
UNIVERSAL GYSIN FORMULAS FOR FLAG BUNDLES

by

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To the memory of Alain Lascoux

Abstract. — We give push-forward formulas for all flag bundles of types A, B, C, D . The formulas (and also the proofs) involve only Segre classes of the original vector bundles and characteristic classes of universal bundles. As an application, we provide new determinantal formulas.

0. Introduction

A proper morphism $F: Y \rightarrow X$ of nonsingular algebraic varieties over an algebraically closed field yields an additive map $F_*: A^*Y \rightarrow A^*X$ of Chow groups induced by push-forward cycles, called the *Gysin map* (see Fulton's book [7]; note that the theory developed in this book allows one to generalize the results of the present paper to singular varieties over a field and their Chow groups; moreover, for complex varieties, one can also use the cohomology rings with integral coefficients). We will alternatively denote F_* by \int_Y^X .

Push-forward formulas show how the classes of algebraic cycles on Y go via the Gysin map to classes of algebraic cycles on X . In the present paper, we are interested in push-forwards in flag bundles. We shall give formulas for the classical types A, B, C, D . These have a *universal* character in three aspects:

- these involve characteristic classes of universal vector bundles;
- these universally hold for *any* polynomial in such classes;
- these use in a universal way only the Segre classes of the original vector bundles.

The starting point of our argument is a reformulation of the classical formula for push-forward of powers of the hyperplane class in a projective bundle, that we recall. Let $E \rightarrow X$ be a vector bundle of rank n . Let $\pi: \mathbf{P}(E) \rightarrow X$ be the projective bundle of lines in E and let $\xi := c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ be the hyperplane class. For any i , the i -th *Segre class* of E is

$$s_i(E) := \pi_* (\xi^{i+n-1}).$$

This is a definition in [7] and a lemma in preceding intersection theory (see e.g. [13, Lemma 1]).

Then, consider a polynomial $f(\xi) = \sum_i \alpha_i \xi^i$ with coefficients in the Chow ring of X (here, we identify A^*X with a subring of $A^*\mathbf{P}(E)$ and throughout the text, we will often omit pullback notation for vector bundles and algebraic cycles). Using the projection formula

$$(1) \quad \pi_*(f(\xi)) = \sum_i \alpha_i s_{i-n+1}(E).$$

This formula can be transformed using the point of view of generating series. Before going further, let us introduce some notation. For a monomial m and a Laurent polynomial f , we will

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denote by $[m](f)$ the coefficient of m in f and we will call m the *extracted monomial*. It is clear that for any *shifting monomial* \tilde{m}

$$[\tilde{m}m](\tilde{m}f) = [m](f).$$

We will use this property repeatedly.

Coming back to formula (1), if $\dim(X) = d$, let

$$s_x(E) := 1 + xs_1(E) + \cdots + x^d s_d(E),$$

be the *Segre polynomial* of E . Thus, by definition

$$s_{i-n+1}(E) = [x^{i-n+1}](s_x(E)) = [x^{-n+1}](x^{-i} s_x(E)).$$

In order to have non negative powers we will use the change of variables $t = 1/x$. We get

$$s_{i-n+1}(E) = [t^{n-1}](t^i s_{1/t}(E)).$$

Remark that in the later expression, the extracted monomial does *not* depend on i , whence, by linearity, for any polynomial $f \in A^\bullet X[t]$, the push-forward formula (1) becomes

$$(2) \quad \pi_*(f(\xi)) = [t^{n-1}]\left(\sum_i \alpha_i t^i s_{1/t}(E)\right) = [t^{n-1}](f(t) s_{1/t}(E)).$$

Notably, this expression (2) involves only the Segre polynomial, that behave better than individual Segre classes with respect to relations in the Grothendieck ring of the base variety. This will play a significant role in our induction strategy to generalize the formula (2) to all partial flag bundles of types A, B, C, D .

This is done in Theorems 1.1, 1.2, 2.1, 3.1. The looked at push-forwards are presented as suitable coefficients of some polynomials in the ring $A^\bullet X[t_1, \dots, t_q]$, where $\{t_i\}$ are some auxiliary variables and q is an integer determined by the flag data. The proof relies on the idea to iterate formula (2) on chains of projective bundles (for types B and D , we assume that there exist isotropic subbundles of maximal possible dimension, *cf.* [6, 8] for a discussion on that assumption). This allows us to give a formula for full flag bundles. The case of a general bundle is then obtained similarly as in Damon's paper [4], without using Grassmann bundles.

To give an insight, we give three first examples. All unexplained notation can be found in Sections 1.1, 1.2, 2.1, 2.2, 3.1, 3.2. Let us however quickly mention that throughout this text, when we consider a certain flag bundle $F \rightarrow X$, the ξ_i are the Chern roots of (the dual of) the tautological subbundles on F ; the letter f denotes a polynomial in the indicated number of variables with coefficients in $A^\bullet X$ (recall that we identify $A^\bullet X$ with subrings of the Chow rings of flag bundles); and by tacit assumption $f(\xi_1, \dots, \xi_q) \in A^\bullet F$, *i.e.* the polynomial f has the appropriate symmetries. These examples are:

– for the Grassman bundle $\pi: \mathbf{F}(q)(E) \rightarrow X$ of a rank n vector bundle,

$$\pi_* f(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{(n-1)-(q-i)} \right] \left(f(t_1, \dots, t_q) \prod_{i=1}^q s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right);$$

– for the complete flag bundle $\pi: \mathbf{F}(E) \rightarrow X$ of a rank n vector bundle,

$$\pi_* f(\xi_1, \dots, \xi_{n-1}) = \left[\prod_{i=1}^{n-1} t_i^{(n-1)} \right] \left(f(t_1, \dots, t_{n-1}) \prod_{i=1}^{n-1} s_{1/t_i}(E) \prod_{1 \leq i < j \leq n-1} (t_j - t_i) \right);$$

– for the Lagrange Grassmann bundle $\pi: \mathbf{F}^\omega(n)(E) \rightarrow X$ of a rank $2n$ symplectic vector bundle (assuming here that the symplectic form ω has values in \mathcal{O}_X),

$$\pi_* f(\xi_1, \dots, \xi_n) = \left[\prod_{i=1}^n t_i^{(n-1+i)} \right] \left(f(t_1, \dots, t_n) \prod_{i=1}^n s_{1/t_i}(E) \prod_{1 \leq i < j \leq n-1} (t_j^2 - t_i^2) \right).$$

The reformulation (2) and the idea to iterate the obtained formula on chains (or “towers”) of projective bundles originally appeared in the paper [5] by the first author. The idea of generalizing this formula to flag bundles was signaled by Bérczi, and it became clear that this suggestion was relevant as we recovered a formula for type A of Ilori from [12]. After the first version of the paper was completed, Manivel informed us that, independently, in their recent paper “Degree

Formulae for Grassman Bundles, II", Kaji and Terasoma prove a similar formula to that of Illori in type A (in the particular case of full flag bundles).

There exist various approaches to push-forward formulas for flag bundles:

- using Grothendieck residues (Akyildiz–Carrell [1], Damon [4], Quillen [21]);
- using localization and residues at infinity (Bérczi–Szenes [2], Tu [22], Zielenkiewicz [24, 23]);
- using symmetrizing operators (Brion [3], the second author, *e.g.* [18, 19] and Ratajski [17]);
- using Schur functions and Grassmann extensions (Józefiak, Lascoux and the second author [13]; for supersymmetric functions see [18, 9]);
- using residues and Grassmann extensions (Kazarian [14, 15]), which leads to formulas showing similarities with ours.

Also, in the recent paper [20] by the second author, a deformation of a push-forward formula from [13] is shown for Hall–Littlewood polynomials.

In the present paper we use only elementary linear algebra of polynomials, whose coefficients yield the sought push-forwards. A use of Segre polynomials $s_x(E)$, specialized with $x = 1/t$, leads to remarkable compact expressions. Moreover, it allows us to generalize the formula for type A uniformly to other classical types B, C, D , for which, to the best of our knowledge, no general Gysin formula was known.

To give a first illustration of the usefulness of our formulas, in Section 4, we provide new determinantal formulas for the push-forward of monomials and of Schur classes. In a forthcoming paper, we will also apply our formulas in order to compute the fundamental classes of Schubert varieties in the four classical types.

1. Universal push-forward formulas for type A

We first consider type A .

1.1. Definition of partial flag bundles of type A . — Let $E \rightarrow X$ be a rank n vector bundle. Let $1 \leq q_1 < \dots < q_m \leq n - 1$ be a sequence of integers. We denote by $\pi: \mathbf{F}(q_1, \dots, q_m)(E) \rightarrow X$ the bundle of flags of subspaces of dimensions q_1, \dots, q_m in the fibres of E . On $\mathbf{F}(q_1, \dots, q_m)(E)$, there is a universal flag $U_{q_1} \subseteq \dots \subseteq U_{q_m}$ of subbundles of π^*E , where $\text{rank}(U_{q_k}) = q_k$ (the fiber of U_{q_k} over the point $(V_{q_1} \subseteq \dots \subseteq V_{q_m} \subset E_x)$, where $x \in X$, is equal to V_{q_k}). For a foundational account on flag bundles, see [11].

1.2. Step-by-step construction of full flag bundles. — In order to use the fundamental formula (2), we recall the construction of the flag bundles $\mathbf{F}(1, 2, \dots, q)(E)$ for $q = 1, 2, \dots, n - 1$ as the chains of projective bundles of lines [10] (see also [9, §2.2]). We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{n-1} \subsetneq V_n = E,$$

such that for each i , the dimension of V_i is i . For V_1 , we can take any line in E . It follows that $\mathbf{F}(1)(E) \simeq \mathbf{P}(E)$. Next, we consider $\mathbf{F}(1, 2)(E) \rightarrow \mathbf{F}(1)(E)$ above V_1 . In order to get a 2-dimensional subspace $V_1 \subsetneq V_2 \subsetneq E$, it suffices to pick one more line in $\mathbf{P}(E/V_1)$. We iterate this construction,

$$\begin{array}{ccc} \mathbf{F}(1, \dots, i, i+1)(E) & & \\ \downarrow & \text{(fiber = } \mathbf{P}(E/V_i)) & \\ \mathbf{F}(1, \dots, i)(E) & & \end{array}$$

picking one line in $\mathbf{P}(E/V_i)$ at each step, until E/V_{n-1} is 1-dimensional, so $\mathbf{P}(E/V_{n-1})$ is a point.

Globalizing this construction over X , we obtain a chain of projective bundles of lines

$$\mathbf{F}(E) := \mathbf{F}(1, \dots, n-1)(E) \rightarrow \mathbf{F}(1, \dots, n-2)(E) \rightarrow \dots \rightarrow \mathbf{F}(1, 2)(E) \rightarrow \mathbf{F}(1)(E) \rightarrow X,$$

which is the same as

$$\mathbf{F}(E) := \mathbf{P}(E/U_{n-1}) \rightarrow \mathbf{P}(E/U_{n-2}) \rightarrow \dots \rightarrow \mathbf{P}(E/U_1) \rightarrow \mathbf{P}(E) \rightarrow X.$$

In this paper, the flag bundles $\mathbf{F}(1, \dots, q)(E)$, for $q = 1, \dots, n-1$, are termed *full*, since these involve all first consecutive integers up to a certain q and we call *complete* flag bundle, denoted $\mathbf{F}(E)$, the full flag bundle if $q = n-1$. Note that this terminology may vary in the literature.

For $i = 1, \dots, n$, we set

$$\xi_i := -c_1(U_i/U_{i-1}).$$

We shall call ξ_1, \dots, ξ_n the roots of E (these are the Chern roots of the dual bundle of E). The i -th root generates the cohomology of the fiber of $\mathbf{F}(1, \dots, i)(E) \rightarrow \mathbf{F}(1, \dots, i-1)(E)$ in the above chain of projective bundles.

1.3. Universal push-forward formula for flag bundles. — We prove the following result.

Theorem 1.1. — *Let $E \rightarrow X$ be a vector bundle of rank n . Given any increasing sequence of integers $0 = q_0 < q_1 < \dots < q_m = q \leq n-1$, the following push-forward formula holds for the partial flag bundle $\pi: \mathbf{F}(q_1, \dots, q_m)(E) \rightarrow X$: for any rational equivalence class $f(\xi_1, \dots, \xi_q) \in A^*(\mathbf{F}(q_1, \dots, q_m)(E))$, one has*

$$\pi_* f(\xi_1, \dots, \xi_q) = [t_1^{e_1} \cdots t_q^{e_q}] \left(f(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right),$$

where for $q_{k-1} < i \leq q_k$, we denote $e_i = (n-1) - (q_k - i)$.

In the last part of this section, we also give a push-forward formula for $f(\xi_1, \dots, \xi_n)$.

Proof. — The idea of the proof is to iterate formula (2) on the above chain of projective bundles of lines.

We first consider full flags. The case of $\mathbf{F}(1)(E) = \mathbf{P}(E) \rightarrow X$ was already treated in the introduction. We now generalize formula (2) to the case of full flag bundles $\mathbf{F}(1, \dots, q)(E) \rightarrow X$, by induction on $q \geq 1$.

By construction, the flag bundle $\mathbf{F}(1, \dots, q)(E)$ is the total space of the projective bundle of lines of the tautological quotient bundle $E/U_{q-1} \rightarrow \mathbf{F}(1, \dots, q-1)(E)$. This quotient bundle having rank $(n-q+1)$, a plain application of the fundamental formula (2) yields

$$\int_{\mathbf{F}(1, \dots, q)(E)}^X f(\xi_1, \dots, \xi_q) = [t_q^{n-q}] \left(\int_{\mathbf{F}(1, \dots, q-1)(E)}^X f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(E/U_{q-1}) \right).$$

In order to use induction, we need to express the total Segre class $s(E/U_{q-1})$ in terms of the remaining roots ξ_1, \dots, ξ_{q-1} , which is easily done by use of the Whitney sum formula for the following relation in the Grothendieck group of $\mathbf{F}(1, \dots, q-1)(E)$

$$[E/U_{q-1}] = [E] - \sum_{i=1}^{q-1} [U_i/U_{i-1}],$$

which yields

$$s(E/U_{q-1}) = s(E) \prod_{i=1}^{q-1} (1 - \xi_i).$$

The reformulation of this formula in terms of generating series is

$$s_{1/t_q}(E/U_{q-1}) = s_{1/t_q}(E) \prod_{i=1}^{q-1} \frac{(t_q - \xi_i)}{t_q}.$$

Hence the above formula becomes

$$\int_{\mathbf{F}(1, \dots, q)(E)}^X f(\xi_1, \dots, \xi_{q-1}, \xi_q) = [t_q^{n-q}] \left(\int_{\mathbf{F}(1, \dots, q-1)(E)}^X f(\xi_1, \dots, \xi_{q-1}, t_q) \frac{1}{t_q^{q-1}} s_{1/t_q}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right),$$

which it is good to reshape by multiplication of both the extracted monomial and the series by the shifting monomial t_q^{q-1}

$$\int_{\mathbf{F}(1, \dots, q)(E)}^X f(\xi_1, \dots, \xi_{q-1}, \xi_q) = [t_q^{n-1}] \left(\int_{\mathbf{F}(1, \dots, q-1)(E)}^X f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(E) \prod_{1 \leq i < j \leq q} (t_j - \xi_i) \right).$$

After the first step, we have to integrate the polynomial in the variables ξ_1, \dots, ξ_{q-1}

$$f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(E) \prod_{1 \leq i < q} (t_q - \xi_i)$$

along the fiber of the projective bundle of lines $\mathbf{F}(1, \dots, q-1)(E) \rightarrow \mathbf{F}(1, \dots, q-2)(E)$.

Iterating the same reasoning until we have replaced all roots ξ_i by some formal variables t_i , we obtain the announced expression

$$\int_{\mathbf{F}(1, \dots, q)(E)}^X f(\xi_1, \dots, \xi_q) = [t_1^{q-1} \dots t_q^{q-1}] \left(f(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right).$$

A useful particular case. — In the particular case of the complete flag bundle $\pi: \mathbf{F}(E) \rightarrow X$ of a rank r vector bundle $E \rightarrow X$ on a variety X , and for a polynomial $g \in \mathbb{Z}[t_1, \dots, t_{r-1}]$ having degree $r(r-1)/2$ the formula reads

$$\int_{\mathbf{F}(E)}^X g(\xi_1, \dots, \xi_{r-1}) = [t_1^{r-1} \dots t_{r-1}^{r-1}] \left(g(t_1, \dots, t_{r-1}) \prod_{1 \leq i \leq r-1} s_{1/t_i}(E) \prod_{1 \leq i < j \leq r-1} (t_j - t_i) \right).$$

We have chosen the degree of g so that the homogeneous degree of the polynomial

$$g(t_1, \dots, t_{r-1}) \prod_{1 \leq i < j \leq r-1} (t_j - t_i)$$

is exactly the homogeneous degree of the extracted monomial. As a consequence, only the constant term of the second factor $\prod s_{1/t_i}(E)$ can contribute. This coefficient is $1 = [X] \in A^0 X$, whence

$$\int_{\mathbf{F}(E)}^X g(\xi_1, \dots, \xi_{r-1}) = [t_1^{r-1} \dots t_{r-1}^{r-1}] \left(g(t_1, \dots, t_{r-1}) \prod_{1 \leq i < j \leq r-1} (t_j - t_i) \right) [X] \in A^0 X \simeq \mathbb{Z}.$$

One even more particular case will prove to be useful. Take

$$g(t_1, \dots, t_{r-1}) := t_{r-1}^1 t_{r-2}^2 \dots t_1^{r-1},$$

then the associated push-forward formula for the flag bundle $\mathbf{F}(E) \rightarrow X$ is

$$(3) \quad \int_{\mathbf{F}(E)}^X g(\xi_1, \dots, \xi_{r-1}) = [t_1^{r-1} \dots t_{r-1}^{r-1}] \left(t_{r-1}^1 t_{r-2}^2 \dots t_1^{r-1} \prod_{1 \leq i < j \leq r-1} (t_j - t_i) \right) [X] = [X].$$

The proof of this formula is a easy induction on r . We only sketch the arguments. Shifting both the extracted monomial and the series by the monomial t_1^{r-1} yields

$$\begin{aligned} [t_1^{r-1} \dots t_{r-1}^{r-1}] (t_1^{r-1} \dots t_{r-1}^1 \prod_{1 \leq i < j \leq r-1} (t_j - t_i)) &= [t_1^0 t_2^{r-1} \dots t_{r-1}^{r-1}] (t_2^{r-2} \dots t_{r-1}^1 \prod_{1 \leq i < j \leq r-1} (t_j - t_i)) \\ &= [t_2^{r-1} \dots t_{r-1}^{r-1}] (t_2^{r-2} \dots t_{r-1}^1 \prod_{2 \leq j \leq r-1} t_j \prod_{2 \leq i < j \leq r-1} (t_j - t_i)) \\ &= [t_2^{r-2} \dots t_{r-1}^{r-2}] (t_2^{r-2} \dots t_{r-1}^1 \prod_{2 \leq i < j \leq r-1} (t_j - t_i)), \end{aligned}$$

which is the adequate induction formula. And for $r = 2$, the last expression is equal to 1.

From full flags to partial flags. — We follow here [4]. Let $0 = q_0 < q_1 < \dots < q_m = q$ be an increasing sequence of integers. Recall that on $F := \mathbf{F}(q_1, \dots, q_m)(E)$, there is the universal flag of vector bundles

$$0 \subsetneq U_{q_1} \subsetneq \dots \subsetneq U_{q_m} \subsetneq E,$$

where $\text{rank}(U_{q_i}) = q_i$. The fiber product

$$\mathbf{Y} := \mathbf{F}(U_{q_1}) \times_F \mathbf{F}(U_{q_2}/U_{q_1}) \times_F \dots \times_F \mathbf{F}(U_{q_m}/U_{q_{m-1}})$$

is isomorphic to $\mathbf{F}(1, \dots, q)(E)$ with the natural projection map $\mathbf{F}(1, \dots, q)(E) \rightarrow F$, and we get a commutative diagram

$$(4) \quad \begin{array}{ccc} \mathbf{F}(1, \dots, q)(E) & \xrightarrow[\simeq]{\theta} & \mathbf{Y} \\ \pi'' \downarrow & & \downarrow \pi' \\ X & \xleftarrow{\pi} & F \end{array}$$

The fiber of π' over the point $(V_{q_1} \subseteq V_{q_2} \subseteq \dots \subseteq V_{q_m} \subseteq E_x) \in F$ is the product of complete flag varieties

$$\mathbf{F}(V_{q_1}) \times \mathbf{F}(V_{q_2}/V_{q_1}) \times \dots \times \mathbf{F}(V_{q_m}/V_{q_{m-1}}).$$

Applying the splitting principle (cf. [10]) to each graded piece $(U_{q_k}/U_{q_{k-1}})_k$ of the “universal” filtration, we infer that the pullbacks $\theta^* \eta_{q_{k-1}+1}, \dots, \theta^* \eta_{q_k}$ of the roots of $U_{q_k}/U_{q_{k-1}}$ are the corresponding roots $\xi_{q_{k-1}+1}, \dots, \xi_{q_k}$ of E .

Our goal is to compute the push-forward $\pi_* = \int_F^X$. Since π'_* is surjective, it is enough to understand $\pi_* \circ \pi'_*$, which is equal to $\pi''_* \circ (\theta^{-1})_*$. But from the full flag case, we know π'_* and also π''_* , so we can proceed.

We are now in position to prove the generalization of (2) to the case of partial flag bundles. Let

$$g(t_1, \dots, t_q) := \prod_{k=1}^m \left(t_{q_k-1}^1 t_{q_k-2}^2 \dots t_{q_{k-1}+1}^{q_k-(q_{k-1}+1)} \right).$$

It follows from (3) that $\pi'_* g(\eta_1, \dots, \eta_q) = [X]$, whence, by the projection formula for π' , we have

$$\pi'_* \left(f(\xi_1, \dots, \xi_q) g(\eta_1, \dots, \eta_q) \right) = f(\xi_1, \dots, \xi_q).$$

This implies

$$\pi_* f(\xi_1, \dots, \xi_q) = \pi_* \pi'_* \left(f(\xi_1, \dots, \xi_q) g(\eta_1, \dots, \eta_q) \right).$$

Now, from the commutativity of (4), we get

$$\begin{aligned} \pi_* \pi'_* \left(f(\xi_1, \dots, \xi_q) g(\eta_1, \dots, \eta_q) \right) &= \pi''_* (\theta^{-1})_* \left(f(\xi_1, \dots, \xi_q) g(\eta_1, \dots, \eta_q) \right) \\ &= \pi''_* \left(f(\xi_1, \dots, \xi_q) g(\xi_1, \dots, \xi_q) \right). \end{aligned}$$

This last expression, using the formula for full flag bundles, is equal to

$$[t_1^{n-1} \dots t_q^{n-1}] \left(f(t_1, \dots, t_q) g(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right).$$

In order to get the announced formula, it suffices to shift both the series and the extracted monomial by the monomial $g(t_1, \dots, t_q)$. Since

$$\frac{t_1^{n-1} \dots t_q^{n-1}}{g(t_1, \dots, t_q)} = \prod_{k=1}^m \left(t_{q_k}^{(n-1)} t_{q_k-1}^{(n-1)-1} t_{q_k-2}^{(n-1)-2} \dots t_{q_{k-1}+1}^{(n-1)-(q_k-(q_{k-1}+1))} \right) = \prod_{i=1}^q t_i^{e_i},$$

with $e_i = (n-1) - (q_k - i)$ for $q_{k-1} < i \leq q_k$, this concludes the proof. \square

As noticed by Ilori [12, pp. 629–630], it can be useful to have a formula with ξ_1, \dots, ξ_n , and not only ξ_1, \dots, ξ_q .

Proposition 1.2. — *With the above notation, one has*

$$\int_{\mathbf{F}(q_1, \dots, q_m)(E)} f(\xi_1, \dots, \xi_n) = [t_1^{e_1} \dots t_n^{e_n}] \left(f(t_1, \dots, t_n) \prod_{1 \leq i \leq n} s_{1/t_i}(E) \prod_{1 \leq i < j \leq n} (t_j - t_i) \right),$$

where for $q_{k-1} < i \leq q_k$, we denote $e_i := (n-1) - (q_k - i)$. Here we take $q_{m+1} = n$.

Proof. — Firstly, we treat the complete flag bundle $\mathbf{F}(E) \rightarrow X$. In the Grothendieck group of $\mathbf{F}(E)$,

$$[E/U_{n-1}] = [E] - \sum_{i=1}^{n-1} [U_i/U_{i-1}],$$

which yields

$$s(E/U_{n-1}) = \sum_{j \geq 0} \xi_n^j = s(E) \prod_{i=1}^{n-1} (1 - \xi_i).$$

As a consequence (we skip the detail)

$$\xi_n^j = [t_n^{n-1}] \left(t_n^j s_{1/t_n}(E) \prod_{1 \leq i < n} (t_n - \xi_i) \right).$$

Now the application of the formula already proved to the polynomial in $n - 1$ variables

$$f(\xi_1, \dots, \xi_{n-1}, \xi_n) = [t_n^{n-1}] \left(f(\xi_1, \dots, \xi_{n-1}, t_n) s_{1/t_n}(E) \prod_{1 \leq i < n} (t_n - \xi_i) \right)$$

yields a similar formula with one more formal variable

$$\int_{\mathbf{F}(E)}^X f(\xi_1, \dots, \xi_n) = [t_1^{n-1} \cdots t_n^{n-1}] \left(f(t_1, \dots, t_n) \prod_{1 \leq i \leq n} s_{1/t_i}(E) \prod_{1 \leq i < j \leq n} (t_j - t_i) \right).$$

To get the formula for partial flag bundles, we adopt the same strategy as above, but consider the projection $\mathbf{F}(E) \rightarrow \mathbf{F}(q_1, \dots, q_m)(E)$. The fiber over a point $(V_{q_1} \subsetneq V_{q_2} \subsetneq \cdots \subsetneq V_{q_m} \subseteq E_x)$ is the product of complete flag varieties

$$\mathbf{F}(V_{q_1}) \times \mathbf{F}(V_{q_2}/V_{q_1}) \times \cdots \times \mathbf{F}(E_x/V_{q_m}).$$

One infers the stated formula, with shifted exponents. \square

2. Universal push-forward formulas for symplectic flag bundles

Let now us deal with the symplectic setting. It can be regarded as the most simple case after the A case, as any line is isotropic for the symplectic form. This last fact will no longer be true in orthogonal setting.

2.1. Definition of partial isotropic flag bundles of type C. — Let $E \rightarrow X$ be a rank $2n$ vector bundle equipped with a non-degenerate symplectic form $\omega: E \otimes E \rightarrow L$ (with values in a certain line bundle $L \rightarrow X$). We say that a subbundle S of E is isotropic if S is a subbundle of its symplectic complement S^ω , where

$$S^\omega := \{w \in E \mid \forall v \in S: \omega(w, v) = 0\}.$$

Let $1 \leq q_1 < \cdots < q_m \leq n$ be a sequence of integers. We denote by $\pi: \mathbf{F}^\omega(q_1, \dots, q_m)(E) \rightarrow X$ the bundle of flags of isotropic subspaces of dimensions q_1, \dots, q_m in the fibers of E . On $\mathbf{F}^\omega(q_1, \dots, q_m)(E)$, there is a universal flag $U_{q_1} \subsetneq \cdots \subsetneq U_{q_m}$ of subbundles of π^*E , where $\text{rank}(U_{q_k}) = q_k$.

2.2. Step-by-step construction of isotropic full flag bundles. — Let us recall the construction of $\mathbf{F}^\omega(1, 2, \dots, q)(E)$ for $q = 1, 2, \dots, n$ as chains of projective bundles of lines [9, §6.1]. We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subsetneq E,$$

of E such that for each i , the dimension of V_i is i and V_i is isotropic. In particular, $V_n = V_n^\omega$ is a maximal isotropic subspace of (E, ω) . For V_1 , we can take any line in E , since ω is skew-symmetric. It follows that $\mathbf{F}^\omega(1)(E) \simeq \mathbf{P}(E)$. Next, we consider $\mathbf{F}^\omega(1, 2)(E) \rightarrow \mathbf{F}^\omega(1)(E)$ above V_1 . In order to

get an isotropic subspace $V_1 \subseteq V_2 \subseteq V_1^\omega$, it suffices to pick one more line in $\mathbf{P}(V_1^\omega/V_1)$. Iterating this construction,

$$\begin{array}{c} \mathbf{F}^\omega(1, \dots, i, i+1)(E) \\ \downarrow \quad (\text{fiber} = \mathbf{P}(V_i^\omega/V_i)), \\ \mathbf{F}^\omega(1, \dots, i)(E) \end{array}$$

picking one line in $\mathbf{P}(V_i^\omega/V_i)$ at each step, one ends up with V_n^ω/V_n , which is zero dimensional, thus obtaining a maximal isotropic subspace V_n .

Globalizing this construction over X , we obtain a chain of projective bundles of lines

$$\mathbf{F}^\omega(1, \dots, n)(E) \rightarrow \mathbf{F}^\omega(1, \dots, n-1)(E) \rightarrow \dots \rightarrow \mathbf{F}^\omega(1, 2)(E) \rightarrow \mathbf{F}^\omega(1)(E) \rightarrow X,$$

which is the same as

$$\mathbf{P}(U_{n-1}^\omega/U_{n-1}) \rightarrow \mathbf{P}(U_{n-2}^\omega/U_{n-2}) \rightarrow \dots \rightarrow \mathbf{P}(U_1^\omega/U_1) \rightarrow \mathbf{P}(E) \rightarrow X.$$

For $i = 1, \dots, n$ we set

$$\xi_i = -c_1(U_i/U_{i-1}).$$

2.3. Useful relations in the Grothendieck group. — Since ω is everywhere non-degenerate, one can consider the isomorphism $\iota_\omega: v \in E \mapsto \omega(v, \cdot) \in \text{Hom}(E, L)$. Restricting the map $\omega(v, \cdot)$ to U_1 , one obtains an isomorphism

$$\iota_\omega: v \in E/U_1^\omega \mapsto \omega(v, \cdot) \in \text{Hom}(U_1, L),$$

which yields the relation in the Grothendieck group of $\mathbf{F}^\omega(1)(E)$

$$[U_1^\omega] = [E] - [U_1^\vee \otimes L].$$

It follows that

$$[U_1^\omega/U_1] = [E] - [U_1] - [U_1^\vee \otimes L].$$

For $i = 2, \dots, n$, doing all the same reasoning on U_{i-1}^ω instead of E , one obtains, in the Grothendieck group of $\mathbf{F}^\omega(1, \dots, i)(E)$

$$[U_i^\omega/U_i] = [U_{i-1}^\omega/U_{i-1}] - [U_i/U_{i-1}] - [(U_i/U_{i-1})^\vee \otimes L].$$

Note that this formula also applies to $i = 1$ with the natural convention $U_0 = 0$. An induction yields

$$(5) \quad [U_j^\omega/U_j] = [E] - \sum_{i=1}^j ([U_i/U_{i-1}] + [(U_i/U_{i-1})^\vee \otimes L]).$$

2.4. Universal push-forward formula for the symplectic case. — We prove the following result.

Theorem 2.1. — *Let $(E, \omega) \rightarrow X$ be a vector bundle having even rank $2n$, equipped with a everywhere non-degenerate symplectic form $\omega: E \otimes E \rightarrow L$, for a certain line bundle $L \rightarrow X$.*

Given any increasing sequence of integers $0 = q_0 < q_1 < \dots < q_m = q \leq n$, the following push-forward formula holds for the isotropic partial flag bundle $\pi: \mathbf{F}^\omega(q_1, \dots, q_m)(E) \rightarrow X$: for any rational equivalence class $f(\xi_1, \dots, \xi_q) \in A^\bullet(\mathbf{F}^\omega(q_1, \dots, q_m)(E))$, one has

$$\pi_* f(\xi_1, \dots, \xi_q) = \left[t_1^{e_1} \cdots t_q^{e_q} \left(f(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i)(t_j + t_i + c_1(L)) \right) \right],$$

where for $q_{k-1} < i \leq q_k$, we denote $e_i = (2n - 1) - (q_k - i)$.

Proof. — We prove the formula for type C in the very same way as we already did for type A.

We first consider full flags. We will prove the formula for $\mathbf{F}^\omega(1, \dots, q)(E) \rightarrow X$ by induction on $q = 1, \dots, n$. Note that for $q = 1$ the sought formula is exactly the same as the fundamental formula

(2). Thus the result holds. Then for $1 < q \leq n$, we consider the projection $\mathbf{F}^\omega(1, \dots, q)(E) \rightarrow \mathbf{F}^\omega(1, \dots, q-1)(E)$. Using the formula (2) one states

$$\int_{\mathbf{F}^\omega(1, \dots, q-1)(E)}^{f(\xi_1, \dots, \xi_q)} = [t_q^{2(n-q)+1}] (f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(U_{q-1}^\omega/U_{q-1})).$$

In order to use induction, it remains to express the total Segre class $s(U_{q-1}^\omega/U_{q-1})$ as a polynomial in ξ_1, \dots, ξ_{q-1} . In that aim, we use the relation (5) in the Grothendieck group of $\mathbf{F}(1, \dots, q)(E)$

$$[U_{q-1}^\omega/U_{q-1}] = [E] - \sum_{i=1}^{q-1} ([U_i/U_{i-1}] + [(U_i/U_{i-1})^\vee \otimes L]),$$

which yields

$$s(U_{q-1}^\omega/U_{q-1}) = s(E) \prod_{i=1}^{q-1} (1 - \xi_i)(1 + \xi_i + c_1(L)).$$

Here, it might be useful to recall our convention $s(E) = 1/c(E)$ and the notation $\xi_i = -c_1(U_i/U_{i-1})$. Using this expression of the Segre class in the above formula, one gets

$$\begin{aligned} \int_{\mathbf{F}^\omega(1, \dots, q-1)(E)}^{f(\xi_1, \dots, \xi_q)} &= [t_q^{2(n-q)+1}] \left(f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(E) \prod_{i=1}^{q-1} \frac{(t_q - \xi_i)(t_q + \xi_i + c_1(L))}{t_q} \right) \\ &= [t_q^{2n-1}] \left(f(\xi_1, \dots, \xi_{q-1}, t_q) s_{1/t_q}(E) \prod_{i=1}^{q-1} (t_q - \xi_i)(t_q + \xi_i + c_1(L)) \right). \end{aligned}$$

One infers, using induction, that

$$\int_{\mathbf{F}^\omega(1, \dots, q)(E)}^{f(\xi_1, \dots, \xi_q)} = [t_1^{2n-1} \dots t_q^{2n-1}] (f(t_1, \dots, t_q) \prod_{1 \leq j \leq q} s_{1/t_j}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i)(t_j + t_i + c_1(L))).$$

From full flags to partial flags. — Clearly, any flag inside an isotropic subbundle is an isotropic flag. Let $0 = q_0 < q_1 < \dots < q_m = q$ be an increasing sequence of integers. Recall that on $F := \mathbf{F}^\omega(q_1, \dots, q_m)(E)$, there is the universal flag of vector bundles

$$0 \subsetneq U_{q_1} \subsetneq \dots \subsetneq U_{q_m} \subsetneq E$$

where $\text{rank}(U_{q_k}) = q_k$. The fiber product

$$\mathbf{Y} := \mathbf{F}(U_{q_1}) \times_F \mathbf{F}(U_{q_2}/U_{q_1}) \times_F \dots \times_F \mathbf{F}(U_{q_m}/U_{q_{m-1}})$$

is isomorphic to $\mathbf{F}^\omega(1, \dots, q)(E)$ with the natural projection map $\mathbf{F}^\omega(1, \dots, q)(E) \rightarrow F$, and we get a commutative diagram

$$\begin{array}{ccc} \mathbf{F}^\omega(1, \dots, q)(E) & \xrightarrow[\simeq]{\theta} & \mathbf{Y} \\ \pi'' \downarrow & & \downarrow \pi' \\ X & \xleftarrow{\pi} & F \end{array}$$

The arguments developed in case *A* still apply, resulting in the same shift of the extracted monomial in the general formula. This concludes the proof. \square

3. Universal push-forward formulas for orthogonal flag bundles

We will need only few modifications to deal with the orthogonal setting. We will do almost the same reasoning, replacing the projective bundle of lines $\mathbf{P}(E)$ by the quadric bundle of isotropic lines $\mathbf{Q}(E)$.

3.1. Definition of partial isotropic flag bundles of types B and D. — Let $E \rightarrow X$ be a vector bundle of rank $2n$ or $2n + 1$ equipped with a non-degenerate orthogonal form $Q: E \otimes E \rightarrow L$ (with values in a certain line bundle $L \rightarrow X$). We say that a subbundle S of E is isotropic if S is a subbundle of its orthogonal complement S^\perp , where

$$S^\perp := \{w \in E \mid \forall v \in S: Q(w, v) = 0\}.$$

In particular, if the dimension of E is even, $S_n = S_n^\perp$.

Let $1 \leq q_1 < \dots < q_m \leq n$ be a sequence of integers. We denote by $\pi: \mathbf{F}^Q(q_1, \dots, q_m)(E) \rightarrow X$ the bundle of flags of isotropic subspaces of dimensions q_1, \dots, q_m in the fibers of E . On $\mathbf{F}^Q(q_1, \dots, q_m)(E)$, there is a universal flag $U_{q_1} \subsetneq \dots \subsetneq U_{q_m}$ of subbundles of π^*E , where $\text{rank}(U_{q_k}) = q_k$.

3.2. Step-by-step construction of isotropic full flag bundles. — Let us recall the construction of $\mathbf{F}^Q(1, 2, \dots, q)(E)$ for $q = 1, 2, \dots, n$ as chains of quadric bundles of isotropic lines [6, §6]. We proceed over $x \in X$ and write $E = E_x$. Consider a flag of subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n \subsetneq E$$

such that for each i , the rank of V_i is i and V_i is isotropic. In particular, if the dimension of E is even, $V_n = V_n^\perp$. For V_1 , we can take any isotropic line in E . Thus, by definition $\mathbf{F}^Q(1)(E) \simeq \mathbf{Q}(E)$, the codimension one subbundle of $\mathbf{P}(E)$ cut out by Q . Next, we consider $\mathbf{F}^Q(1, 2)(E) \rightarrow \mathbf{F}^Q(1)(E)$ above V_1 . In order to get an isotropic subspace $V_1 \subseteq V_2 \subseteq V_1^\perp$, it suffices to pick one more line in $\mathbf{Q}(V_1^\perp/V_1)$. Iterating this construction,

$$\begin{array}{c} \mathbf{F}^Q(1, \dots, i, i+1)(E) \\ \downarrow \quad (\text{fiber} = \mathbf{Q}(V_i^\perp/V_i)) \\ \mathbf{F}^Q(1, \dots, i)(E) \end{array}$$

picking one isotropic line in $\mathbf{Q}(V_i^\perp/V_i)$ at each step, one ends up with V_n^\perp/V_n , which is either zero dimensional or one dimensional. In the first case, when the rank of E is even, it is usual to take only one of the two connected components, but we will not, as we do not want to treat this case separately.

Globalizing this construction over X , we obtain a chain of quadric bundles of isotropic lines

$$\mathbf{F}^Q(1, \dots, n)(E) \rightarrow \mathbf{F}^Q(1, \dots, n-1)(E) \rightarrow \dots \rightarrow \mathbf{F}^Q(1, 2)(E) \rightarrow \mathbf{F}^Q(1)(E) \rightarrow X,$$

which is the same as

$$\mathbf{Q}(U_{n-1}^\perp/U_{n-1}) \rightarrow \mathbf{Q}(U_{n-2}^\perp/U_{n-2}) \rightarrow \dots \rightarrow \mathbf{Q}(U_1^\perp/U_1) \rightarrow \mathbf{Q}(E) \rightarrow X.$$

For $i = 1, \dots, n$ we set

$$\xi_i := -c_1(U_i/U_{i-1}).$$

3.3. A push-forward formula for the quadric bundle. — The quadric bundle $\iota: \mathbf{Q}(E) \hookrightarrow \mathbf{P}(E)$ of isotropic lines of E , cut out by the quadratic form $Q: E \otimes E \rightarrow L$, is the zero set of a section of the line bundle

$$\text{Hom}(\mathcal{O}_{\mathbf{P}(E)}(-1) \otimes \mathcal{O}_{\mathbf{P}(E)}(-1), L) \simeq \mathcal{O}_{\mathbf{P}(E)}(2) \otimes L.$$

Its fundamental class in $\mathbf{P}(E)$ is thus $[\mathbf{Q}(E)] = 2\xi + c_1(L)$. Let $\pi: \mathbf{P}(E) \rightarrow X$ be the natural projection, and $\rho = \pi \circ \iota$. Let $\tilde{\xi} := \iota^* \xi$ denote the restriction of $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$. One has

$$\rho_* (\tilde{\xi}^i \rho^* \alpha) = \pi_* ([\mathbf{Q}(E)] \cdot \xi^i \pi^* \alpha) = \pi_* ((2\xi + c_1(L)) \xi^i \pi^* \alpha) = (2s_{i-r+2}(E) + c_1(L) s_{i-r+1}(E)) \alpha,$$

where $r = \text{rank}(E)$. Similarly as we get (2) from (1), we now infer

$$(6) \quad \rho_* f(\tilde{\xi}) = [t^{r-1}] (f(t)(2t + c_1(L)) s_{1/t}(E)).$$

Since we are working with towers of quadric bundles in the orthogonal case, this formula will play the analog role as the fundamental formula (2) in this section.

3.4. Universal push-forward formula for orthogonal flag bundles. — We prove the following result.

Theorem 3.1. — *Let $(E, Q) \rightarrow X$ be a vector bundle of rank $2n$ or $2n + 1$, equipped with a everywhere non-degenerate quadratic form $Q: E \otimes E \rightarrow L$, for a certain line bundle $L \rightarrow X$.*

Given any increasing sequence of integers $0 = q_0 < q_1 < \dots < q_m = q \leq n$, the following push-forward formula holds for the isotropic partial flag bundle $\pi: \mathbf{F}^Q(q_1, \dots, q_m)(E) \rightarrow X$: for any rational equivalence class $f(\xi_1, \dots, \xi_q) \in A^\bullet(\mathbf{F}^Q(q_1, \dots, q_m)(E))$, one has

$$\pi_* f(\xi_1, \dots, \xi_q) = \left[t_1^{e_1} \cdots t_q^{e_q} \right] \left(f(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i \leq q} (2t_i + c_1(L)) \prod_{1 \leq i < j \leq q} (t_j - t_i)(t_j + t_i + c_1(L)) \right),$$

where for $q_{k-1} < i \leq q_k$, we denote $e_i = (\text{rank}(E) - 1) - (q_k - i)$.

Note that, if the rank is $2n$ and $q = n$, we consider *both* of the two isomorphic connected components of the flag bundle. Thus, if one is interested in only one of the two components, the result should be divided by 2. When $c_1(L) = 0$, this makes appear the usual coefficient 2^{n-1} .

Proof. — We first prove the formula for full flags. Since the quadratic form Q is everywhere non-degenerate, one can consider the isomorphism $\iota_Q: v \in E \mapsto Q(v, \cdot) \in \text{Hom}(E, L)$. As in the symplectic case, this isomorphism yields the relation in the Grothendieck group of $\mathbf{F}^Q(1, \dots, j)(E)$

$$[U_j^\perp/U_j] = [E] - \sum_{i=1}^j ([U_i/U_{i-1}] + [(U_i/U_{i-1})^\vee \otimes L]).$$

The proof goes in the very same way as for type C, replacing the fundamental formula for $\mathbf{P}(E)$ by this for $\mathbf{Q}(E)$. So we skip the details.

Then, to go from full flags to partial flags, the argument is the same as in the symplectic case. Clearly, any flag inside an isotropic subbundle is an isotropic flag. It yields the expected formulas with shifted exponents. \square

Note that in the basic case where the quadratic form Q takes values in the trivial line bundle $L = \mathcal{O}_X$ the theorem has a simpler form, and reads as follows.

Corollary 3.2. — *Let $(E, Q) \rightarrow X$ be a vector bundle of rank $2n$ or $2n + 1$, equipped with a everywhere non-degenerate quadratic form $Q: E \otimes E \rightarrow \mathcal{O}_X$.*

Given any increasing sequence of integers $0 = q_0 < q_1 < \dots < q_m = q \leq n$, the following push-forward formula holds for the isotropic partial flag bundle $\pi: \mathbf{F}^Q(q_1, \dots, q_m)(E) \rightarrow X$: for any rational equivalence class $f(\xi_1, \dots, \xi_q) \in A^\bullet(\mathbf{F}^Q(q_1, \dots, q_m)(E))$, one has

$$\pi_* f(\xi_1, \dots, \xi_q) = 2^q \left[t_1^{e_1} \cdots t_q^{e_q} \right] \left(f(t_1, \dots, t_q) \prod_{1 \leq i \leq q} s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j^2 - t_i^2) \right),$$

where for $q_{k-1} < i \leq q_k$, we denote $e_i = (\text{rank}(E) - 2) - (q_k - i)$.

4. Applications: new determinantal formulas

To finish, we would like to compute new determinantal formulas, in types A, B, C, D , in order to illustrate the usefulness and the efficiency of our approach.

We shall use the following linearity result, whose proof is left to the reader.

Lemma 4.1. — (i) *For any $f_{ij} \in A^\bullet X[t_i]$ where $1 \leq i, j \leq q$, and for any exponents $e_1, \dots, e_q \in \mathbb{N}$*

$$\left[\prod_{1 \leq i \leq q} t_i^{e_i} \right] \det(f_{ij}) = \det([t_i^{e_i}] f_{ij}).$$

(ii) *For any $f_{ij} \in A^\bullet X[t_j]$ where $1 \leq i, j \leq q$, and for any exponents $e_1, \dots, e_q \in \mathbb{N}$*

$$\left[\prod_{1 \leq j \leq q} t_j^{e_j} \right] \det(f_{ij}) = \det([t_j^{e_j}] f_{ij}).$$

4.1. Schur functions. — For any partition $\lambda = (\lambda_1, \dots, \lambda_q)$, recall ([16, 7]) that the Schur polynomial $s_\lambda \in \mathbb{Z}[t_1, \dots, t_q]$ can be defined by the formula:

$$(7) \quad s_\lambda(t_1, \dots, t_q) := \frac{\det(t_{q-j+1}^{\lambda_i-i+q})_{1 \leq i, j \leq q}}{\prod_{1 \leq i < j \leq q} (t_j - t_i)}.$$

Note that in particular for $\lambda = (i)$, one has $s_i = h_i$ the complete symmetric function of degree i , and accordingly $s_i(E) = s_i(\xi_1, \dots, \xi_q)$. Then the Jacobi–Trudi identity states

$$s_\lambda(t_1, \dots, t_q) = \det(s_{\lambda_i-i+j}(t_1, \dots, t_q))_{1 \leq i, j \leq q},$$

where $s_i = 0$ for $i < 0$. This identity allows one to generalize the Segre classes of E for any $I \in \mathbb{Z}^q$ in a natural way, by setting

$$(8) \quad s_I(E) := \det(s_{I_i+(j-i)}(E))_{1 \leq i, j \leq q}.$$

Note that there is a dichotomy for sequences $I \in \mathbb{Z}^q$:

- either for all $i = 1, \dots, q$, one has $I_i \geq i - q$, then (7) holds for s_I ;
- or for some $i \in 1, \dots, q$, one has $I_i < i - q$, then for $j = 1, \dots, q$ one has $s_{I_i-i+j}(E) = 0$ and accordingly $s_I(E) = 0$.

4.2. Formulas using monomials. — These Schur functions naturally appear in type A when one uses the additive basis of monomials, as illustrated below.

Proposition 4.2. — *Let $E \rightarrow X$ be a rank n vector bundle. Consider the partial flag bundle $\pi: \mathbf{F}(q_1, \dots, q_m)(E) \rightarrow X$, and a rational equivalence class*

$$f(\xi_1, \dots, \xi_q) := \sum \alpha_I \xi_1^{I_1} \cdots \xi_q^{I_q} \in A^\bullet(\mathbf{F}(q_1, \dots, q_m)(E)).$$

One has the Gysin formula:

$$\pi_* f(\xi_1, \dots, \xi_q) = \sum \alpha_I s_{I-\rho}(E),$$

where for $q_{k-1} < i \leq q_k$ the i th part of ρ is

$$\rho_i := n - q_k.$$

Proof. — The pushforward formula for type A yields

$$\pi_* f(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{e_i} \right] \left(\sum \alpha_I t_1^{I_1} \cdots t_q^{I_q} \prod_{i=1}^q s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right),$$

where $e_i := n - 1 - q_k + i$, for $q_{k-1} < i \leq q_k$.

Now, it is well known (and due to Vandermonde) that $\prod_{1 \leq i < j \leq q} (t_j - t_i) = \det(t_i^{j-1})_{1 \leq i, j \leq q}$, and then by Lemma 4.1(i)

$$\pi_* f(\xi_1, \dots, \xi_q) = \sum \alpha_I \det([t_i^{e_i}](t_i^{j-1+I_i} s_{1/t_i}(E)))_{1 \leq i, j \leq q} = \sum \alpha_I \det(s_{I_i-e_i+j-1}(E))_{1 \leq i, j \leq q},$$

which is exactly the announced expression, since for $q_{k-1} < i \leq q_k$

$$I_i - e_i + j - 1 = I_i - (n - q_k) + (j - i). \quad \square$$

In the symplectic case and in the orthogonal case, we will now assume that $L = \mathcal{O}_X$ is the trivial bundle. In this case

$$\prod_{1 \leq i < j \leq q} (t_j - t_i)(t_j + t_i + c_1(L)) = \prod_{1 \leq i < j \leq q} (t_j^2 - t_i^2) = \det(t_i^{2(j-1)})_{1 \leq i, j \leq q}.$$

Hence, the proof of Proposition 4.2 is easily adapted in order to obtain determinantal formulas in types B, C, D . We first introduce the classes

$$(9) \quad s_I^{(2)}(E) := \det(s_{I_i+2(j-i)}(E))_{1 \leq i, j \leq q},$$

for all $I \in \mathbb{Z}^d$. It seems adequate to call them *quadratic Schur functions* (compare with (8)). They are closely related to the classes $s_\lambda^{[2]}(E)$ defined in [17], as it will soon appear (see Section 4.3).

Proposition 4.3. — Let $E \rightarrow X$ be a symplectic vector bundle of rank $2n$, equipped with the symplectic form $\omega: E \otimes E \rightarrow \mathcal{O}_X$. Consider the isotropic partial flag bundle $\pi: \mathbf{F}^\omega(q_1, \dots, q_m)(E) \rightarrow X$ and a rational equivalence class

$$f(\xi_1, \dots, \xi_q) := \sum \alpha_I \xi_1^{I_1} \cdots \xi_q^{I_q} \in A^\bullet(\mathbf{F}^\omega(q_1, \dots, q_m)(E)).$$

One has the Gysin formula

$$\pi_* f(\xi_1, \dots, \xi_q) = \sum \alpha_I s_{I-\rho}^{(2)}(E),$$

where for $q_{k-1} < i \leq q_k$ the i th part of ρ is

$$\rho_i := 2n - q_k - i + 1.$$

Proposition 4.4. — Let $E \rightarrow X$ be an orthogonal vector bundle, equipped with the quadratic form $Q: E \otimes E \rightarrow \mathcal{O}_X$. Consider the isotropic partial flag bundle $\pi: \mathbf{F}^Q(q_1, \dots, q_m)(E) \rightarrow X$ and a rational equivalence class

$$f(\xi_1, \dots, \xi_q) := \sum \alpha_I \xi_1^{I_1} \cdots \xi_q^{I_q} \in A^\bullet(\mathbf{F}^Q(q_1, \dots, q_m)(E)).$$

One has the Gysin formula

$$\pi_* f(\xi_1, \dots, \xi_q) = 2^q \sum \alpha_I s_{I-\rho}^{(2)}(E),$$

where for $q_{k-1} < i \leq q_k$ the i th part of ρ is

$$\rho_i := \text{rank}(E) - q_k - i.$$

Remark that in both statements, since $L = \mathcal{O}_X$, one has $s_{2p+1}(E) = 0$ for all p . Thus if one $I_i - \rho_i$ is odd, the corresponding quadratic Schur function $s_{I-\rho}^{(2)}(E)$ is 0.

4.3. Formulas using Schur functions. — We now focus on the Grassmann bundles and use the additive basis of Schur functions. We start with type A , and show an alternative deduction of [13, Corollary 1].

Proposition 4.5. — Let $E \rightarrow X$ be a rank n vector bundle. Consider the Grassmann bundle $\pi: \mathbf{F}(q)(E) \rightarrow X$. For any $I \in \mathbb{Z}^q$, one has the Gysin formula

$$\pi_* s_I(\xi_1, \dots, \xi_q) = s_{I-\rho}(E),$$

where ρ is the partition $(n - q, \dots, n - q)$.

Proof. — The pushforward formula for the polynomial $s_I(\xi_1, \dots, \xi_q)$ is

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{e_i} \right] \left(s_I(t_1, \dots, t_q) \prod_{i=1}^q s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \right),$$

where for $i = 1, \dots, q$, the exponents are $e_i := n - 1 - q + i$.

If for some $i \in 1, \dots, q$, one has $I_i < i - q$, then one checks that the stated formula becomes $\pi_* 0 = 0$. So we can assume that definition (7) holds:

$$s_I(t_1, \dots, t_q) \prod_{1 \leq i < j \leq q} (t_j - t_i) = \det \left(t_{q-j+1}^{I_i - i + q} \right)_{1 \leq i, j \leq q},$$

and then by linearity of the determinant with respect to the columns

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \det \left([t_{q-j+1}^{e_{q-j+1}}] t_{q-j+1}^{I_i - i + q} s_{1/t_{q-j+1}}(E) \right)_{1 \leq i, j \leq q} = \det \left(s_{I_i - i + q - e_{q-j+1}}(E) \right)_{1 \leq i, j \leq q},$$

which is the announced Schur function since

$$I_i - i + q - e_{q-j+1} = I_i - (n - q) + (j - i). \quad \square$$

We now treat types B, C, D . Here the quadratic Schur functions $s_\lambda^{(2)}$ will appear again. The comparison with the results of [17] (for the case of maximal rank $q = n$; see also [23]), reveals a deep connection between the Schur type functions $s_\lambda^{(2)}(E)$ of this work and the Schur type functions $s_\lambda^{[2]}(E)$ of [17], that should probably be investigated.

But first, we need to prove the following combinatorial result.

Lemma 4.6. — For $e \in \mathbb{Z}$ and $\Lambda(t_1, \dots, t_q)$ antisymmetric, one has

$$\left[\prod_{i=1}^q t_i^{e+i} \right] \left(\Lambda(t_1, \dots, t_q) \prod_{1 \leq i < j \leq q} (t_j + t_i) \right) = \left[\prod_{i=1}^q t_i^{e+i} \right] \left(\Lambda(t_1, \dots, t_q) \prod_{i=1}^q t_i^{q-i} \right).$$

Proof. — We prove this formula by induction. Denote by $(i, j) > (i_0, j_0)$ the set of pairs of integers $i < j$ such that (i, j) is larger than (i_0, j_0) in lexicographic order. Notice that

$$\prod_{1 \leq i < j \leq q} (t_i + t_j) = \sum_{p=2}^q \left(t_1^{p-2} t_p \prod_{(i,j) > (1,p)} (t_i + t_j) \right) + t_1^{q-1} \prod_{2 \leq i < j \leq q} (t_i + t_j).$$

Hence:

$$\begin{aligned} \left[\prod_{i=1}^q t_i^{e+i} \right] \left(\Lambda(t_1, \dots, t_q) \prod_{1 \leq i < j \leq q} (t_i + t_j) \right) = \\ \sum_{p=2}^q \left[\prod_{i=1}^q t_i^{e+i} \right] \left(t_1^{p-2} t_p \Lambda(t_1, \dots, t_q) \prod_{(i,j) > (1,p)} (t_i + t_j) \right) + \left[\prod_{i=1}^q t_i^{e+i} \right] \left(t_1^{q-1} \Lambda(t_1, \dots, t_q) \prod_{2 \leq i < j \leq q} (t_i + t_j) \right). \end{aligned}$$

All the terms in the sum are $= 0$, since after shifting by t_p , one takes the coefficient of $t_{p-1}^{e+p-1} t_p^{e+p-1}$ in an antisymmetric function in t_{p-1}, t_p . Thus

$$\left[\prod_{i=1}^q t_i^{e+i} \right] \left(\Lambda(t_1, \dots, t_q) \prod_{1 \leq i < j \leq q} (t_i + t_j) \right) = \left[\prod_{i=1}^q t_i^{e+i} \right] \left(t_1^{q-1} \Lambda(t_1, \dots, t_q) \prod_{2 \leq i < j \leq q} (t_i + t_j) \right).$$

One concludes by induction on q . \square

We can now proceed.

Proposition 4.7. — Let $E \rightarrow X$ be a symplectic vector bundle of rank $2n$, equipped with the symplectic form $\omega: E \otimes E \rightarrow \mathcal{O}_X$. Consider the isotropic Grassmann bundle $\pi: \mathbf{F}^\omega(q)(E) \rightarrow X$, for $q = 1, \dots, n$. For any $I \in \mathbb{Z}^d$, one has the Gysin formula

$$\pi_* s_I(\xi_1, \dots, \xi_q) = s_{I-\rho}^{(2)}(E),$$

where $\rho_i = 2n - q - i + 1$ for $i = 1, \dots, q$.

Proof. — The pushforward formula for the polynomial $s_I(\xi_1, \dots, \xi_q)$ is

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{e_i} \right] \left(s_I(t_1, \dots, t_q) \prod_{i=1}^q s_{1/t_i}(E) \prod_{1 \leq i < j \leq q} (t_j - t_i) \prod_{1 \leq i < j \leq q} (t_j + t_i) \right),$$

where for $i = 1, \dots, q$, the exponents are $e_i := 2n - 1 - q + i$.

If for some $i \in 1, \dots, q$, one has $I_i < i - q$, then one checks that the stated formula becomes $\pi_* 0 = 0$. So we can assume that definition (7) holds:

$$s_I(t_1, \dots, t_q) \prod_{1 \leq i < j \leq q} (t_j - t_i) = \det \left(t_{q-j+1}^{I_i - i + q} \right)_{1 \leq i, j \leq q'}$$

and then by linearity of the determinant with respect to the columns

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{e_i} \right] \left(\det \left(t_{q-j+1}^{I_i - i + q} s_{1/t_{q-j+1}}(E) \right)_{1 \leq i, j \leq q} \prod_{1 \leq i < j \leq q} (t_j + t_i) \right),$$

which according to the preceding lemma is the same as

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \left[\prod_{i=1}^q t_i^{e_i} \right] \left(\det \left(t_{q-j+1}^{I_i - i + q} s_{1/t_{q-j+1}}(E) \right)_{1 \leq i, j \leq q} \prod_{i=1}^q t_i^{q-i} \right).$$

Thus, using again the linearity with respect to the columns

$$\pi_* s_I(\xi_1, \dots, \xi_q) = \det \left([t_{q-j+1}^{e_{q-j+1}}] \left(t_{q-j+1}^{I_i - i + q + j - 1} s_{1/t_{q-j+1}}(E) \right) \right)_{1 \leq i, j \leq q} = \det \left(s_{I_i - i + q + j - 1 - e_{q-j+1}}(E) \right)_{1 \leq i, j \leq q},$$

which is the announced quadratic Schur function since

$$I_i - i + q + j - 1 - e_{q-j+1} = I_i - (2n - q - i + 1) + 2(j - i). \quad \square$$

As already noticed in the case $q = n$ in [17], if one of the $I_i - \rho_i$ is odd, one obtains 0, since all odd Segre classes $s_{2p+1}(E)$ are = 0 when $L = \mathcal{O}_X$, and in one row, the degree jumps by 2 between two columns.

The proof is easily adapted in the orthogonal case, in order to get the following statement.

Proposition 4.8. — *Let $E \rightarrow X$ be an orthogonal vector bundle, equipped with the quadratic form $Q: E \otimes E \rightarrow \mathcal{O}_X$. Consider the isotropic Grassmann bundle $\pi: \mathbf{F}^Q(q)(E) \rightarrow X$, for $q = 1, \dots, \lfloor \text{rank}(E)/2 \rfloor$. For $I \in \mathbb{Z}^q$, one has the following Gysin formula*

$$\pi_* s_I(\xi_1, \dots, \xi_q) = 2^q s_{I-\rho}^{(2)}(E),$$

where $\rho_i = \text{rank}(E) - q - i$ for $i = 1, \dots, q$.

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References

- [1] E. AKYILDIZ & J. B. CARRELL – “An algebraic formula for the Gysin homomorphism from G/B to G/P ”, *Illinois Journal of Mathematics* **31** (1987), no. 2, p. 312–320.
- [2] G. BÉRCZI & A. SZENES – “Thom polynomials of Morin singularities”, *Annals of Mathematics. Second Series* **175** (2012), no. 2, p. 567–629.
- [3] M. BRION – “The push-forward and Todd class of flag bundles”, in *Parameter spaces (Warsaw, 1994)*, Banach Center Publ., vol. 36, Polish Acad. Sci., Warsaw, 1996, p. 45–50.
- [4] J. DAMON – “The Gysin homomorphism for flag bundles”, *American Journal of Mathematics* **95** (1973), p. 643–659.
- [5] L. DARONDEAU – “Fiber integration on the Demailly tower”, *Annales de l’Institut Fourier* **66** (2016), no. 1, p. 29–54.
- [6] D. EDIDIN & W. GRAHAM – “Characteristic classes and quadric bundles”, *Duke Mathematical Journal* **78** (1995), no. 2, p. 277–299.
- [7] W. FULTON – *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984.
- [8] ———, “Schubert varieties in flag bundles for the classical groups”, in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, p. 241–262.
- [9] W. FULTON & P. PRAGACZ – *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics, vol. 1689, Springer-Verlag, Berlin, 1998.
- [10] A. GROTHENDIECK – “La théorie des classes de Chern”, *Bulletin de la Société Mathématique de France* **86** (1958), p. 137–154.
- [11] ———, “Fibrés vectoriels: fibrés projectifs, fibrés en drapeaux”, in *Seminaire H. Cartan 13e*, vol. 1, 1960.
- [12] S. A. ILORI – “A generalization of the Gysin homomorphism for flag bundles”, *American Journal of Mathematics* **100** (1978), no. 3, p. 621–630.
- [13] T. JÓZEFIAK, A. LASCoux & P. PRAGACZ – “Classes of determinantal varieties associated with symmetric and skew-symmetric matrices”, *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* **45** (1981), no. 3, p. 662–673.
- [14] M. KAZARIAN – “On Lagrange and symmetric degeneracy loci”, preprint available at <http://www.newton.ac.uk/preprints/NI00028.pdf>, 2000.
- [15] ———, “Gysin homomorphism formula and the Thom polynomials for $\Sigma^{i,j}$ singularities”, Talk given at the Conference IMPANGA 15 in Będlewo, 2015.
- [16] I. G. MACDONALD – *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
- [17] P. PRAGACZ & J. RATAJSKI – “Formulas for Lagrangian and orthogonal degeneracy loci; \tilde{Q} -polynomial approach”, *Compositio Mathematica* **107** (1997), no. 1, p. 11–87.
- [18] P. PRAGACZ – “Enumerative geometry of degeneracy loci”, *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* **21** (1988), no. 3, p. 413–454.
- [19] ———, “Symmetric polynomials and divided differences in formulas of intersection theory”, in *Parameter spaces (Warsaw, 1994)*, Banach Center Publ., vol. 36, Polish Acad. Sci., Warsaw, 1996, p. 125–177.
- [20] ———, “A Gysin formula for Hall-Littlewood polynomials”, *ArXiv e-prints* (2015), arXiv:1403.0788.

- [21] D. QUILLEN – “On the formal group laws of unoriented and complex cobordism theory”, *Bulletin of the American Mathematical Society* **75** (1969), p. 1293–1298.
- [22] L. W. TU – “Computing the Gysin map using fixed points”, *ArXiv e-prints* (2015), arXiv:1507.00283.
- [23] M. ZIELENKIEWICZ – “Integrating Schur polynomials using iterated residues at infinity”, *ArXiv e-prints* (2014), arXiv:1406.1721.
- [24] ———, “Integration over homogeneous spaces for classical Lie groups using iterated residues at infinity”, *Cent. Eur. J. Math.* **12** (2014), no. 4, p. 574–583.

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