CONDITIONALITY CONSTANTS OF QUASI-GREEDY BASES IN BANACH SPACES OF NON-TRIVIAL TYPE

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Abstract. We show that for every $2 < q < \infty$ there is a Banach space of type 2 and cotype $q$ possessing a quasi-greedy basis whose conditionality constant sequence is of the same order as $(\log(m))^\infty_{m=2}$.

1. Introduction and background

Let $(x_n)_{n=1}^\infty$ be a (Schauder) basis for a Banach space $X$, i.e., a sequence in $X$ with the property that for every $f \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ such that

$$f = \sum_{n=1}^\infty a_n x_n. \quad (1.1)$$

Associated to a basis $(x_n)_{n=1}^\infty$ we have the sequence $(x^*_n)_{n=1}^\infty$ in $X^*$ of its biorthogonal functionals defined by $x^*_n(f) = a_n$, where $(a_n)_{n=1}^\infty$ and $f$ are as in (1.1). The basis $(x_n)_{n=1}^\infty$ is said to be semi-normalized if

$$0 < \inf_{n \in \mathbb{N}} \|x_n\| \leq \sup_{n \in \mathbb{N}} \|x_n\| < \infty.$$

Let us recall the following simple and well known result.

Lemma 1.1 ([11, Corollary 3.1]). A basis $(x_n)_{n=1}^\infty$ is semi-normalized if and only if

$$\sup_{n \in \mathbb{N}} \max\{\|x_n\|, \|x^*_n\|\} < \infty.$$

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Let $\mathcal{B} = (x_n)_{n=1}^{\infty}$ be a semi-normalized basis of a Banach space $X$. Then by Lemma 1.1 we have a continuous linear operator $\mathcal{F}: X \to c_0$ given by

$$f \mapsto (x_n^* (f))_{n=1}^{\infty}.$$ 

Hence, for any $f \in X$ there is an injective map $\rho: \mathbb{N} \to \mathbb{N}$ (an ordering of $\mathbb{N}$) such that

$$|x_{\rho(k)}^*(f)| \geq |x_{\rho(n)}^*(f)| \quad \text{if} \quad k \leq n. \quad (1.2)$$

If the sequence $(x_n^*(f))_{n=1}^{\infty}$ contains several terms with the same absolute value then such an ordering is not uniquely determined. In order to get uniqueness, we impose the additional condition

$$\rho(k) \leq \rho(n) \quad \text{whenever} \quad |x_{\rho(k)}^*(f)| = |x_{\rho(n)}^*(f)|. \quad (1.3)$$

If $f$ is infinitely supported there is a unique ordering $\rho$ of $\mathbb{N}$ that verifies (1.2) and (1.3); moreover such an ordering verifies $\rho(\mathbb{N}) = \text{supp}(f)$. In case that $f$ is finitely supported there is a unique ordering that verifies (1.2), (1.3) and $\rho(\mathbb{N}) = \mathbb{N}$. In both cases we will refer to such a unique ordering as the greedy ordering for $f$.

If the series in (1.1) converges unconditionally for every $f \in X$, i.e., for any $f \in X$ and any permutation $\pi$ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\pi(n)}^*(f)x_{\pi(n)}$ converges to $f$, we say that $(x_n)_{n=1}^{\infty}$ is an unconditional basis. We will call conditional bases those bases that are not unconditional. Konyagin and Temlyakov [8] defined a semi-normalized basis to be quasi-greedy if for any $f \in X$ the series $\sum_{n=1}^{\infty} x_{\rho(n)}^*(f)x_{\rho(n)}$ converges to $f$, where $\rho$ is the greedy ordering for $f$. Note that being quasi-greedy is formally a weaker condition than being semi-normalized and unconditional. Indeed, Wojtaszczyk [12] proved that in a wide class of separable Banach spaces there exist conditional quasi-greedy bases, showing this way that those two concepts are different. Since then, the problem of finding estimates for the conditionality constants of quasi-greedy bases attracted the attention of the specialists. Recall that the conditionality constant of order $m$ of a basis $\mathcal{B} = (x_n)_{n=1}^{\infty}$ in a Banach space $X$ is defined by

$$k_m[\mathcal{B}, X] = \sup_{|A|\leq m} \|P_A\|,$$

where for each finite set $A \subseteq \mathbb{N}$ the operator $P_A$ is the natural projection

$$P_A: X \to X, \quad f \mapsto \sum_{n \in A} x_n^*(f)x_n.$$ 

Of course, a basis $\mathcal{B}$ is unconditional if and only if $\sup_m k_m[\mathcal{B}, X] < \infty$ (see e.g. [2, Proposition 3.1.5]). Hence the “order of growth” of the sequence $(k_m[\mathcal{B}, X])_{m=1}^{\infty}$ of conditionality constants provides a natural quantitative measure of the conditionality of the basis $\mathcal{B}$.
Note that every semi-normalized basis $B$ of a Banach space $X$ verifies the estimate
\[ k_m[B, X] \lesssim m \quad \text{for } m \in \mathbb{N}. \]
Here and throughout this note, given families of positive real numbers $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that $\sup_{i \in I} \alpha_i / \beta_i < \infty$, while $\alpha_i \approx \beta_i$ for $i \in I$ means that $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$.

There are semi-normalized bases $B$ in Banach spaces $X$ for which the above estimate is sharp, i.e., $k_m[B, X] \approx m$ for $m \in \mathbb{N}$. The simplest example of such a basis is the summing basis of $c_0$. However, the situation is quite different if we restrict our attention to quasi-greedy bases:

**Theorem 1.2** ([4, Lemma 8.2]). For any quasi-greedy basis $B$ in a Banach space $X$ we have
\[ k_m[B, X] \lesssim \log(m) \quad \text{for } m \geq 2. \tag{1.4} \]

Garrigós and Hernández [6] provided examples of quasi-greedy bases for which the estimate (1.4) is sharp, i.e., $k_m[B, X] \approx \log(m)$ for $m \geq 2$. The role of the underlying Banach space in these examples was revealed in [7], where it is proved that the conditionality constants of any quasi-greedy basis for a separable $L_p$-space ($1 < p < \infty$) verifies a better estimate. Later on, Albiac, Ansorena, Garrigós, Hernández, and Raja extended Garrigós-Wojtaszczyk result to superreflexive Banach spaces. To be precise, we have the following result:

**Theorem 1.3** ([1, Theorem 1.1]). For any quasi-greedy basis $B$ in a superreflexive Banach space $X$ there is a constant $0 < \alpha < 1$ such that
\[ k_m[B, X] \lesssim (\log(m))^{\alpha} \quad \text{for } m \geq 2. \]

Moreover, the estimate given in Theorem 1.3 can not be improved even in Hilbert spaces. This fact, proved also in [7], relies on the following result:

**Theorem 1.4** ([7, Theorem 3.1]). Let $X$ be a Banach space and let $\rho: [0, \infty) \to [1, \infty)$ be an increasing function. Assume that there is a semi-normalized basis $B$ is $X$ such that $\rho(m) \lesssim k_m[B, X]$ for $m \in \mathbb{N}$. Then there is a quasi-greedy basis $B_0 = (x_n)_{n=1}^\infty$ in $X \oplus \ell_2$ such that
\[
\begin{align*}
(a) \quad \rho(\log(m)) &\lesssim k_m[B_0, X \oplus \ell_2] \quad \text{for } m \in \mathbb{N}, \\
(b) \quad \| \sum_{n \in A} x_n \| &\approx |A|^{1/2} \quad \text{for every finite set } A \subseteq \mathbb{N}.
\end{align*}
\]

At this point, one can wonder if there is wider class of Banach spaces for which the thesis of Theorem 1.3 holds. In [1] it is showed that it is hopeless to try with reflexive Banach spaces. Another attempt is to
focus the attention in the notion of (Rademacher) type of the space. Indeed, as an immediate consequence of a theorem of Maurey and Pisier theorem ([9]), every superreflexive Banach space has both non-trivial type and a non-trivial cotype. Notice also that all the Banach spaces constructed in [1,7] with a quasi-greedy basis whose conditionality constant sequence is of the same order as $(\log(m))_{m=2}^\infty$ are of type 1. Thus it seems natural to investigate if Theorem 1.3 can be extended to Banach spaces with non-trivial type. In this paper we solve this question in the negative:

**Theorem 1.5** (Main Theorem). Let $2 < q < \infty$ (respectively, $1 < q < 2$). There is a quasi-greedy basis $\mathcal{B}$ for a Banach space $X$ of type 2 and cotype $q$ (respectively, type $q$ and cotype 2) such that

$$k_m[\mathcal{B}, X] \approx \log(m) \text{ for } m \geq 2.$$ 

Section 2 is devoted to develop the machinery that will lead to the proof of Theorem 1.5 in Section 3. This proof relies, on the one hand, on a construction by Pisier and Xu [10] of non-reflexive Banach spaces having non-trivial type and cotype, and on the other hand on a result which allows us tailor quasi-greedy bases with large conditionality constants in certain Banach spaces.

We end this preliminary section by singling out some additional terminology on greedy-type bases. A basis $\{x_n\}_{n=1}^\infty$ in a Banach space $X$ is said to be democratic if $\left\| \sum_{j \in A} x_n \right\| \approx \left\| \sum_{j \in B} x_n \right\|$ for any $A, B$ finite subsets of $\mathbb{N}$ with $|A| = |B|$. The basis is said to be almost greedy if there is a constant $C \geq 1$ such that for all $m \in \mathbb{N}$ and $f \in X$,

$$\|f - G_m(f)\| \leq C \inf \{\|f - P_A(f)\| : |A| = m\}$$

Dilworth et al. [5] characterized almost greedy basis as those bases that are simultaneously quasi-greedy and democratic. Note that, in light of this, the basis $\mathcal{B}_0$ for $X \oplus \ell_2$ obtained in Theorem 1.4 is not only quasi-greedy but almost greedy.

If necessary the reader will find more background on Banach space theory and greedy-like bases in [2].

2. Preliminaries

Given a basis $\{x_n\}_{n=1}^\infty$ of a Banach space $X$ over the field $\mathbb{F}$ (where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) and $m \in \mathbb{N}$, the partial sum operator $S_m : X \to X$ is defined by

$$f \mapsto S_m(f) = \sum_{n=1}^m a_n x_n.$$
It is well known that $K = \sup_m \|S_m\| < \infty$, and the number $K$ is called the **basis constant**. Schauder bases can be characterized in terms of the partial sum operators:

**Theorem 2.1** (see [11, Theorem 4.1]). Let $(x_n)_{n=1}^{\infty}$ be a sequence in Banach space $\mathbb{X}$ whose closed linear span is the whole space $\mathbb{X}$. Assume that there is a sequence $(x_n^*)_{n=1}^{\infty}$ in $\mathbb{X}^*$ such that $x_n^*(x_k) = \delta_{n,k}$ for all $n, k \in \mathbb{N}$ and the sequence of operators $(S_m)_{m=1}^{\infty}$ given by $S_m(\cdot) = \sum_{n=1}^{m} x_n^*(\cdot)x_n$ is uniformly bounded. Then $(x_n)_{n=1}^{\infty}$ is a basis for $\mathbb{X}$ whose biorthogonal functionals are $(x_n^*)_{n=1}^{\infty}$ and whose partial sum operators are $(S_m)_{m=1}^{\infty}$.

In many situations Theorem 2.1 is the only tool to show that a system is a Schauder basis, and it will be heavily used in this manuscript. For example it can be used to prove the following few elementary lemmas, which we state for expositional ease.

**Lemma 2.2** (see [11, Proposition 4.2]). Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be bases for the Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively. Then the sequence $(z_n)_{n=1}^{\infty}$ given by

$$z_{2n-1} = (x_n, 0), \quad z_{2n} = (0, y_n), \quad n \in \mathbb{N},$$

is a basis for $\mathbb{X} \oplus \mathbb{Y}$. Moreover, if both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are semi-normalized then so is $(z_n)_{n=1}^{\infty}$.

**Lemma 2.3.** Let $(x_n)_{n=1}^{\infty}$ be a semi-normalized basis for a Banach space $\mathbb{X}$. Then the sequence $(y_n)_{n=1}^{\infty}$ given by

$$y_{2n-1} = x_{2n-1} + x_{2n}, \quad y_{2n} = x_{2n-1} - x_{2n}, \quad n \in \mathbb{N},$$

is a semi-normalized basis for $\mathbb{X}$.

**Proof.** Let $(S_m)_{m=1}^{\infty}$ be the partial sum projections associated to $(x_n)_{n=1}^{\infty}$ and let $K$ be its basis constant. By Lemma 1.1, $C = \sup_n \{\|x_n\|, \|x_n^*\|\} < \infty$. Let us define $(y_n^*)_{n=1}^{\infty}$ by

$$y_{2n-1}^* = \frac{x_{2n-1}^* + x_{2n}^*}{2}, \quad y_{2n}^* = \frac{x_{2n-1}^* - x_{2n}^*}{2}, \quad n \in \mathbb{N},$$

so that $y_n^*(y_k) = \delta_{n,k}$ for every $k, n \in \mathbb{N}$. Let $m \in \mathbb{N}$. The operator $T_m: \mathbb{X} \to \mathbb{X}$ given by $T_m(\cdot) = \sum_{n=1}^{m} y_n^*(\cdot)y_n$ verifies

$$T_{2m-1}(\cdot) = S_{2m-2}(\cdot) + \frac{1}{2} \sum_{i,j \in \{0,1\}} x_{2m-i}^*(\cdot)x_{2m-j},$$

$$T_{2m}(\cdot) = S_{2m}.$$
Consequently \( \sup_n \|T_m\| \leq K + 2C \). By Theorem 2.1, \( (y_n)_{n=1}^\infty \) is a basis. Moreover

\[
\frac{1}{K} \inf_n \|x_n\| \leq \|y_k\| \leq 2 \sup_n \|x_n\|
\]

for every \( k \in \mathbb{N} \), i.e., \( (y_n)_{n=1}^\infty \) is semi-normalized.

**Lemma 2.4.** Let \( (x_n)_{n=1}^\infty \) be a sequence in a Banach space \( X \) such that \( \inf_n \|x_n\| > 0 \). For \( n \in \mathbb{N} \) put \( y_n = \sum_{k=1}^n x_k \).

(a) Assume that \( (x_n)_{n=1}^\infty \) is a semi-normalized basis for \( X \). Then \( (y_n)_{n=1}^\infty \) is a basis for \( X \) (in which case, it is semi-normalized) if and only if \( \sup_n \|y_n\| < \infty \).

(b) Assume that \( (y_n)_{n=1}^\infty \) is a semi-normalized basis for \( X \) with biorthogonal functionals \( (y_n^*)_{n=1}^\infty \). Then \( (x_n)_{n=1}^\infty \) is a basis for \( X \) (in which case, it is semi-normalized) if and only if \( \sup_n \|\sum_{k=1}^n y_k^*\| < \infty \).

**Proof.** Part (a) is proved in [11, Proposition 4.3], so we only deal with Part (b). Let \( K \) be the constant of the basis \( (y_n)_{n=1}^\infty \) and \( C = \sup_n \|y_n\| \). Assume that \( (x_n)_{n=1}^\infty \) is basis for \( X \) and that \( (x_n^*)_{n=1}^\infty \) are its biorthogonal functionals. We have \( \|x_n\| = \|y_n - y_{n-1}\| \leq 2C \) for all \( n \geq 2 \). By Lemma 1.1, \( D = \sup_n \|x_n^*\| < \infty \). Let \( n \in \mathbb{N} \). We have \( (x_n^* - x_{n+1}^*)(y_k) = \delta_{n,k} \) for every \( k \) and, hence, \( x_n^* - x_{n+1}^* = y_n^* \). Therefore, \( \|\sum_{k=1}^n y_k^*\| = \|x_1^* - x_{n+1}^*\| \leq 2D \).

Assume that \( E = \sup_n \|\sum_{k=1}^n y_k^*\| < \infty \). Use Banach-Alaoglu theorem to pick \( y^* \in X^* \) a weak* cluster point of \( \{\sum_{k=1}^n y_k^* : n \in \mathbb{N}\} \). For \( n \in \mathbb{N} \) put \( x_n^* = y^* - \sum_{k=1}^{n-1} y_k^* \). Since \( y^*(y_n) = 1 \) for all \( n \in \mathbb{N} \), we have \( x_n^*(x_k) = \delta_{n,k} \) for all \( k, n \in \mathbb{N} \). For \( m \in \mathbb{N} \) let \( S_m(\cdot) = \sum_{n=1}^m x_n^*(\cdot)x_n \) and \( T_m(\cdot) = \sum_{n=1}^m y_n^*(\cdot)y_n \). We have that

\[
S_m(\cdot) = -T_m(\cdot) + \left(y^* - \sum_{n=1}^m y_n^*\right)(\cdot)y_m,
\]

and so \( \|S_m\| \leq K + C(\|y^*\| + E) \). Invoking Theorem 2.1 the proof is over.

We are ready to prove our first result on quasi-greedy bases.

**Theorem 2.5.** Suppose that a Banach space \( X \) has a semi-normalized basis \( (x_n)_{n=1}^\infty \) such that \( \sup_n \|\sum_{k=1}^n x_k\| < \infty \). Then there is an almost greedy basis \( B = (y_n)_{n=1}^\infty \) for \( X \oplus \ell_2 \) such that \( k_m[\mathcal{B}, X \oplus \ell_2] \approx \log(m) \) for \( m \geq 2 \) and \( \|\sum_{n \in A} x_n\| \approx |A|^{1/2} \) for \( A \subseteq \mathbb{N} \) finite.

**Proof.** For \( n \in \mathbb{N} \) put \( y_n = \sum_{k=1}^n x_k \), let \( C = \sup_n \|y_n\| \) and let \( K \) be the basis constant of \( (x_n)_{n=1}^\infty \). Combining Lemma 2.4(a), Lemma 2.2
and Lemma 2.3 we claim that the sequence \( \mathcal{B}_0 = (z_n)_{n=1}^{\infty} \) given by
\[
z_{2n-1} = (x_n, y_n), \quad z_{2n} = (x_n, -y_n), \quad n \in \mathbb{N}
\]
is a semi-normalized basis for \( \mathbb{X} \oplus \mathbb{X} \). Pick \( m \in \mathbb{N} \) and consider
\[
f := \sum_{n=1}^{2m} z_n = \left( 2 \sum_{n=1}^{m} x_n, 0 \right) = 2(y_n, 0), \quad \text{and}
\]
\[
g := \sum_{n=1}^{2m} (-1)^n z_n = \left( 0, 2 \sum_{n=1}^{m} y_n \right) = 2 \left( 0, \sum_{n=1}^{m} (m + 1 - n)x_n \right).
\]

Notice that \( (f + g)/2 = P_A(f) \), where \( A = \{2n: 1 \leq n \leq m\} \). Hence
\[
k_m[\mathcal{B}_0, \mathbb{X} \oplus \mathbb{X}] \geq \frac{\|P_A(f)\|}{\|f\|} \geq \frac{\|g\| - \|f\|}{2\|f\|} \geq \frac{mK\|x_1\| - C}{2C}.
\]
Consequently \( k_m[\mathcal{B}_0, \mathbb{X} \oplus \mathbb{X}] \approx m \) for \( m \in \mathbb{N} \). An appeal to Theorem 1.4 completes the proof.

Given a compatible couple \((\mathbb{X}_0, \mathbb{X}_1)\) of Banach spaces, the real interpolation space of indices \( 0 < \theta < 1 \) and \( 1 \leq q < \infty \) is defined by
\[
(\mathbb{X}_0, \mathbb{X}_1)_{\theta,q} = \left\{ f \in \mathbb{X}_0 + \mathbb{X}_1: \|f\|_{\theta,q} = \left( \int_0^\infty K^q(f, t, \mathbb{X}_0, \mathbb{X}) \frac{dt}{t^{1+\theta q}} \right)^{1/q} < \infty \right\},
\]
where
\[
K(f, t, \mathbb{X}_0, \mathbb{X}_1) = \inf \{ \|f_0\|_{\mathbb{X}_0} + t\|f_1\|_{\mathbb{X}_1}: f_0 \in \mathbb{X}_0, f_1 \in \mathbb{X}_1, f = f_0 + f_1 \},
\]
for \( t > 0 \). Real interpolation will provide an exact interpolation scheme, i.e., if \( T: (\mathbb{X}_0, \mathbb{X}_1) \to (\mathbb{Y}_0, \mathbb{Y}_1) \) is an admissible operator between two compatible couples (that is, \( T \) is linear from \( \mathbb{X}_0 + \mathbb{X}_1 \) into \( \mathbb{Y}_0 + \mathbb{Y}_1 \) and bounded from \( \mathbb{X}_i \) into \( \mathbb{Y}_i, i = 0, 1 \) then
\[
\|T: (\mathbb{X}_0, \mathbb{X}_1)_{\theta,q} \to (\mathbb{Y}_0, \mathbb{Y}_1)_{\theta,q}\| \leq \max_{i=0,1} \|T: \mathbb{X}_i \to \mathbb{Y}_i\|.
\]
Here, at it is costumary, \( \|T: \mathbb{X} \to \mathbb{Y}\| \) denotes the norm of the operator \( T \) when regarded as an operator from \( \mathbb{X} \) into \( \mathbb{Y} \). In particular we have that
\[
\mathbb{X}_0 \cap \mathbb{X}_1 \subset (\mathbb{X}_0, \mathbb{X}_1)_{\theta,q} \subset \mathbb{X}_0 + \mathbb{X}_1,
\]
with norm-one inclusions. Recall that the norms in \( \mathbb{X}_0 \cap \mathbb{X}_1 \) and \( \mathbb{X}_0 + \mathbb{X}_1 \) are given by
\[
\|f\|_{\mathbb{X}_0 \cap \mathbb{X}_1} = \max\{\|f\|_{\mathbb{X}_0}, \|f\|_{\mathbb{X}_1}\}, \quad \|f\|_{\mathbb{X}_0 + \mathbb{X}_1} = K(f, 1, \mathbb{X}_0, \mathbb{X}_1).
\]
The next lemma describes the behavior of bases with respect to the real interpolation method.
Lemma 2.6. Let \((X_0, X_1)\) be a compatible couple of Banach spaces and let \(B = (x_n)_{n=1}^\infty\) be a sequence in \(X_0 \cap X_1\) whose closed linear span is the whole space \(X_0 \cap X_1\). Assume that \(B\) is a basis for \(X_i\), \(i = 0, 1\). Then, for any \(0 < \theta < 1\) and any \(1 \leq q < \infty\), \(B\) is a basis for \((X_0, X_1)_{\theta,q}\). Moreover, if \(B\) is semi-normalized in both \(X_0\) and \(X_1\), then \(B\) is semi-normalized in \((X_0, X_1)_{\theta,q}\).

Proof. For \(i = 0, 1\), denote by \((x^*_n)_{n=1}^\infty\) the biorthogonal sequence and by \(K_i\) the basis constant in the space \(X_i\). Let \(n \in \mathbb{N}\). We have that \(x_n^*(x_k) = x_n^*(1) x_k\) for any \(k \in \mathbb{N}\) and that \(x_n^*(\cdot)\) is continuous on \(X_0 \cap X_1\), \(i = 0, 1\). Consequently \(x_n^*(f) = x_n^*(1) f\) for all \(f \in X_0 \cap X_1\) and so there is an admissible operator \(x_n^* : (X_0, X_1) \to (\mathbb{F}, \mathbb{F})\) that verifies \(x_n^*(x_k) = \delta_{n,k}\) for every \(k \in \mathbb{N}\). Therefore, given \(m \in \mathbb{N}\), we can safely define an admissible operator \(S_m : (X_0, X_1) \to (X_0, X_1)\) by \(S_m(f) = \sum_{n=1}^m x_n^*(f) x_n\). By interpolation,

\[
\|S_m : (X_0, X_1)_{\theta,q} \to (X_0, X_1)_{\theta,q}\| \leq \max_{i=0, 1} \|S_m : X_i \to X_i\| \leq \max\{K_0, K_1\}.
\]

Since the linear span of \(B\) is dense in \(X_0 \cap X_1\) and \(X_0 \cap X_1\) is dense in \((X_0, X_1)_{\theta,q}\) (see [3, Theorem 2.9]) we infer from Lemma 2.1 that \(B\) is a Schauder basis for \((X_0, X_1)_{\theta,q}\).

Assume now that \(B\) is semi-normalized in \(X_i\), \(i = 0, 1\). Then, by Lemma 1.1,

\[
C = \sup_{n \in \mathbb{N}, i=0, 1} \{\|x_n\|_{X_i}, \|x_n^* : X_i \to \mathbb{F}\|\} < \infty.
\]

Hence \(\|x_n^* : (X_0, X_1)_{\theta,q} \to \mathbb{F}\| \leq \max_{i=0, 1} \|x_n^* : X_i \to \mathbb{F}\| \leq C\) and \(\|x_n\|_{\theta,q} \leq \|x_n\|_{X_0 \cap X_1} \leq C\) for all \(n \in \mathbb{N}\). By Lemma 1.1, the basis is semi-normalized in \((X_0, X_1)_{\theta,q}\).

\[\square\]

3. Towards the proof of the Main Theorem

Let by \(v_1\) denote the space of all sequences of scalars of bounded variation, i.e.,

\[
v_1 = \{f = (a_n)_{n=1}^\infty : \|f\|_{v_1} = |a_1| + \sum_{n=1}^\infty |a_{n+1} - a_n| < \infty\}.
\]

The Banach space \(v_1\) is nothing but \(\ell_1\) in a rotated position. Indeed, the linear bijection

\[Q : \mathbb{F}^\mathbb{N} \to \mathbb{F}^\mathbb{N}, \quad (a_n)_{n=1}^\infty \mapsto \left(\sum_{k=1}^n a_k\right)_{n=1}^\infty\]
restricts to an isometry from $\ell_1$ onto $v_1$. Via this isometry the functional on $\ell_1$ given by $(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n$ induces a functional on $v_1$ given by $(a_n)_{n=1}^\infty \mapsto \lim_n a_n$. Therefore

$$v_1 \subseteq c := \{(a_n)_{n=1}^\infty : \exists \lim_n a_n \in \mathbb{F}\} \subseteq \ell_\infty.$$ 

Let $v_1^0$ be the subspace of codimension one of $v_1$ that corresponds to

$$\ell_1^0 = \{(a_n)_{n=1}^\infty \in \ell_1 : \sum_{n=1}^\infty a_n = 0\}$$

under the isometry $Q$, that is,

$$v_1^0 = v_1 \cap c_0 = \{(a_n)_{n=1}^\infty \in v_1 : \lim_n a_n = 0\}.$$

Pisier and Xu [10] investigated the interpolated spaces $(v_1, \ell_\infty)_{\theta,q}$. However, the space $(v_1^0, c_0)_{\theta,q}$ is more fit for our purposes.

**Lemma 3.1.** The linear mapping $R: \mathbb{F}^N \to \mathbb{F}^N$ given by

$$(a_n)_{n=1}^\infty \mapsto (a_{n+1} - a_1)_{n=1}^\infty$$

restricts to an isomorphism both from $c_0$ onto $c$ and from $v_1^0$ onto $v_1$.

**Proof.** It is clear that $R$ is a bijection from $c_0$ onto $c$ whose inverse is given by by $(b_n)_{n=1}^\infty \mapsto (b_{n-1} - \lim_n b_n)_{n=1}^\infty$ (with the convention that $b_0 = 0$). Hence $R$ is an isomorphism between both spaces. It is straightforward to check that the mapping $U = Q^{-1} \circ R \circ Q$ is given by $(a_n)_{n=1}^\infty \mapsto (a_{n+1})_{n=1}^\infty$. We infer that $U$ is an isomorphism from $\ell_1^0$ onto $\ell_1$ whose inverse is given by $(b_n)_{n=1}^\infty \mapsto (-\sum_{n=1}^\infty b_n, b_1, b_2, \ldots)$. Consequently, $R$ is an isomorphism from $v_1^0$ onto $v_1$. \hfill \Box

**Lemma 3.2.** Let $0 < \theta < 1$ and $1 \leq q < \infty$. We have

(a) $(v_1, \ell_\infty)_{\theta,q} = (v_1, c)_{\theta,q}$ isometrically.

(b) $(v_1^0, c_0)_{\theta,q} \approx (v_1, \ell_\infty)_{\theta,q}$.

(c) $(v_1^0, c_0)_{\theta,q}$ is a subspace of codimension one of $(v_1, \ell_\infty)_{\theta,q}$.

**Proof.** In order to prove (a) it suffices to show that $(v_1, \ell_\infty)_{\theta,q} \subseteq c$. Let $f = (a_n)_{n=1}^\infty \in \ell_\infty \setminus c$. There exists $\varepsilon > 0$ such that for every $j \in \mathbb{N}$ there are $n \geq k \geq j$ such that $|a_n - a_k| \geq \varepsilon$. Let $f = g + h$ with $g = (b_n)_{n=1}^\infty \in v_1$ and $h = (c_n)_{n=1}^\infty \in \ell_\infty$. Since $g \in c$, there is $j \in \mathbb{N}$ such that $|b_n - b_k| \leq \varepsilon/2$ for every $n \geq k \geq j$. We infer that $|c_n - c_k| \geq \varepsilon/2$ for some $n \geq k \geq j$. Therefore $\|h\|_\infty \geq \varepsilon/4$. Consequently, $K(f, t, v_1, \ell_\infty) \geq t\varepsilon/4$ for every $t > 0$, thus $\|f\|_{\theta,q} = \infty$.

By using interpolation it follows from Lemma 3.1 that the mapping $R$ is an isomorphism from $(v_1^0, c_0)_{\theta,q}$ onto $(v_1, c)_{\theta,q}$. Hence, (b) holds.
It is clear that the mapping

\[ V: c \to c_0 \oplus F, \quad (a_n)_{n=1}^\infty \mapsto (a_n - \lim_{n} a_n)_{n=1}^\infty, \lim a_n \]

is an isomorphism both from \( v_1 \) onto \( v_0^1 \oplus F \) and from \( c \) onto \( c_0 \oplus F \). By interpolation, \( V \) is an isomorphism from \( (v_1, c)_{\theta,q} \) onto \( (v_0^1, c_0)_{\theta,q} \oplus F \). Therefore, (c) holds. □

**Completion of the proof of Theorem 1.5.** Let \((e_n)_{n=1}^\infty\) denote the standard unit vector basis of a given sequence space. Via the isometry \( Q \), we get that the sequence \((x_n)_{n=1}^\infty\) defined by \( x_n = \sum_{k=n}^\infty e_k \) is a basis for \( v_1 \) equivalent to the unit vector basis of \( \ell_1 \). Via the isomorphism \( R \) described in Lemma 3.1, the sequence \((y_n)_{n=1}^\infty\) defined by \( y_n = \sum_{k=1}^n e_k \) is a basis for \( v_0^0 \) equivalent to the unit vector basis of \( \ell_1 \). Notice that \( \|e_n\|_{v_1} = 2 \) for all \( n \in \mathbb{N} \) and that \( \sup_n \| \sum_{k=1}^n e_n \|_{\ell_1} < \infty \). Hence, by Lemma 2.4(b), \((e_n)_{n=1}^\infty\) is also a semi-normalized basis for \( v_0^1 \).

We have that \((e_n)_{n=1}^\infty\) is a semi-normalized basis for \( c_0 \) and, by Lemma 2.4(a), so is the summing basis \((y_n)_{n=1}^\infty\). By Lemma 2.6 both \((e_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) are semi-normalized bases for \( X := (v_0^1, c_0)_{\theta,q} \) \((0 < \theta < 1, 1 \leq p < \infty)\). By Theorem 2.5 the space \( X \oplus X \oplus \ell_2 \) has a quasi-greedy as desired. Finally, if \( 2 < (1-\theta)^{-1} \leq q \), then \( X \) has type 2 and cotype \( q \), while if \( q \leq (1-\theta)^{-1} < 2 \) then \( X \) has type \( q \) and cotype 2 (see [10, Theorem 1.1]). Obviously, \( X \oplus X \oplus \ell_2 \) has the same type and cotype properties as \( X \). □

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