On a certain family of $U(\mathfrak{b})$-modules

Piotr Pragacz*
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-656 Warszawa, Poland
P.Pragacz@impan.pl

Abstract

We report on results of Kraśkiewicz and the author, and Watanabe on KP modules materializing Schubert polynomials, and filtrations having KP modules as their subquotients. We discuss applications of the bundles $S_w(E)$ for filtered ample bundles E and KP filtrations to positivity due to Fulton and Watanabe respectively.

To IMPANGA on the occasion of her 15th birthday

Contents

1 Introduction

In the present article, we survey a recent chapter of representation theory of $U(\mathfrak{b})$-modules: the theory of KP modules.

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There is a really famous family of $U(\mathfrak{b})$-modules: these are Demazure modules (see [7, 8], [14], [2], and [15]). They are given by the spaces of sections of line bundles on the Schubert varieties in flag manifolds. Invented in the 1970s by Demazure, the theory was developed in the 1980s by Joseph, Andersen, Polo, Mathieu, van der Kallen et al.

The origins of KP modules, though also related to Schubert varieties, are different. In the beginning of the 1980s, Lascoux and Schützenberger [20] discovered Schubert polynomials — certain polynomial lifts of cohomology classes of Schubert varieties in flag manifolds (see [3]). Schubert polynomials are described in Sect. 3. It was a conjecture of Lascoux (Oberwolfach, June 1983) that there should exist a functorial version of this construction (similarly as to Schur functions there correspond Schur functors, cf. Sect. 2). The Lascoux conjecture was solved affirmatively by Kraśkiewicz and the author [16, 17]. The so-obtained modules were called *Kraśkiewicz–Pragacz modules*, in short *KP modules* by Watanabe, who is the author of further developments of the theory of the KP modules and KP filtrations in the spirit of highest weight categories [5]. His work [33, 34, 35, 36] is surveyed in Sect. 5. The aforementioned KP modules and related modules, e.g., Schur flagged modules are discussed in Sect. 4.

The last two sections are devoted to studying positivity.

In Section 6, we discuss a recent result of Watanabe, showing that a Schur function specialized with the monomials $x_1^{\alpha_1}x_2^{\alpha_2}\dots$ of a Schubert polynomial is a nonnegative combination of Schubert polynomials. The method relies on KP filtrations.

For a monomial $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots$, set $l(x^\alpha) := \alpha_1 x_1 + \alpha_2 x_2 + \dots$. In Section 7, we discuss a result of Fulton from the 1990s, asserting that a Schur function (or Schubet polynomial) specialized with the expressions $l(x^\alpha)$ associated to the monomials x^α of a Schubert polynomial is a nonnegative combination of Schubert polynomials. The method relies on ample vector bundles.

Perhaps a couple of words about comparison of Demazure and KP modules is in order. KP modules are in some way similar to Demazure modules (of type (A)), the modules generated by an extremal vector in an irreducible representation of $\mathfrak{gl}_n$: they are both cyclic $U(\mathfrak{b})$-modules parametrized by the weights of the generators, and if the permutation is vexillary, then the KP module coincides with Demazure module with the same weight of the generator. If a permutation is not vexillary, then there exists a strict surjection from the KP module to the Demazure module of the corresponding lowest weight.

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2 Schur functors

Throughout this paper, let K be a field of characteristic zero.

In his dissertation [30] (Berlin, 1901), Schur gave a classification of irreducible polynomial representations of the general linear group $GL_n(K)$, i.e., homomorphisms

$$GL_n(K) \to GL_N(K)$$

sending an $n \times n$-matrix X to an $N \times N$-matrix $[P_{ij}(X)]$, where P_{ij} is a polynomial in the entries of X. Let Σ_n denotes the symmetric group of all bijections of $\{1, \dots, n\}$. Consider the following two actions on $E^{\otimes n}$, where E is a finite dimensional vector space over K:

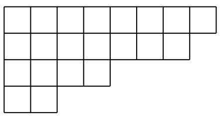
- of the symmetric group Σ_n via permutations of the factors;
- the diagonal action of $GL(E)$.

Irreducible representations S^λ of the symmetric group Σ_n are labeled by partitions of n, see, e.g., [4]. By a *partition* of n, we mean a sequence

$$\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$$

of integers such that $|\lambda| = \lambda_1 + \ldots + \lambda_k = n$. A partition is often represented graphically by its diagram with λ_i boxes in the ith row (cf. [24]).

Example 1. The diagram of the partition $(8, 7, 4, 2)$ is



Let R be a commutative ring at let E be an R-module. We define the *Schur module* associated with a partition λ as follows:

$$V_\lambda(E) := \mathrm{Hom}_{\mathbb{Z}[\Sigma_n]}(S^\lambda, E^{\otimes n}) \,,$$

where $\mathbb{Z}[\Sigma_n]$ denotes the group ring of Σ_n with integer coefficients (cf. [27]).

In fact, $V_\lambda(-)$ is a *functor*: if E, F are R-modules over a commutative ring R, and $f : E \to F$ is an R-homomorphism, then f induces an R-homomorphism $V_\lambda(E) \to V_\lambda(F)$. In this way, we get all irreducible polynomial representations of $GL_n(K)$.

Let us label the boxes of the diagram of a partition of n with numbers $1, \dots, n$, and call such an object a *tableau*. For example, a tableau for the diagram of the partition $(8, 7, 4, 2)$ is

1	15	19	3	10	5	21	13
11	8	18	9	6	17	4	
7	20	12	16				
16	2						

Consider the following two elements of $\mathbb{Z}[\Sigma_n]$ associated with a tableau:

- P:= sum of $w \in \Sigma_n$ preserving the rows of the tableau;
- N:= sum of $w \in \Sigma_n$ with their signs, preserving the columns of the tableau.

The following element of $\mathbb{Z}[\Sigma_n]$:

$$e(\lambda) := N \circ P$$

is called a *Young symmetrizer*. For more on Young symmetrizers, see [4]. We have an alternative presentation of Schur module:

$$V_\lambda(E) = e(\lambda) E^{\otimes n}.$$

There is still another way of getting Schur modules $V_\lambda(E)$ (see [18] and [1]) as the images of natural homomorphisms between the tensor products of symmetric and exterior powers of a module:

$$S_{\lambda_1}(E) \otimes \ldots \otimes S_{\lambda_k}(E) \to E^{\otimes|\lambda|} \to \wedge^{\mu_1}(E) \otimes \ldots \otimes \wedge^{\mu_l}(E). \tag{1}$$

(here $\mu = \lambda^\sim$ is the conjugate partition of λ, see [24], where a different notation is used; the first map is the diagonalization in the symmetric algebra, and the second map is the multiplication in the exterior algebra), and

$$\wedge^{\mu_1}(E) \otimes \ldots \otimes \wedge^{\mu_l}(E) \to E^{\otimes|\lambda|} \to S_{\lambda_1}(E) \otimes \ldots \otimes S_{\lambda_k}(E)$$

(here the first map is the diagonalization in the exterior algebra, and the second map is multiplication in the symmetric algebra).

Example 2. We have $V_{(n)}(E) = S_n(E)$ and $V_{(1^n)}(E) = \wedge^n(E)$, the nth symmetric and exterior power of E.

Let T be the subgroup of diagonal matrices in $GL_n(K)$:

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & 0 \\ & & x_3 & \\ & 0 & & \ddots \end{pmatrix}. \tag{2}$$

Let E be a finite dimensional vector space over K. Consider the action of T on the Schur module $V_\lambda(E)$ associated with a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, induced from the action of $GL_n(K)$ via restriction.

Theorem 3 (Main result of Schur's Thesis)**.** *The trace of the action of T on $V_\lambda(E)$ is equal to the Schur function:*

$$s_\lambda(x) = \det\left(s_{\lambda_p - p + q}(x_1, \ldots, x_n)\right)_{1 \le p,q \le k},$$

where $s_i(x_1, \ldots, x_n)$, $i \in \mathbb{Z}$, is the complete symmetric function of degree i if $i \geq 0$, and zero otherwise.

For more on Schur functions, see [24] and [19].

Among the most important formulas in the theory of symmetric functions, they are the following Cauchy formulas:

$$\prod_{i,j}(1-x_iy_j)^{-1}=\sum_{\lambda}s_\lambda(x)s_\lambda(y)$$

and

$$\prod_{i,j}(1+x_iy_j)=\sum_{\lambda}s_\lambda(x)s_{\lambda^\sim}(y)\,.$$

Schur functors were brought to the attention of algebraists in [18] together with the materializations[1] of the Cauchy formulas: for free R-modules E, F one has:

$$S_n(E\otimes F)=\bigoplus_{|\lambda|=n}V_\lambda(E)\otimes V_\lambda(F)\,,$$

and

$$\wedge^n(E\otimes F)=\bigoplus_{|\lambda|=n}V_\lambda(E)\otimes V_{\lambda^\sim}(F)\,.$$

Whereas the former formula gives the irreducible $GL(E)\times GL(F)$-representations of the decomposition of the space of polynomial functions on the space of $\dim(E)\times\dim(F)$-matrices, the importance of the latter comes from the fact that it describes the *Koszul syzygies* of the ideal generated by the entries of a generic $\dim(E)\times\dim(F)$-matrix. Then, using suitable derived functors (following a method introduced in Kempf's 1971 Thesis), this allows one to describe syzygies of determinantal ideals [18]. In fact, there is a natural extension of Schur functors to complexes with many applications (see [27] and [1]).

3 Schubert polynomials

In the present article, by a *permutation* $w=w(1),w(2),\ldots,$ we shall mean a bijection $\mathbb{N}\to\mathbb{N}$, which is the identity off a finite set. The group of permutations will be denoted by Σ_∞. The symmetric group Σ_n is identified with the subgroup of Σ_∞ consisting of permutations w such that $w(i)=i$ for $i>n$. We set

$$A:=\mathbb{Z}[x_1,x_2,\ldots]\,.$$

We define a linear operator $\partial_i : A\to A$ as follows:

$$\partial_i(f):=\frac{f(x_1,\ldots,x_i,x_{i+1},\ldots)-f(x_1,\ldots,x_{i+1},x_i,\ldots)}{x_i-x_{i+1}}\,.$$

These are classical Newton's divided differences. For more on divided differences, see [20, 21], [23], and [19].

For a simple reflection $s_i=1,\ldots,i-1,i+1,i,i+2,\ldots,$ we put $\partial_{s_i}:=\partial_i$.

Lemma 4. *Suppose that $w=s_1\ldots s_k=t_1\ldots t_k$ are two reduced words for a permutation w. Then we have $\partial_{s_1}\circ\ldots\circ\partial_{s_k}=\partial_{t_1}\circ\ldots\circ\partial_{t_k}$.*

(See [3] and [6].)

[1] We say that a formula on the level of representations "materializes" a polynomial formula if the latter is the character of the former.

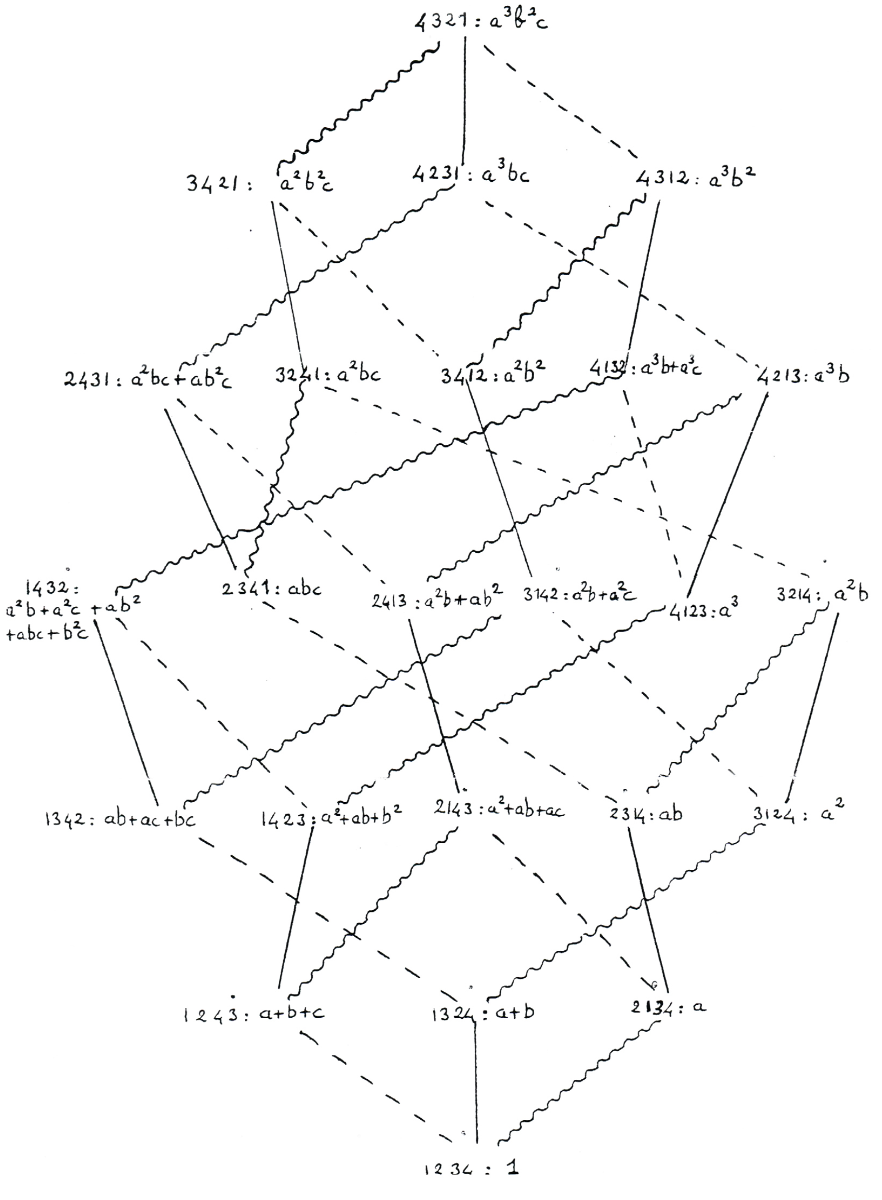


Figure 1: Schubert polynomials for Σ_4

Thus for any $w \in \Sigma_\infty$, we can define ∂_w as $\partial_{s_1} \circ \ldots \circ \partial_{s_k}$ independently of a reduced word of w.

Let $w \in S_\infty$ and let n be a natural number such that $w(k) = k$ for $k > n$. We define the *Schubert polynomial of Lascoux and Schützenberger* [20] associated with a permutation w, by setting

$$\mathfrak{S}_w := \partial_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}^1 x_n^0)\,,$$

where w_0 is the permutation $(n, n-1, \ldots, 2, 1), n+1, n+2, \ldots$. Observe that this definition does not depend on the choice of n, because

$$\partial_n \circ \ldots \circ \partial_2 \circ \partial_1(x_1^n x_2^{n-1} \ldots x_n) = x_1^{n-1}x_2^{n-2}\ldots x_{n-1}\,.$$

In the picture of Schubert polynomials for the symmetric group Σ_4, displayed in Figure 1, we have: $a = x_1$, $b = x_2$, $c = x_3$; to a permutation there is attached its Schubert polynomial; the line $\sim\!\sim\!\sim$ means ∂_1, the straight continuous line means ∂_2, and the line $---$ means ∂_3. (The author got this picture from Lascoux in February 1982 together with a preliminary version of [20].)

Also, we define the kth *inversion set* of a permutation w as follows:

$$I_k(w) := \{l : l > k,\ w(k) > w(l)\} \qquad k = 1, 2, \ldots\,.$$

By the *code* of w (notation: $c(w)$), we shall mean the sequence $(i_1, i_2, \ldots)$, where $i_k = |I_k(w)|$, $k = 1, 2, \ldots$. A code determines the corresponding permutation in a unique way.

Example 5. The code $c(4, 2, 1, 6, 3, 5, 7, 8, \ldots)$ is equal to $(3, 1, 0, 2, 0, \ldots)$.

Let us mention the following properties of Schubert polynomials:

- A Schubert polynomial $\mathfrak{S}_w$ is symmetric in x_k and x_{k+1} if and only if $w(k) < w(k+1)$ (or equivalently if $i_k \le i_{k+1}$);
- If $w(1) < w(2) < \ldots < w(k) > w(k+1) < w(k+2) < \ldots$ (or equivalently $i_1 \le i_2 \le \ldots \le i_k$, $0 = i_{k+1} = i_{k+2} = \ldots$), then $\mathfrak{S}_w$ is equal to the Schur function $s_{i_k,\ldots,i_2,i_1}(x_1, \ldots, x_k)$.
- If $i_1 \ge i_2 \ge \ldots$, then $\mathfrak{S}_w = x_1^{i_1}x_2^{i_2}\ldots$ is a monomial.

Example 6. The Schubert polynomials of degree 1 are $\mathfrak{S}_{s_i} = x_1 + \ldots + x_i$, $i = 1, 2, \ldots$.

For properties of Schubert polynomials, including those mentioned above, the reader may consult [23] and [19].

Note that the Schubert polynomials indexed by Σ_n do not generate additively $\mathbb{Z}[x_1, \ldots, x_n]$. We have (see [23])

$$\sum_{w \in \Sigma_n} \mathbb{Z}\mathfrak{S}_w = \bigoplus_{0 \le \alpha_i \le n-i} \mathbb{Z}x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}\,.$$

If the sets $I_k(w)$ form a chain (with respect to the inclusion), then the permutation w is called *vexillary*. Equivalently, there are no $i < j < k < l$ with $w(j) < w(i) < w(l) < w(k)$ (see, e.g., [23]).

The following result stems from [20] and [32]:

Theorem 7 (Lascoux–Schützenberger, Wachs)**.** *If w is a vexillary permutation with code $(i_1, i_2, \ldots, i_n > 0, 0, \ldots)$, then*

$$\mathfrak{S}_w = s_{(i_1,\ldots,i_n)^{\geq}}\left(\min I_1(w) - 1, \ldots, \min I_n(w) - 1\right)^{\leq} .$$

(By $(-)^{\geq}$ and $(-)^{\leq}$, we mean the respectively ordered sets.)

Here, for two sequences of natural numbers

$$i_1 \geq \ldots \geq i_k \quad \text{and} \quad 0 < b_1 \leq \ldots \leq b_k ,$$

$$s_{i_1,\ldots,i_k}(b_1, \ldots, b_k) := \det\left(s_{i_p - p + q}(x_1, \ldots, x_{b_p})\right)_{1 \leq p,q \leq k}$$

is a *flagged Schur function* (see [23]).

4 Functors asked by Lascoux

Let R be a commutative $\mathbb{Q}$-algebra, and $E_\bullet : E_1 \subset E_2 \subset \ldots$ a flag of R-modules. Suppose that $\mathcal{I} = [i_{k,l}]$, $k, l = 1, 2, \ldots$, is a matrix of 0's and 1's such that

- $i_{k,l} = 0$ for $k \geq l$;
- $\sum_l i_{k,l}$ is finite for any k;
- $\mathcal{I}$ has a finite number of nonzero rows.

Such a matrix $\mathcal{I}$ is called a *shape*. We shall represent a shape graphically by replacing each unit by "$\times$" and by omitting zeros on the diagonal and under it. Moreover, we shall omit all the columns which are right to the last nonzero column. For example, the matrix

$$\begin{array}{cccccccccc} 0&0&0&0&0&0&0&0&0&\ldots \\ 0&0&0&0&0&0&0&1&0&\ldots \\ 0&0&0&1&0&0&0&0&0&\ldots \\ 0&0&0&0&1&0&1&0&0&\ldots \\ 0&0&0&0&0&0&1&0&0&\ldots \\ 0&0&0&0&0&0&0&0&0&\ldots \\ &&&\ldots\ldots&&&&&& \end{array} \quad \text{will be represented as} \quad \begin{array}{ccccccc} 0&0&0&0&0&0&0 \\ &0&0&0&0&0&\times \\ &&\times&0&0&0&0 \\ &&&\times&0&\times&0 \\ &&&&0&\times&0 \\ &&&&&0&0 \end{array} .$$

Let $i_k := \sum_{l=1}^{\infty} i_{k,l}$, $\widetilde{i}_l := \sum_{k=1}^{\infty} i_{k,l}$. We define $S_{\mathcal{I}}(E_\bullet)$ as the image of the following composition:

$$\Phi_{\mathcal{I}}(E_\bullet) : \bigotimes_k S_{i_k}(E_k) \xrightarrow{\Delta_S} \bigotimes_k \bigotimes_l S_{i_{k,l}}(E_k) \xrightarrow{m_\wedge} \bigotimes_l \bigwedge^{\widetilde{i}_l} E_l , \tag{3}$$

where Δ_S is the diagonalization in the symmetric algebra and $m_\wedge$ is the multiplication in the exterior algebra.

For example, if all the 1's occur in $\mathcal{I}$ only in the kth row, then we get $S_p(E_k)$, where p is the number of 1's. Also, if all the 1's occur in $\mathcal{I}$ in one column in consecutive p rows, then we get $\wedge^p(E_k)$, where k is the number of the lowest row with 1.

Remark 8. Note that $S_{\mathcal{I}}(-)$ is a *functor*: if $E_\bullet$ and $F_\bullet$ are flags of R-modules,

$$f : \bigcup_n E_n \to \bigcup_n F_n$$

is an R-homomorphism such that $f(E_n) \subset F_n$, then f induces an R-homomorphism $S_{\mathcal{I}}(E_\bullet) \to S_{\mathcal{I}}(F_\bullet)$.

Let $w \in \Sigma_\infty$. By the *shape* of w we mean the matrix:

$$\mathcal{I}_w = [i_{k,l}] := [\chi_k(l)]\,, \quad k,l = 1,2,\ldots\,,$$

where χ_k is the characteristic function of $I_k(w)$.

Example 9. For $w = 4,2,1,6,3,5,7,8,\ldots$, the shape $\mathcal{I}_w$ is equal to

$$\begin{array}{ccccc}
\times & \times & 0 & \times & 0 \\
 & \times & 0 & 0 & 0 \\
 & & 0 & 0 & 0 \\
 & & & \times & \times \\
 & & & & 0
\end{array}$$

We define a module $S_w(E_\bullet)$, associated with a permutation w and a flag $E_\bullet$ as $S_{\mathcal{I}_w}(E_\bullet)$; this leads to a functor $S_w(-)$. These modules were defined by Kraśkiewicz and the author in [16, 17].

Example 10. Invoking the previous example, we see that $S_{42163578\ldots}(E_\bullet)$ is the image of the map

$$\Phi_{\mathcal{I}_{42163578\ldots}} : S_3(E_1) \otimes S_1(E_2) \otimes S_2(E_4) \to E_1 \otimes \wedge^2(E_2) \otimes \wedge^2(E_4) \otimes E_5\,.$$

From now on, let $E_\bullet$ be a flag of K-vector spaces with $\dim E_i = i$. Let B be the Borel group of linear endomorphisms of $E := \bigcup E_i$, which preserve $E_\bullet$. The modules used in the definition of $S_{\mathcal{I}}(E_\bullet)$ are $\mathbb{Z}[B]$-modules, and the maps are homomorphisms of $\mathbb{Z}[B]$-modules. Let $\{u_i : i = 1,2,\ldots\}$ be a basis of E such that $u_1, u_2, \ldots, u_k$ span E_k. Then the module $S_{\mathcal{I}}(E_\bullet)$ as a cyclic $\mathbb{Z}[B]$-submodule in $\bigotimes_l \bigwedge^{\widetilde{i}_l} E_l$, is generated by the element

$$u_{\mathcal{I}} := \bigotimes_l u_{k_{1,l}} \wedge u_{k_{2,l}} \wedge \ldots \wedge u_{k_{i_l,l}}\,,$$

where $k_{1,l} < k_{2,l} < \ldots < k_{i_l,l}$ are precisely those indices for which $i_{k_{r,l},l} = 1$. In particular, $S_w(E_\bullet)$ is a cyclic $\mathbb{Z}[B]$-module generated by $u_w = u_{\mathcal{I}_w}$. The modules $S_w(E_\bullet)$ were called by Watanabe *Kraśkiewicz–Pragacz modules*, in short *KP modules* [33, v3].

Example 11. The KP module $S_{42163578\ldots}(E_\bullet)$ is generated over $\mathbb{Z}[B]$ by the element

$$u_{42163578\ldots} = u_1 \otimes u_1 \wedge u_2 \otimes u_1 \wedge u_4 \otimes u_4\,.$$

KP modules give a substantial generalization of Schur modules discussed in Sect. 2. Note that any Schur module $V_\lambda(E_m)$, where $\lambda = (\lambda_1, \ldots, \lambda_k > 0)$ and $k \le m$, can be realized as $S_w(E_\bullet)$ for some w. We claim that the permutation w with the code $(0^{m-k}, \lambda_k, \ldots, \lambda_1, 0, \ldots)$ determines the desired KP module. Take for example $m = 4$, $\lambda = (4,3,1)$. The permutation with the code $(0,1,3,4,0,\ldots)$ is $1,3,6,8,2,4,5,7,9,\ldots$, and has the shape

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & \times & 0 & 0 & 0 \\
 & & 0 & \times & \times & \times & 0 \\
 & & & \times & \times & \times & \times \\
 & & & & 0 & 0 & 0 \\
 & & & & & 0 & 0 \\
 & & & & & & 0
\end{array}$$

Its KP module is given by the image of the map

$$\Phi_{\mathcal{I}_{136824579\ldots}} : E_2 \otimes S_3(E_3) \otimes S_4(E_4) \to \wedge^3(E_4) \otimes \wedge^2(E_4) \otimes \wedge^2(E_4) \otimes E_4\,.$$

We invoke at this point a standard basis theorem for Schur modules (see [1] and [31]), which allows us to replace the flag $E_1 \subset \ldots \subset E_m$ by $E_m = \ldots = E_m$ without change of the image of $\Phi_{\mathcal{I}_w}$. We obtain that the image of $\Phi_{\mathcal{I}_{136824579\ldots}}$ is equal to the image of the map (1)

$$E_4 \otimes S_3(E_4) \otimes S_4(E_4) \to \wedge^3(E_4) \otimes \wedge^2(E_4) \otimes \wedge^2(E_4) \otimes E_4,$$

defining $V_{(4,3,1)}(E_4)$.

We shall now study the character of $S_w(E_\bullet)$. In Section 7, we shall show an application of the following theorem to algebraic geometry. In fact, this theorem will tell us what are the Chern roots of the bundle $S_w(E)$ as functions of the Chern roots of the original bundle E.

Consider the maximal torus (2) $T \subset B$ consisting of diagonal matrices with $x_1, x_2, \ldots$ on the diagonal, with respect to the basis $\{u_i : i = 1, 2, \ldots\}$.

Theorem 12 (Kraśkiewicz–Pragacz)**.** *Let $w \in \Sigma_\infty$. The trace of the action of T on $S_w(E_\bullet)$ is equal to the Schubert polynomial $\mathfrak{S}_w$.*

About the proof: we study the multiplicative properties of $S_w(E_\bullet)$'s, comparing them with those of the $\mathfrak{S}_w$'s.

Let $t_{p,q}$ be the permutation:

$$1, \ldots, p-1, q, p+1, \ldots, q-1, p, q+1, \ldots\,.$$

We now record the following formula for multiplication by $\mathfrak{S}_{s_k}$:

Theorem 13 (Monk)**.** *Let $w \in \Sigma_\infty$. We have*

$$\mathfrak{S}_w \cdot (x_1 + \ldots + x_k) = \sum \mathfrak{S}_{w \circ t_{p,q}}\,,$$

where the sum is over p, q such that $p \le k$, $q > k$ and $l(w \circ t_{p,q}) = l(w) + 1$.

(See [26], [21], and [23].)

Example 14. We have

$$\mathfrak{S}_{135246\ldots} \cdot (x_1 + x_2) = \mathfrak{S}_{235146\ldots} + \mathfrak{S}_{153246\ldots} + \mathfrak{S}_{145236\ldots}\,.$$

In fact, we shall use more efficiently the following result of Lascoux and Schützenberger (see [22]).

Theorem 15 (Transition formula)**.** *Let $w \in \Sigma_\infty$. Suppose that (j, s) is a pair of positive integers such that*

1) *$j < s$ and $w(j) > w(s)$,*

2) *for any $i \in]j, s[$, $w(i) \notin [w(s), w(j)]$*

3) *for any $r > j$, if $w(s) < w(r)$ then there exists $i \in]j, r[$ such that $w(i) \in [w(s), w(r)]$.*

Then

$$\mathfrak{S}_w = \mathfrak{S}_v \cdot x_j + \sum_{p=1}^{m} \mathfrak{S}_{v_p} ,$$

where $v = w \circ t_{j,s}$, $v_p = w \circ t_{j,s} \circ t_{k_p,j}$, $p = 1, \ldots, m$, *say. Here the numbers* k_p *are such that*

4) $k_p < j$ *and* $w(k_p) < w(s)$,

5) *if* $i \in]k_p, j[$ *then* $w(i) \notin [w(k_p), w(s)]$.

Note that if (j, s) is the maximal pair (in the lexicographical order) satisfying 1), then conditions 2)–3) are also fulfilled. A transition corresponding to this pair will be called *maximal*. In particular, for any nontrivial permutation, there is at least one transition.

Example 16. For the permutation $521863479\ldots$,

$$\mathfrak{S}_{521843679\ldots} \cdot x_5 + \mathfrak{S}_{524813679\ldots} + \mathfrak{S}_{541823679\ldots}$$

is the maximal transition, and other transitions are

$$\mathfrak{S}_{521763489\ldots} \cdot x_4 + \mathfrak{S}_{527163489\ldots} + \mathfrak{S}_{571263489\ldots} + \mathfrak{S}_{721563489\ldots}$$

and

$$\mathfrak{S}_{512864379\ldots} \cdot x_2 .$$

We prove that for the maximal transition for w, there exists a filtration of $\mathbb{Z}[B]$-modules

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m \subset \mathcal{F} = S_w(E_\bullet)$$

together with isomorphisms

$$\mathcal{F}/\mathcal{F}_m \simeq S_v(E_\bullet) \otimes E_j/E_{j-1} \qquad \text{and} \qquad \mathcal{F}_p/\mathcal{F}_{p-1} \simeq S_{v_p}(E_\bullet)$$

where $p = 1, \ldots, m$. This implies an isomorphism of T-modules

$$S_w(E_\bullet) \simeq S_v(E_\bullet) \otimes E_j/E_{j-1} \oplus \bigoplus_{p=1}^{m} S_{v_p}(E_\bullet) .$$

By comparing this with the transition formula for Schubert polynomials, the assertion of Theorem 12 follows by a suitable induction.
(For details see [17, Sect. 4].)

There exist *flagged Schur modules* $S_\lambda(-)$, associated with suitable shapes (see [17, p. 1330]). Suppose that $E_\bullet$ is a flag of free R-modules with $E_1 = R$ and such that the ith inclusion in the flag is given by $E_i \hookrightarrow E_i \oplus R \simeq E_{i+1}$. We record

Theorem 17 (Kraśkiewicz–Pragacz)**.** *If w is a vexillary permutation with code* $(i_1, i_2, \ldots, i_n > 0, 0, \ldots)$, *then*

$$S_w(E_\bullet) = S_{(i_1,\ldots,i_n)^{\geq}} \left(E_{\min I_1(w)-1}, \ldots, E_{\min I_n(w)-1}\right)^{\leq} .$$

(See [16, 17].)

There is a natural extension of KP modules to complexes with interesting applications (see [29]).

5 KP filtrations of weight modules

All the results of this section (with just a few exceptions) are due to Watanabe in [33, 34]. In [33], the author studied the structure of KP modules using the theory of highest weight categories [5]. From the results in [33], in particular, one obtains a certain highest weight category whose standard modules are KP modules. In [34], the author investigated the tensor multiplication properties of KP modules. At the end of this section, we shall summarize [35, 36].

In this section, we shall use the language and techniques of enveloping algebras, see, e.g., [9].

Let $\mathfrak{b}$ be the Lie algebra of $n \times n$ upper matrices over K, $\mathfrak{t}$ that of diagonal matrices, and $U(\mathfrak{b})$ the enveloping algebra of $\mathfrak{b}$. Suppose that M is a $U(\mathfrak{b})$-module and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, Denote by

$$M_\lambda = \{m \in M : hm = \langle \lambda, h \rangle m\}$$

the *weight space* of λ, $\langle \lambda, h \rangle = \sum \lambda_i h_i$. If M is a direct sum of its weight spaces and each weight space has finite dimension, then M is called a *weight module*. For a weight module, we set

$$ch(M) := \sum_\lambda \dim M_\lambda x^\lambda \,,$$

where $x^\lambda = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$.

Let e_{ij} be the matrix with 1 at the (i,j)-position and 0 elsewhere.

Let K_λ be a one-dimensional $U(\mathfrak{b})$-module, where h acts by $\langle \lambda, h \rangle$ and the matrices e_{ij}, where $i < j$, acts by zero. Any finite-dimensional weight module admits a filtration by these one-dimensional modules.

Fix $n \in \mathbb{N}$. In this section, we shall mainly work with permutations from

$$\Sigma^{(n)} := \{w : w(n+1) < w(n+2) < \ldots\} \,.$$

Observe that the codes of permutations in $\Sigma^{(n)}$ are in $\mathbb{Z}^n_{\geq 0}$; in fact, they exhaust $\mathbb{Z}^n_{\geq 0}$. Moreover, we have

$$\sum_{w \in \Sigma^{(n)}} \mathbb{Z}\mathfrak{S}_w = \mathbb{Z}[x_1, \ldots, x_n] \,.$$

Write $E = \bigoplus_{1 \leq i \leq n} K u_i$. For each $j \in \mathbb{N}$, let $l_j = l_j(w)$ be the cardinality of the set

$$\{i < j : w(i) > w(j)\} = \{i_1 < \ldots < i_{l_j}\} \,,$$

and write

$$u_w^{(j)} = u_{i_1} \wedge \ldots \wedge u_{i_{l_j}} \in \Lambda^{l_j}(E) \,.$$

We have $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \ldots$ and $S_w = U(\mathfrak{b}) u_w$. The weight of u_w is $c(w)$. Observe that Theorem 12 can be restated as

Theorem 18. *For any $w \in \Sigma^{(n)}$, S_w is a weight module and $ch(S_w) = \mathfrak{S}_w$.*

A natural question arises: What is the annihilator of u_w?

Let $w \in \Sigma^{(n)}$. Consider the following assignment:

$$(1 \leq i < j \leq n) \quad \longrightarrow \quad m_{ij}(w) = \#\{k > j : w(i) < w(k) < w(j)\} \,.$$

Then $e_{ij}^{m_{ij}+1}$ annihilates u_w. Let $I_w \subset U(\mathfrak{b})$ be the ideal generated by $h - \langle c(w), h\rangle$, $h \in \mathfrak{t}$, and $e_{ij}^{m_{ij}(w)+1}$, $i < j$. Then there exists a surjection

$$U(\mathfrak{b})/I_w \twoheadrightarrow S_w$$

such that 1 mod $I_w \mapsto u_w$.

Theorem 19 (Watanabe)**.** *This surjection is an isomorphism.*

(See [33], Sect. 4.)

For $\alpha, \beta \in \mathbb{Z}^n$, we shall write $\alpha \pm \beta$ for $(\alpha_1 \pm \beta_1, \dots, \alpha_n \pm \beta_n)$.

For $\lambda \in \mathbb{Z}^n_{\geq 0}$, we set $S_\lambda := S_w$, where $c(w) = \lambda$. For $\lambda \in \mathbb{Z}^n$ take k such that $\lambda + k\mathbf{1} \in \mathbb{Z}^n_{\geq 0}$ ($\mathbf{1} = (1, \dots, 1)$ n times), and set $S_\lambda = K_{-k\mathbf{1}} \otimes S_{\lambda+k\mathbf{1}}$. We shall use a similar notation for Schubert polynomials.

A *KP fitration* of a weight module is a sequence

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of weight modules such that each subquotient M_i/M_{i-1} is isomorphic to some KP module.

One can ask the following questions:

1. When a weight module admits a KP filtration?
2. Does $S_\lambda \otimes S_\mu$ have a KP filtration?

Write $\rho = (n-1, n-2, \dots, 1, 0)$. The module K_ρ will play a role of a "dualizing module".

Let $\mathcal{C}$ denote the category of all weight modules. For $\Lambda \subset \mathbb{Z}^n$, let $\mathcal{C}_\Lambda$ be the full subcategory of $\mathcal{C}$ consisting of all weight modules whose weights are in Λ. If $|\Lambda| < \infty$ and $\Lambda' = \{\rho - \lambda : \lambda \in \Lambda\}$, then the map $M \mapsto M^* \otimes K_\rho$ yields an isomorphism $C_{\Lambda'} \cong \mathcal{C}_\Lambda^{op}$.

Lemma 20. *For any $\Lambda \subset \mathbb{Z}^n$, $\mathcal{C}_\Lambda$ has enough projectives.*

(See [33, Sect. 6].)

We now define some useful orders on Σ_∞. For $w, v \in \Sigma_\infty$, $w \leq_{lex} v$ if $w = v$ or there exists $i > 0$ such that $w(j) = v(j)$ for $j < i$ and $w(i) < v(i)$.

For $\lambda \in \mathbb{Z}^n$, define $|\lambda| = \sum \lambda_i$. If $\lambda = c(w)$, $\mu = c(v)$, we write $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $w^{-1} \leq_{lex} v^{-1}$. For general $\lambda, \mu \in \mathbb{Z}^n$ take k such that $\lambda + k\mathbf{1}, \mu + k\mathbf{1} \in \mathbb{Z}^n_{\geq 0}$, and define $\lambda \geq \mu$ iff $\lambda + k\mathbf{1} \geq \mu + k\mathbf{1}$.

For $\lambda \in \mathbb{Z}^n$, set $\mathcal{C}_{\leq\lambda} := \mathcal{C}_{\{\nu : \nu \leq \lambda\}}$.

Proposition 21. *For $\lambda \in \mathbb{Z}^n$, the modules S_λ and $S^*_{\rho-\lambda} \otimes K_\rho$ are in $\mathcal{C}_{\leq\lambda}$. Moreover, S_λ is projective and $S^*_{\rho-\lambda} \otimes K_\rho$ is injective.*

(See [33, Sect. 6].)

For the definitions and properties of Ext's, we refer to [25]. All Ext's will be taken over $U(\mathfrak{b})$, in $\mathcal{C}_{\leq\lambda}$.

Theorem 22 (Watanabe)**.** *For $\mu, \nu \leq \lambda$, $\mathrm{Ext}^i(S_\mu, S^*_{\rho-\nu} \otimes K_\rho) = 0$ if $i \geq 1$.*

(See [33, Sect. 7]. This can be regarded a "Strong form of Polo's theorem" [15, Theorem 3.2.2].)

Theorem 23 (Watanabe)**.** *Let $M \in \mathcal{C}_{\leq\lambda}$. If $\mathrm{Ext}^1(M, S^*_{\rho-\mu} \otimes K_\rho) = 0$ for all $\mu \leq \lambda$, then M has a KP filtration such that each of its subquotients is isomorphic to some S_ν with $\nu \leq \lambda$.*

(See [33, Sect. 8] and also [34, Theorem 2.5].)

Corollary 24.

(1) *If $M = M_1 \oplus \ldots \oplus M_r$, then M has a KP filtration iff each M_i does.*

(2) *If $0 \to L \to M \to N \to 0$ is exact and M, N have KP filtrations, then L also does.*

Proof. Assertion (1) follows from $\mathrm{Ext}^1(M,N) = \oplus\,\mathrm{Ext}^1(M_i, N)$ for any N.

Assertion (2) follows from the exactness of the sequence

$$\mathrm{Ext}^1(M,A) \to \mathrm{Ext}^1(L,A) \to \mathrm{Ext}^2(N,A)$$

for any A. □

Proposition 25. *Let $w \in \Sigma^{(n)}$, $1 \leq k \leq n-1$. Then $S_w \otimes S_{s_k}$ has a KP filtration.*

This result was established in [17, Sect. 5] for $k = 1$ and in [34, Sect. 3] in general.

Theorem 26 (Watanabe)**.** *$S_w \otimes S_v$ has a KP filtration for any $w, v \in \Sigma^{(n)}$.*

In order to outline a proof, set $l_i = l_i(w)$ and consider an $U(\mathfrak{b})$-module

$$T_w = \bigotimes_{2\leq i\leq n} \Lambda^{l_i}(K^{i-1})\,.$$

The module T_w is a direct sum component of

$$\bigotimes_{2\leq i\leq n} \big(S_{s_{i-1}} \otimes \ldots \otimes S_{s_{i-1}}\big)\,,$$

where $S_{s_{i-1}}$ appears l_i times.

Proposition 27. *Suppose that $w \in \Sigma_n$. Then there is an exact sequence*

$$0 \to S_w \to T_w \to N \to 0\,,$$

where N has a filtration whose subquotients are S_u with $u^{-1} >_{lex} w^{-1}$.

Granting this proposition, a proof of the theorem consists of considering the following exact sequence of $U(\mathfrak{b})$-modules

$$0 \quad\to\quad S_w \otimes S_v \quad\to\quad T_w \otimes S_v \quad\to\quad N \otimes S_v \quad\to\quad 0\,.$$

Here, the middle module has a KP filtration by the proposition and the module on the right one by induction on $\mathrm{lex}(w)$. Consequently the module on the left has a KP filtration by Corollary 24.

To show the proposition, we need the following

Lemma 28.
(i) *If the coefficient of $x^{c(v)}$ in $\mathfrak{S}_w$ is nonzero, then we have $v^{-1} \geq_{lex} w^{-1}$.*
(ii) *If $\mathrm{Ext}^1(S_w, S_u) \neq (0)$, then $u^{-1} <_{lex} w^{-1}$.*

(See [33, Sect. 6].)

We come back to the proposition. Define the integers m_{wu} by

$$\sum_{u \in \Sigma_n} m_{wu} \mathfrak{S}_u = \prod_{2 \leq i \leq n} e_{l_i}(x_1, \ldots, x_{i-1}),$$

where e_k denotes the elementary symmetric function of degree k. Thus m_{wu} is the number of times S_u appears as a subquotient of any KP filtration of T_w. Let us invoke the following Cauchy-type formula:

$$\prod_{i+j \leq n} (x_i + y_j) = \sum_{w \in \Sigma_n} \mathfrak{S}_w(x) \mathfrak{S}_{ww_0}(y) \tag{4}$$

(see [23] and [19]). Consider the bilinear form $\langle , \rangle$ on the free abelian group generated by Schubert polynomials $\mathfrak{S}_w$, where $w \in \Sigma_n$, corresponding to this Cauchy identity, i.e., such that $\langle \mathfrak{S}_u, \mathfrak{S}_{u'w_0} \rangle = \delta_{uu'}$ for any $u, u' \in \Sigma_n$. We have

$$m_{wu} = \Big\langle \mathfrak{S}_{uw_0}, \prod_{2 \leq i \leq n} e_{l_i}(x_1, \ldots, x_{i-1}) \Big\rangle .$$

We now use an additional property of the bilinear form: for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ with $\alpha_i, \beta_i \leq n - i$,

$$\Big\langle x^{\rho - \alpha}, \prod_{1 \leq i \leq n-1} e_{\beta_i}(x_1, \ldots, x_{n-i}) \Big\rangle = \delta_{\alpha, \beta} .$$

Using these two mutually orthogonal bases of monomials and products of elementary symmetric functions, we infer that m_{wu} is the coefficient of $x_1^{n-1-l_n} x_2^{n-2-l_{n-1}} \cdots$ in $\mathfrak{S}_{uw_0}$. It is not hard to see that for any k,

$$n - k - l_{n+1-k} = c(ww_0)_k ,$$

and thus m_{wu} is equal to the coefficient of $x^{c(ww_0)}$ in $\mathfrak{S}_{uw_0}$. By Lemma 28(i), this coefficient is nonzero only if

$$u^{-1} \geq_{lex} w^{-1} .$$

If $u = w$, then $m_{wu} = 1$; thus the subquotients of any KP filtration of T_w are the KP modules S_u, where $u^{-1} >_{lex} w^{-1}$, together with S_w which occurs just once. Since, by Lemma 28(ii), $\mathrm{Ext}^1(S_w, S_u) = 0$ if $u^{-1} >_{lex} w^{-1}$, one can take a filtration such that S_w occurs as a submodule of T_w.

The proposition gives a materialization of (4).

Theorem 29 (Watanabe)**.** *Let $\lambda \in \mathbb{Z}^n$ and $M \in \mathcal{C}_{\leq \lambda}$. Then we have*

$$ch(M) \leq \sum_{\nu \leq \lambda} \dim_K \mathrm{Hom}_{U(\mathfrak{b})}(M, S^*_{\rho - \nu} \otimes K_\rho) \mathfrak{S}_\nu .$$

(Here $\sum a_\alpha x^\alpha \leq \sum b_\alpha x^\alpha$ means that $a_\alpha \leq b_\alpha$ for any α.) The equality holds if and only if M has a KP filtration with all subquotients isomorphic to S_μ, where $\mu \leq \lambda$.

(See [33, Sect. 8].)

As a byproduct, we get a formula for the coefficient of $\mathfrak{S}_w$ in $\mathfrak{S}_u \mathfrak{S}_v$:

Corollary 30. *This coefficient is equal to the dimension of*

$$\mathrm{Hom}_{U(\mathfrak{b})}(S_u \otimes S_v, S^*_{w_0 w} \otimes K_\rho) = \mathrm{Hom}_{U(\mathfrak{b})}(S_u \otimes S_v \otimes S_{w_o w}, K_\rho)\,.$$

Proof. We use ch together with its multiplicativity property, and infer

$$\mathfrak{S}_u \mathfrak{S}_v = ch(S_u \otimes S_v) = \sum_w \dim_K \mathrm{Hom}_{U(\mathfrak{b})}(S_u \otimes S_v, S^*_{\rho-\lambda} \otimes K_\rho)\mathfrak{S}_w\,. \qquad \square$$

Note 31. In [35], Watanabe showed that the highest weight category from [33] is self Ringel-dual and that the tensor product operation on $U(\mathfrak{b})$-modules is compatible with Ringel duality functor. Also, an interesting formula: $\mathrm{Ext}^i(S_w, S_v) \simeq \mathrm{Ext}^i(S_{w_0 v w_0}, S_{w_0 w w_0})$ was established there. In a recent paper [36], Watanabe gave explicit KP filtrations materializing Pieri-type formulas for Schubert polynomials.

6 An application of KP filtrations to positivity

Let V_σ denote the Schur functor associated to a partition σ (cf. Sect. 2), and let S_λ be the KP module associated with a sequence λ.

Proposition 32 (Watanabe)**.** *The module $V_\sigma(S_\lambda)$ has a KP filtration.*

Proof. The module $S_\lambda^{\otimes k}$ has a KP filtration for any λ and any k. Hence

$$\mathrm{Ext}^1(S_\lambda^{\otimes k}, S^*_\nu \otimes K_\rho) = 0$$

for any ν by Theorem 23. The module $V_\sigma(S_\lambda)$ is a direct sum factor of $S_\lambda^{\otimes|\sigma|}$. Hence

$$\mathrm{Ext}^1(V_\sigma(S_\lambda), S^*_\nu \otimes K_\rho) = 0$$

for any ν, and $V_\sigma(S_\lambda)$ has a KP filtration. $\square$

Corollary 33. *If $\mathfrak{S}_w$ is a sum of monomials $x^\alpha + x^\beta + \ldots$, then the following specialization of a Schur function: $s_\sigma(x^\alpha, x^\beta, \ldots)$ is a sum of Schubert polynomials with nonnegative coefficients.*

7 An application of the bundles $S_w(E)$ to positivity

A good reference for the notions of algebraic geometry needed in this section is [13]. For simplicity, by X, we shall denote a nonsingular projective variety.

Given a vector bundle E on X and a partition λ, by the *Schur polynomial of* E, denoted by $s_\lambda(E)$, we shall mean $s_\lambda(\alpha_1, \ldots, \alpha_n)$, where α_i are the Chern roots of E.

Let us recall that a vector bundle E on X is *ample* if for any coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(\mathcal{F})$ such that for any $m \geq m_0$ the sheaf $S_m(E) \otimes \mathcal{F}$ is generated by its global sections (cf. [12]). Let $\mathcal{O}(1)$ be the twisting sheaf of Serre on $\mathbf{P}(E)$ (cf. [13, II]). Then the ampleness of E is equivalent to the ampleness of the line bundle $\mathcal{O}(1)$ (see [12, Sect. 3]).

In [10, p. 632], Fulton showed that a Schur (or Schubert) polynomial of the bundle $S_w(E)$ associated to a filtered bundle E is a nonnegative combination of Schubert polynomials of E. The method relies on ample vector bundles.

We shall say that a weighted homogeneous polynomial $P(c_1, c_2, \ldots)$ of degree d, where the variables c_i are of degree i, is *numerically positive for ample vector bundles* if for any variety X of dimension d and any ample vector bundle E on X,

$$\int_X P(c_1(E), c_2(E), \ldots) > 0\,.$$

(Here $\int_X$ denotes the degree of zero-cycles.)

Note that under the identification of c_i with the ith elementary symmetric function in $x_1, x_2, \ldots$, any weighted homogeneous polynomial $P(c_1, c_2, \ldots)$ of degree d is a $\mathbb{Z}$-combination of Schur polynomials $\sum_{|\lambda|=d} a_\lambda s_\lambda$. In their study of numerically positive polynomials for ample vector bundles, the authors of [11] showed

Theorem 34 (Fulton–Lazarsfeld)**.** *Such a nonzero polynomial $\sum_{|\lambda|=d} a_\lambda s_\lambda$ is numerically positive for ample vector bundles if and only if all the coefficients a_λ are nonnegative.*

Schur functors give rise to Schur bundles $V_\lambda(E)$ associated to vector bundles E. In [28, Cor. 7.2], the following result was shown

Corollary 35 (Pragacz)**.** *In a $\mathbb{Z}$-combination $s_\lambda(V_\mu(E)) = \sum_\nu a_\nu s_\nu(E)$ of Schur polynomials, all the coefficients a_ν are nonnegative.*

Indeed, assuming that E is ample, combining the theorem and the fact that the Schur bundle of an ample bundle is ample (see [12]), the assertion follows.

To proceed further, we shall need a variant of functors $S_{\mathcal{I}}(-)$ associated with sequences of surjections of modules. Suppose that

$$F_0 = F \twoheadrightarrow F_1 \twoheadrightarrow F_2 \twoheadrightarrow \ldots \tag{5}$$

is a sequence of surjections of R-modules, where R is a commutative ring. Let $\mathcal{I} = [i_{k,l}]$ be a shape (see Sect. 4), $i_k := \sum_{l=1}^{\infty} i_{k,l}$, and $\widetilde{i}_l := \sum_{k=1}^{\infty} i_{k,l}$. We define $S'_{\mathcal{I}}(F_\bullet)$ as the image of the following composition:

$$\Psi_{\mathcal{I}}(F_\bullet) : \bigotimes_k \wedge^{i_k}(F_k) \xrightarrow{\Delta_\wedge} \bigotimes_k \bigotimes_l \wedge^{i_{k,l}}(F_k) \xrightarrow{m_S} \bigotimes_l S_{\widetilde{i}_l}(F_l)\,, \tag{6}$$

where $\Delta_\wedge$ is the diagonalization in the exterior algebra and m_S is the multiplication in the symmetric algebra. For $w \in \Sigma_\infty$, we define

$$S'_w(F_\bullet) = S'_{\mathcal{I}_w}(F_\bullet)\,.$$

Suppose that R is K, and F_i is spanned by $f_1, f_2, \ldots, f_{n-i}$. Consider the maximal torus (2) $T \subset B$ consisting of diagonal matrices with $x_1, x_2, \ldots$ on the diagonal, with respect to the basis $\{f_i : i = 1, 2, \ldots\}$. If $w \in \Sigma_n$, then the character of S'_w, i.e., the trace of the action of T on $S'_w(F_\bullet)$, is equal to $\mathfrak{S}_{w_0 w w_0}$.

We now pass to filtered vector bundles. By a *filtered bundle*, we shall mean a vector bundle E of rank n, equipped with a flag of subbundles

$$0 = E_n \subset E_{n-1} \subset \ldots \subset E_0 = E \,,$$

where $\mathrm{rank}(E_i) = n - i$ for $i = 0, 1, \ldots, n$.

For a polynomial P of degree d, and a filtered vector bundle E on a variety X, by substituting $c_1(E_0/E_1)$ for x_1, $c_1(E_1/E_2)$ for x_2,..., $c_1(E_{i-1}/E_i)$ for $x_i, \ldots$, we get a codimension d class denoted $P(E_\bullet)$. Such a polynomial P will be called *numerically positive for filtered ample vector bundles*, if for any filtered ample vector bundle E on any d-dimensional variety X,

$$\int_X P(E_\bullet) > 0 \,.$$

Write P as a $\mathbb{Z}$-combination of Schubert polynomials $P = \sum a_w \mathfrak{S}_w$ with the unique coefficients a_w. We then have (see [10])

Theorem 36 (Fulton)**.** *Such a nonzero polynomial $P = \sum a_w \mathfrak{S}_w$ is numerically positive for filtered ample bundles if and only if all the coefficients a_w are nonnegative.*

In [10], in fact, a more general result for **r**-filtered ample vector bundles is proved.

Suppose that (5) is a sequence of vector bundles of ranks $n, n-1, \ldots, 1$. We record

Lemma 37. *If F is an ample vector bundle, then the bundle $S'_w(F_\bullet)$ is ample.*

Proof. The bundle $S'_w(F_\bullet)$ is the image of the map (6), thus it is a quotient of a tensor product of exterior powers of the bundles F_k which are quotients of F. The assertion follows from the facts (see [12]) that a quotient of an ample bundle is ample, an exterior power of an ample vector bundle is ample, and the tensor product of ample bundles is ample. □

Let $x_i = c_1(\mathrm{Ker}(F_{n-i} \to F_{n-i+1}))$. Write $s_\lambda(S'_w(F_\bullet))$ as a $\mathbb{Z}$-combination

$$s_\lambda(S'_w(F_\bullet)) = \sum_v a_v \mathfrak{S}_v(x_1, x_2, \ldots) \,. \tag{7}$$

Then assuming that F is ample, and observing that F admits a filtration by $\mathrm{Ker}(F \twoheadrightarrow F_i)$, $i = 0, \ldots, n$, the following result holds true by the theorem and the lemma.

Corollary 38 (Fulton)**.** *The coefficients a_v in* (7) *are nonnegative.*

Let E be a filtered vector bundle with Chern roots $x_1, x_2, \ldots$ and w be a permutation. Then using the notation from the introduction, the Chern roots of the vector bundle $S_w(E_\bullet)$ are expressions of the form $l(x^\alpha)$, where x^α are monomials of $\mathfrak{S}_w$. This follows from the character formula for KP modules (cf. Theorem 12). Therefore we can restate the corollary in the following way.

Corollary 39. *A Schur function specialized with the expressions $l(x^\alpha)$ associated to the monomials x^α of a Schubert polynomial is a nonnegative combination of Schubert polynomials.*

Note 40. In [17, Sect. 6], the reader can find a discussion of some developments related to KP modules. It would be interesting to find further applications of KP modules and construct analogues of KP modules for other types.

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