On certain family of B-modules

$n \in \mathbb{Z}_{>0}$ char $K = 0$

b = Lie alg of all $n \times n$ upper triang. matrices, $t \subset b$ diag. matrices

$U(L) = T(L)/I$ $a\otimes b - b \otimes a - [a,b]$

$U(b)$ – enveloping algebra of b.

M $U(b)$-module $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ $M_\lambda = \{m \in M : hm = \langle \lambda, h \rangle m\}$

weight space of weight λ | $\langle \lambda, h \rangle = \sum \lambda_i h_i$

If M is a direct sum of its weight spaces and each weight space has finite dimension, then M is called weight module, and

$ch(M) = \sum_\lambda \dim M_\lambda \, x^\lambda$, where $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$.

$e_{ij} \in b$ matrix with 1 at the (i,j)-position and others 0.

If $x \in M_\lambda$, then $e_{ij} x \in M_{\lambda + \varepsilon_i - \varepsilon_j}$, $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$.

$\lambda \in \mathbb{Z}^n$ K_λ = 1 dim'l $U(b)$-module, where $h \in t$ acts by $\langle \lambda, h \rangle$ and e_{ij} acts by 0.

Note that any finite-dimensional weight module admits a filtration by these 1-dim'l modules.

Let $w \in S_\infty^{(n)}$ $= \{w : w(n+1) < w(n+2) < \dots\}$. $K^n = \bigoplus_{1 \le i \le n} K u_i$

For each $j \in \mathbb{N}$, let $\{i < j : w(i) > w(j)\} = \{i_1 < \dots < i_{l_j}\}$

$u_w^{(j)} = u_{i_1} \wedge \dots \wedge u_{i_{l_j}} \in \Lambda^{l_j} K^n$. Let $u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \dots \in \bigotimes_{j \in \mathbb{N}} \Lambda^{l_j} K^n$

$S_w = U(b) u_w$.

<u>Thm (KP)</u> For any $w \in S_\infty^{(n)}$, S_w is a weight module and $ch(S_w) =$

$w \in S_\infty^{(n)}$, $1 \le i < j \le n$ $m_{ij}(w) = \#\{k > j : w(i) < w(k) < w(j)\}$

$e_{ij}^{m_{ij}(w)+1}$ annihilates u_w, S_w is gen by an element u_w of weight $c(w)$

let $I_w \subset U(b)$ ideal gen. by $h - \langle c(w), h \rangle$ $(h \in t)$ and $e_{ij}^{m_{ij}(w)+1}$ $i<j$

$\rightsquigarrow$ $\exists!$ surj. morphism of $U(b)$-modules from $U(b)/I_w \to S_w$

$1 \bmod I_w \mapsto u_w$

<u>Thm (W)</u> This surjection is iso.

(Use the transition formula.)

Schubert polys $\mathfrak{S}_\lambda$ $\lambda \in \mathbb{Z}^n$

$\lambda \in \mathbb{Z}^n_{\geq 0}$ $\mathfrak{S}_\lambda = \mathfrak{S}_w$, $S_\lambda = S_w$, where $w = \mathrm{perm}(\lambda)$

$\lambda \in \mathbb{Z}^n$, take $k \in \mathbb{Z}$ $\lambda + k1 \in \mathbb{Z}^n_{\geq 0}$ $1 = (1, \dots, 1)$ n-tuple of ones

define $\mathfrak{S}_\lambda = x^{-k1} \mathfrak{S}_{\lambda + k1}$ $S_\lambda = K_{-k1} \otimes S_{\lambda+k1}$.

S_λ is a weight module, $ch(S_\lambda) = \mathfrak{S}_\lambda$ $\forall \lambda \in \mathbb{Z}^n$.

Schubert filtration of a weight module M: $0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ of weight modules s.t. M_i / M_{i-1} is iso to some $S_{\lambda(i)}$.

$\rho = (n-1, n-2, \dots, 2, 1, 0) \in \mathbb{Z}^n$, K_ρ "dualizing module"

$\mathcal{C}$ - category of all weight modules.

For $\Lambda \subset \mathbb{Z}^n$ let $\mathcal{C}_\Lambda$ be the full subcategory of $\mathcal{C}$ consisting of all weight modules whose weights are in Λ.

$|\Lambda| < \infty$ $\Lambda' = \{\rho - \lambda : \lambda \in \Lambda\}$

$$\mathcal{C}_{\Lambda'} \cong \mathcal{C}^{op}_\Lambda \qquad M \longmapsto M^* \otimes K_\rho$$

<u>Lemma</u> for any $\Lambda \subset \mathbb{Z}^n$, $\mathcal{C}_\Lambda$ has enough projectives.

Orders: $w, v \in S_\infty$ $w \leq_{lex} v$ if $w = v$ or there exists an $i > 0$ s.t. $w(j) = v(j)$ for all $j < i$ and $w(i) < v(i)$.

For $\lambda \in \mathbb{Z}^n$ define $|\lambda| = \sum \lambda_i$. If $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$, $w = \mathrm{perm}(\lambda)$, $v = \mathrm{perm}(\mu)$ we write $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $w^{-1} \leq_{lex} v^{-1}$. For general $\lambda, \mu \in \mathbb{Z}^n$ take k s.t. $\lambda + k1$ and $\mu + k1 \in \mathbb{Z}^n_{\geq 0}$, and define $\lambda \geq \mu \iff \lambda + k1 \geq \mu + k1$.

For $\lambda \in \mathbb{Z}^n$, define $\mathcal{C}_{\leq \lambda} = \mathcal{C}_{\{\nu : \nu \leq \lambda\}}$, $\mathcal{C}_{< \lambda} = \mathcal{C}_{\{\nu : \nu < \lambda\}}$.

<u>Prop</u>. $\lambda \in \mathbb{Z}^n$ The modules S_λ and $S^*_{\rho - \lambda} \otimes K_\rho$ are in $\mathcal{C}_{\leq \lambda}$. Moreover, S_λ is projective and $S^*_{\rho-\lambda} \otimes K_\rho$ is injective in $\mathcal{C}_{\leq \lambda}$.

Thm (W) ("Strong form of Polo thm.")

For $\lambda \in \mathbb{Z}^n$, $\mu, \nu \le \lambda$ and $i \ge 1$, $\mathrm{Ext}^i_{\mathcal{C}_{\le\lambda}}(S_\mu, S^*_{\rho-\nu} \otimes K_\rho) = 0$.

Existence of Schubert filtrations

Thm (W) Let $\lambda \in \mathbb{Z}^n$, $M \in \mathcal{C}_{\le\lambda}$ and assume $\mathrm{Ext}^1(M, S^*_{\rho-\mu} \otimes K_\rho) = 0$ for all $\mu \le \lambda$.

Then M has filtration s.t. each of its subquotients is isomorphic to some S_ν ($\nu \le \lambda$).

Cor (1) If $M = M_1 \oplus \dots \oplus M_r$ then M has S-filtration iff each M_i has.

(2) If $0 \to L \to M \to N \to 0$ is exact and M, N have S-filtrations, then L also has.

Pf (1) $\mathrm{Ext}^1(M, N) = \bigoplus \mathrm{Ext}^1(M_i, N)$ $\forall N$.

(2) $\mathrm{Ext}^1(M, A) \to \mathrm{Ext}^1(L, A) \to \mathrm{Ext}^2(N, A)$ $\forall A$.

Prop. $w \in S^{(n)}_\infty$, $1 \le k \le n-1$. Then $S_w \otimes S_{s_k}$ has S-filtr.

(W; KP for $k=1$) — key point

Main result of this part:

Thm (W) $S_w \otimes S_v$ has S-filtration for $w, v \in S^{(n)}_\infty$.

$\mathfrak{b}$-module $T_w = \bigotimes_{2 \le i \le n} \left(\Lambda^{l_i(w)} K^{i-1}\right)$ $\quad l_i(w) = \#\{j < i : w(j) > w(i)\}$

T_w is a direct sum component of $\bigotimes_{2 \le i \le n} \otimes^{l_i(w)} K^{i-1} \cong \bigotimes_{2 \le i \le n} \overbrace{S_{s_{i-1}} \otimes \dots \otimes S_{s_{i-1}}}^{l_i(w)}$

Prop. $w \in S_n$. Then there is an exact sequence $0 \to S_w \to T_w \to N \to 0$, where N has a filtr. whose subquotients are S_u $\left(u^{-1} \underset{\mathrm{lex}}{>} w^{-1}\right)$.

$\Longrightarrow$ Cauchy formula for $\prod_{i+j \le n} (x_i + y_j) = \dots$

Pf of thm

$$0 \to S_w \otimes S_v \to T_w \otimes S_v \to N \otimes S_v \to 0$$

$$\begin{array}{ccc} \wr\!\uparrow & \uparrow & \uparrow \\ \text{filtr. by} & \text{filtr. by} & \text{filtr by} \\ \text{Cor. } \cancel{\text{Lemma}}\ (2). & \text{prop.} & \text{ind. on } \ell ex(w). \end{array}$$ □

$\boxed{\text{Thm}}$ Let $\lambda \in \mathbb{Z}^n$ and $M \in \mathcal{C}_{\leq\lambda}$. Then

$ch(M) \leq \sum_{\nu \leq \lambda} \dim_K(\mathrm{Hom}_{\mathfrak{b}}(M, S^*_{\rho-\nu} \otimes K_\rho))\mathfrak{S}_\nu$, and equality holds if M has a filtration s.t. each of its subquotients is iso to S_μ, $\mu \leq \lambda$

Cor $u, v, w \in S_n$. Then the coeff. of $\mathfrak{S}_w$ in $\mathfrak{S}_u \mathfrak{S}_v$ is

dim of $\mathrm{Hom}_{\mathfrak{b}}(S_u \otimes S_v, \underline{S^*_{w_0 w}} \otimes K_\rho) = \mathrm{Hom}_{\mathfrak{b}}(S_u \otimes S_v \otimes S_{w_0 w}, K_\rho)$.

Pf $\mathfrak{S}_u \mathfrak{S}_v = ch(S_u \otimes S_v) = \sum_w \dim(S_u \otimes S_v, \underline{S^*_{\rho-\lambda} \otimes K_\rho})\mathfrak{S}_w$. □

Prop. $s_\sigma(S_\lambda)$ has S-filtr. (Schur) (s_σ = Schur functor indexed by partition σ)

Pf For any $\lambda^{(1)}, \dots, \lambda^{(r)} \in \mathbb{Z}^n$, $S_{\lambda^{(1)}} \otimes \cdots \otimes S_{\lambda^{(r)}}$ has S-filtr.

$(S_\lambda)^{\otimes k}$ has S-filtr. $\forall \lambda\ \forall k$. $\Rightarrow \mathrm{Ext}^1((S_\lambda)^{\otimes k}, S^*_\nu \otimes K_\rho) = 0\ \forall \nu$

$s_\sigma(S_\lambda)$ is a direct sum factor of $(S_\lambda)^{\otimes|\sigma|} \Rightarrow$

$\mathrm{Ext}^1(s_\sigma(S_\lambda), S^*_\nu \otimes K_\rho) = 0 \Rightarrow s_\sigma(S_\lambda)$ has S-filtr.

Cor. $\mathfrak{S}_w$ = sum of monomials $x^\alpha + x^\beta + \cdots$, then $s_\sigma(x^\alpha, x^\beta, \dots)$ is a positive sum of Schubert polyn

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