well $E_{2n-i} := E_i{}^{\omega}$.

For a partition $\lambda \subseteq (2n-d)^d$, one defines the *Schubert open cell* $\mathring{\Omega}_\lambda(E_\bullet)$ in $\mathbf{G}^{\omega}_d(E)$ over the point $x \in X$ by the conditions there is/given

$$\mathring{\Omega}_\lambda(E_\bullet) := \Big\{V \in \mathbf{G}^{\omega}_d(E)(x)\colon \dim\big(V \cap E_{2n-d+i-\lambda_i}(x)\big) = i, \text{ for } i = 1,\dots,d\Big\}.$$

Denote $\nu_{d+1-i} := 2n - d + i - \lambda_i$ the dimension of the reference space appearing in the ith condition. The partition indexing the Schubert cell $\mathring{\Omega}_\lambda$ is uniquely defined if one considers only *admissible* partitions λ, *i.e.* partitions such that $\nu_i + \nu_j \neq 2n+1$. For such admissible partitions we define~~s~~ the Schubert bundle $\varpi_\lambda\colon \Omega_\lambda \to X$ as the Zariski-closure of $\mathring{\Omega}_\lambda$, given over a point $x \in X$ by the conditions

$$\Omega_\lambda(E_\bullet) := \Big\{V \in \mathbf{G}^{\omega}_d(E)(x)\colon \dim\big(V \cap E_{2n-d+i-\lambda_i}(x)\big) \geq i, \text{ for } i = 1,\dots,d\Big\}.$$

This is a subvariety of the Grassmann bundle $\mathbf{G}^{\omega}_d(E)$, that is in general singular. In the spirit of Kempf and Laksov **[KL74]**, and also inspired by Kazarian **[Kaz00]**, we will now construct smooth flag bundles $F_\mu(E_\bullet)$ giving for certain partitions ν morphisms $\vartheta_\nu\colon F_\nu(E_\bullet) \to X$ birational to ϖ_λ.

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2. Isotropic Kempf–Laksov flag bundles

Let $\mathbf{F}^\omega(1,\dots,d)(E)$ denote the bundle of flags of nested isotropic subspaces with dimensions $1,\dots,d$ in the fibers of E.

Then, we define the isotropic Kempf–Laksov bundle $\vartheta_\mu \colon F_\mu(E_\bullet) \to \cancel{X}$ over the point $x \in X$ as the dense subset there is/given

$$F_\mu(E_\bullet) \overset{\text{dense}}{\subseteq} \left\{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d \in \mathbf{F}^\omega(1,\dots,d)(E)(x) \colon V_i \subseteq E_{\mu_{d+1-i}}(x)\right\},$$

of flags satisfying the supplementary (generic) conditions

$$(1) \qquad (V_{d-i}/V_{\delta_i}) \cap (E_{2n-\mu_i}(x)/V_{\delta_i}) = \{0\} \quad \text{for } i = 1,\dots,d-1.$$

To a strict partition $\mu = (\mu_1,\dots,\mu_d) \subseteq (2n)^d$ with at most d parts, we associate a d-tuple of integers $\delta = \delta(\mu)$ defined by

$$\delta_i := \#\{j > i \colon E_{\mu_j} \subseteq E_{\mu_i}{}^\omega\} = \#\{j > i \colon \mu_i + \mu_j \leq 2n\}.$$

If $\delta_i \neq 0$, one has

$$E_{\mu_{d+1-\delta_i}} \subseteq E_{2n-\mu_i} = (E_{\mu_i})^\omega.$$

We will call *isotropic Kempf–Laksov flag bundles* such bundles ϑ_μ. The forgetful map $\mathbf{F}^\omega(1,\dots,d) \to \mathbf{F}^\omega(1,\dots,d-1)$ induces (dominant) maps $F_{(\mu_1,\dots,\mu_d)}(E_\bullet) \to F_{(\mu_2,\dots,\mu_d)}(E_\bullet)$.

Notice that in opposition to the ordinary Kempf–Laksov flag bundles in type A, to obtain a pure-dimensional variety, it is necessary to remove the "bad fibers" that do not satisfy (1), as it is illustrated in the following examples.[TODO]

These bundles can be described by the mean of a chain of zero locus in projective bundles of lines, as follows. The idea is to construct the flag $V_1 \subsetneq \cdots \subsetneq V_d$ line-by-line (considering quotients V_i/V_{i-1} of successive spaces) in such way that it satisfies the incidence conditions defining F_μ at each step. loci

We now describe the step $F_{(\mu_1,\mu_2,\dots,\mu_d)}(E_\bullet) \to F_{(\mu_2,\dots,\mu_d)}(E_\bullet)$. Let U_{d-1} be the universal subbundle of rank $d-1$ on $\mathbf{F}^\omega(1,\dots,d-1)(E)$. Note that in restriction to $F_{(\mu_2,\dots,\mu_d)}$:

- the condition $V_{d-1} \subseteq E_{\mu_2}(x)$ yields: $U_{d-1} \subseteq E_{\mu_2} \subseteq E_{\mu_1}$;
- the condition V_{d-1} isotropic yields: $V_{d-1} \oplus \ell(x)$ isotropic $\Leftrightarrow \ell \subset U_{d-1}^\omega$ (recall that a line ℓ is always isotropic).

It thus follows from the definition of $F_\mu(E_\bullet)$ that

$$(2) \qquad F_{(\mu_1,\dots,\mu_d)}(E_\bullet) \simeq \left\{\ell \in P\Big((E_{\mu_1}/U_{d-1})|_{F^\circ_{(\mu_2,\dots,\mu_d)}(E_\bullet)}\Big) \colon \ell \subseteq U_{d-1}^\omega\right\},$$

where one considers only the fibers over

$$F^\circ_{(\mu_2,\dots,\mu_d)}(E_\bullet) := \{V_\bullet \in F_{(\mu_2,\dots,\mu_d)}(E_\bullet) \colon \dim(V_{d-1} \cap E_{2n-\mu_1}) = \delta_1\}.$$

We denote U_d/U_{d-1} be the tautological subbundle of $\mathbf{P}((E_{\mu_1}/U_{d-1})|_{F_{(\mu_2,\dots,\mu_d)}(E_\bullet)})$, so that U_d coincide with the restriction to $F_{(\mu_1,\dots,\mu_d)}(E_\bullet)$ of the universal subbundle of rank d on $\mathbf{F}^\omega(1,\dots,d)(E)$.

Note that since we restrict to $F_{(\mu_2,\dots,\mu_d)}(E_\bullet)$, one has

$$U_{d-1} \subseteq (U_{d-1})^\omega \text{ and } U_{\delta_1} \subseteq E_{\mu_{d+1-\delta_1}} \subseteq E_{2n-\mu_1} = (E_{\mu_1})^\omega.$$

Hence, there is a well-defined global section of the vector bundle

$$\mathrm{Hom}\Big(U_d/U_{d-1}, L \otimes (U_{d-1}/U_{\delta_1})^\vee\Big) \simeq L \otimes (U_d/U_{d-1})^\vee \otimes (U_{d-1}/U_{\delta_1})^\vee$$

defined at the point $\ell = V_d/V_{d-1} \subseteq E_{\mu_1}/V_{d-1}$ by:

$$(3) \qquad s(\ell) := \left\{t \in \ell \mapsto \omega(t,\cdot)|_{V_{d-1}}\right\}.$$

We denote by Z_d the zero-locus of s in $\mathbf{P}((E_{\mu_1}/U_{d-1})|_{F_{(\mu_2,\dots,\mu_d)}(E_\bullet)})$.

Over a point $V_{d-1} \supsetneq \cdots \supsetneq V_1$, the lines in Z_d are these lines that are (symplecticly) orthogonal to V_{d-1} or equivalently the lines ℓ such that the vector space

$$V_d = \ell \oplus V_{d-1}$$

is isotropic. Indeed, both ℓ and V_{d-1} are already isotropic.

Let us now compute fiberwise the codimension of Z_d in $\mathbf{P}((E_{\mu_1}/U_{d-1})|_{F_{(\mu_2,\dots,\mu_d)}(E_\bullet)})$. Over a point $V_{d-1} \supsetneq \cdots \supsetneq V_1$ above a point $x \in X$, the zero-locus Z_d consists of the common zeroes of the linear forms

$$\omega(\cdot,\ell') \colon E_{\mu_1}/V_{d-1} \to L_x,$$

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for $\ell' \subset V_{d-1}$. Such a linear form is trivial if and only if

$$\ell' \subsetneq V_{d-1} \cap E_{\mu_1}^{\omega} = V_{d-1} \cap E_{2n-\mu_1}.$$

As a consequence the codimension of the fibers of Z_d is given by the evaluation at a point $V_{d-1} \supsetneq \cdots \supsetneq V_1 \in F_{(\mu_2,\ldots,\mu_d)}(E_\bullet)$ of

$$\text{(4)} \qquad \operatorname{codim}(Z_d, \mathbf{P}(E_{\mu_1}/U_{d-1})) = (d-1) - \dim(U_{d-1} \cap E_{2n-\mu_1}).$$

Now, recall that $U_{\delta_1} \subsetneq E_{2n-\mu_1}$, so in any fiber $U_{\delta_1} \subseteq U_{d-1} \cap E_{2n-\mu_1}$. One infers from (4) the upper bound on the codimension

$$\operatorname{codim}(Z_d, \mathbf{P}(E_{\mu_1}/U_{d-1})) \leq d - 1 - \delta_1 = \operatorname{rank}\left(L \otimes (U_d/U_{d-1})^\vee \otimes (U_{d-1}/U_{\delta_1})^\vee\right).$$

In the fibers where the equality holds, the zero-locus Z_d has the expected codimension and the section s is transverse to the zero section.

When $\mu_i + \mu_j \neq 2n+1$, such a fiber is given by

$$V_k/V_{k-1} = E_{\mu_{d+1-k}}/E_{\mu_{d+1-k}-1}.$$

The fact that is satisfies the conditions follow almost from definition of δ_i and the fact that it is isotropic follows from $\mu_i + \mu_j \neq 2n+1$.

A classical upper semi-continuity argument yields thus that the set of the base

$$\dim(U_{d-1} \cap E_{2n-\mu_1}) = \delta_1,$$

in which Z_d has the expected codimension is open dense. We restrict to the fibers over this open.

Corollary 2.1. — *If ν is the strict partition with d parts associated to λ,*

$$\operatorname{codim}(\Omega_\lambda, \mathbf{G}_d^\omega(E)) = |\lambda| - |\delta(\nu)|.$$

3. Gysin formulas

Consider a symplectic vector bundle (E, ω) of rank $2n$, carrying a reference flag $E_\bullet$ of isotropic and co-isotropic subbundles of E

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E_n^{\omega} \subsetneq \cdots \subsetneq E_0^{\omega} = E$$

(recall the convention $\operatorname{rank}(E_i) = i$ and the notation $E_{2n-i} := E_i^{\omega}$ for $i \leq n$).

To sum up the previous Section, for a strict partition $\mu \subset (2n)^d$, we get a sequence of Kempf–Laksov flag bundles

$$F_{(\mu_1,\ldots,\mu_d)}(E_\bullet) \longrightarrow F_{(\mu_2,\ldots,\mu_d)}(E_\bullet) \longrightarrow \cdots \longrightarrow F_{(\mu_{d-1},\mu_d)}(E_\bullet) \longrightarrow F_{(\mu_d)}(E_\bullet) \longrightarrow X,$$

induced by forgetful maps, which is the same as the chain of zero locus in projective bundles: loci

$$\text{(5)} \qquad \begin{tikzcd}[column sep=small]
Z_d \arrow[rr] \arrow[d, hook, "\iota_d"'] & & Z_{d-1} \arrow[rr, dashed] \arrow[d, hook, "\iota_{d-1}"'] & & Z_2 \arrow[rr] \arrow[d, hook, "\iota_2"'] & & Z_1 \arrow[r] \arrow[d, equal, "\iota_1"'] & X, \\
\mathbf{P}(\iota_{d-1}^*(E_{\mu_1}/U_{d-1})) \arrow[urr] & & \mathbf{P}(\iota_{d-2}^*(E_{\mu_2}/U_{d-2})) \arrow[urr, dashed] & & \mathbf{P}(\iota_1^*(E_{\mu_{d-1}}/U_1)) \arrow[urr] & & \mathbf{P}(E_{\mu_d}) \arrow[ur] &
\end{tikzcd}$$

where for $i = 1, \ldots, d$, we denote $Z_i := \{\ell \in U_{i-1}{}^{\omega}\} \subset \mathbf{P}(\iota_{i-1}^*(E_{\mu_{d+1-i}}/U_{i-1}))$. c

In the spirit of [DP15, DP16] we shall deduce a Gysin formula for $\vartheta_\mu : F_\mu(E_\bullet) \to X$ from this description.

We fix an integer d and we denote by U the universal subbundle on $\mathbf{G}_d^\omega(E)$, as well as its pullback to $\mathbf{F}^\omega(1,\ldots,d)(E)$ by the natural forgetful map $\mathbf{F}^\omega(1,\ldots,d)(E) \to \mathbf{G}_d^\omega(E)$. We still denote bu U the restrictions of these respective bundles to Schubert bundles in $\mathbf{G}_d^\omega(E)$ or to Kempf–Laksov bundles in $\mathbf{F}^\omega(1,\ldots,d)(E)$. For a symmetric polynomial f in d variables, we write $f(U)$ for the specialization of f with Chern roots of $U^\vee$.

For a Laurent polynomial P in d variables $t_1, \ldots, t_d$, and a monomial m, we denote by $[m](P)$ the coefficient of m in the expansion of P. Clearly, for any second monomial m', one has $[mm'](Pm') = [m](P)$, a property that we will use repeatedly.

Like in our former papers,

***Theorem* 3.1 (Gysin formula).** — *For any strict partition $\mu \subseteq (2n)^d$ and $\vartheta_\mu \colon F_\mu(E_\bullet) \to X$:*

$$(\vartheta_\mu)_* f(U) = \Big[\prod_{j=1}^{d} t_j^{\mu_j-1}\Big]\Big(f(t_1,\ldots,t_d)\prod_{1\le i<j\le d}(t_i-t_j)\prod_{\substack{1\le i<j\le d\\ \mu_i+\mu_j\ge 2n+1}}(c_1(L)+t_i+t_j)\prod_{1\le j\le d} s_{1/t_j}(E_{\mu_j})\Big).$$

Proof. — We will prove this formula by induction. With the notation of (5), for $i = 1,\ldots,d$, let

$$\xi_i := c_1\Big(\mathcal{O}_{\mathbf{P}((E_{\mu_i}/U_{d-i})|_{Z_{d-i}})}(1)\Big)\Big|_{Z_{d-i+1}} = c_1((U_{d+1-i}/U_{d-i})^\vee).$$

Then $\xi_1,\ldots,\xi_d$ is a set of Chern roots for $U^\vee$ on $F_\mu(E_\bullet)$.

We want to compute $(\vartheta_\mu)_* f(\xi_1,\ldots,\xi_d)$. If $d = 1$, this is the Gysin formula along projective bundles of lines. Assume that the formula holds for $d-1$. Since $Z_{d-1} \simeq F_{\mu_2,\ldots,\mu_d}(E_\bullet)$, we know the Gysin formula $A^\bullet(Z_{d-1}) \to A^\bullet(X)$, it is thus sufficient to study the Gysin map $A^\bullet(Z_d) \to A^\bullet(Z_{d-1})$. Considering (5), we decompose this map as

$$A^\bullet(Z_d) \xrightarrow{i_*} A^\bullet\mathbf{P}((E_{\mu_1}/U_{d-1})|_{Z_{d-1}}) \xrightarrow{p_*} A^\bullet(Z_{d-1}).$$

The Gysin formula for p_* is the formula for projective bundles of lines. It remains to study the Gysin formula for i_*.

The map i_* is then given by the cup product with the top Chern class

$$c_{\mathrm{top}}(L\otimes(U_d/U_{d-1})^\vee\otimes(U_{d-1}/U_{\delta_1})^\vee) = \prod_{1<j<d-\delta_1}(c_1(L)+\xi_1+\xi_j) = \prod_{1<j:\ \mu_1+\mu_j\ge 2n+1}(c_1(L)+\xi_1+\xi_j).$$

Composing the Gysin formulas for p_* and i_* (and using the projection formula), we get

$$p_*i_*(f(\xi_1,\xi_2,\ldots,\xi_d)) = [t_1^{\mu_1-d}]\Big(f(t_1,\xi_2,\ldots,\xi_d)\prod_{1<j:\ \mu_1+\mu_j\ge 2n+1}(c_1(L)+t_1+\xi_j)\, s_{1/t_1}(E_{\mu_1}/U_{d-1})\Big).$$

Now

$$s_{1/t_1}(E_{\mu_1}/U_{d-1}) = s_{1/t_1}(E_{\mu_1})c_{1/t_1}(U_{d-1}) = s_{1/t_1}(E_{\mu_1})\prod_{1<j}(1-\xi_j/t_1) = \frac{1}{t_1^{d-1}}s_{1/t_1}(E_{\mu_1})\prod_{1<j}(t_1-\xi_j)$$

Thus (multiplying the extracted monomial and the polynomial by t_1^{d-1}):

$$p_*i_*(f(\xi_1,\xi_2,\ldots,\xi_d)) = [t_1^{\mu_1-1}]\Big(f(t_1,\xi_2,\ldots,\xi_d)\prod_{1<j\le d}(t_1-\xi_j)\prod_{\substack{1<j\le d\\ \mu_1+\mu_j\ge 2n+1}}(c_1(L)+t_1+\xi_j)\, s_{1/t_1}(E_{\mu_1})\Big).$$

The stated formula easily follows, using the induction hypothesis. □

Note that the formula and its proof also holds for general polynomials $f(\xi_1,\ldots,\xi_d)$, without symmetry.

***Corollary* 3.2.** — *For any strict partition $\mu \subseteq (2n)^d$, and $\varpi_\mu \colon \Omega_\mu \to X$:*

$$(\varpi_\mu)_* f(U) = \Big[\prod_{j=1}^{d} t_j^{\mu_j-1}\Big]\Big(f(t_1,\ldots,t_d)\prod_{1\le i<j\le d}(t_i-t_j)\prod_{\substack{1\le i<j\le d\\ \mu_i+\mu_j\ge 2n+1}}(c_1(L)+t_i+t_j)\prod_{1\le j\le d} s_{1/t_j}(E_{\mu_j})\Big).$$

References

[Kaz00] M. Kazarian – "On Lagrange and symmetric degeneracy loci", preprint available at http://www.newton.ac.uk/preprints/NI00028.pdf, 2000.

[KL74] G. Kempf & D. Laksov – "The determinantal formula of Schubert calculus", *Acta Mathematica* **132** (1974), p. 153–162.