ON NONATOMIC BANACH LATTICES AND HARDY SPACES

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Abstract. We are interested in the question when a Banach space $X$ with an unconditional basis is isomorphic (as a Banach space) to an order-continuous nonatomic Banach lattice. We show that this is the case if and only if $X$ is isomorphic as a Banach space with $X(l_2)$. This and results of J. Bourgain are used to show that spaces $H_1(T^n)$ are not isomorphic to nonatomic Banach lattices. We also show that tent spaces introduced in [4] are isomorphic to Rad $H_1$.

1. Introduction

There is a natural distinction between sequence spaces and function spaces (or, between atomic and nonatomic Banach lattices) in functional analysis. As an example, let us point out the subtitles of two volumes of [15] and [16]. However, many classical function spaces (e.g. the spaces $L_p, 1$ for $1 < p < \infty$ [22] or [16]) have unconditional bases and hence are isomorphic as Banach spaces to sequence spaces (atomic Banach lattices). On the other hand, $L_1, 1$ and has no unconditional basis ([22] or [16]) and in the other direction the sequence spaces $\ell_p$, for $p \neq 2$ are not isomorphic to any nonatomic Banach lattice [1]. In this note we discuss a general criterion for deciding whether a Banach space with an unconditional basis (i.e., a sequence space) can be isomorphic to a nonatomic Banach lattice (i.e., a function space). Our main result (Theorem 2.4) gives a simple necessary and sufficient condition for an atomic Banach lattice $X$ to be isomorphic to an order-continuous nonatomic Banach lattice; of course if $X$ contains no copy of $c_0$ every Banach lattice structure on $X$ is order-continuous.

Our main motivation is to study the Hardy space $H_1(T)$. After the discovery that the space $H_1(T)$ has an unconditional basis [17] it become natural to investigate if $H_1(T)$ is isomorphic to a nonatomic Banach lattice. Applying Theorem 2.4 to $H_1$ and using some previous results of Bourgain [2] and [3] we show that $H_1$ is not isomorphic to any nonatomic Banach lattice, and further more that $H_1(T^n)$ is not isomorphic to a nonatomic Banach lattice for any natural number $n$.

We conclude by showing that the space Rad $H_1$ or $H_1(l_2)$ is isomorphic to the tent spaces $T^1$ introduced by Coifman, Meyer and Stein [4].

2. Lattices with unconditional bases

Our terminology about Banach lattices will agree with [16]; we also refer the reader to [9] and [10] for the isomorphic theory of nonatomic Banach lattices.

A (real) Banach lattice $X$ is called order continuous if every order-bounded increasing sequence of positive elements is norm convergent. Any Banach lattice not containing $c_0$ is automatically order-continuous.

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For any order-continuous Banach lattice $X$ we can define an associated Banach lattice $X(\ell_2)$ (using the Krivine calculus [16] pp. 40-42) as the space of sequences $(x_n)_{n=1}^{\infty}$ in $X$ such that $(\sum_{k=1}^{n} |x_k|^2)^{1/2}$ is order-bounded (and hence is a convergent sequence) in $X$. $X(\ell_2)$ becomes an order-continuous Banach lattice when normed by $(x_n) = (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$.

If $X$ has nontrivial cotype then $X(\ell_2)$ is naturally isomorphic to the space $\text{Rad} X$ which is the subspace of $L_2(0, 1; X)$ of functions of the form $\sum_{n=1}^{\infty} x_n r_n$ where $(r_n)$ is the sequence of Rademacher functions. The space $\text{Rad} X$ is clearly an isomorphic invariant of $X$, and so if two Banach lattices $X$ and $Y$ with nontrivial cotype are isomorphic it follows easily that $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic. However, this result holds in general by a result of Krivine [13] or [16] Theorem 1.f.14.

**Theorem 2.1.** If $X, Y$ are order-continuous Banach lattices and $T : X \rightarrow Y$ is a bounded linear operator, then if $(x_n) \in X(\ell_2)$ we have $(Tx_n) \in Y(\ell_2)$ and

$$(T(x_n))_{\ell_2} \leq K_G(x_n),$$

where, as usual, $K_G$ denotes the Grothendieck constant.

**Proof.** Essentially this is Krivine’s theorem, but we do need to show that if $(x_n) \in X(\ell_2)$ then $(Tx_n) \in Y(\ell_2)$. To see this we show that $(\sum_{k=1}^{n} |Tx_k|^2)^{1/2}$ is norm-Cauchy. In fact if $m > n$ then

$$\left(\sum_{k=1}^{m} |Tx_k|^2\right)^{1/2} - \left(\sum_{k=1}^{n} |Tx_k|^2\right)^{1/2} \leq \left(\sum_{k=n+1}^{m} |Tx_k|^2\right)^{1/2} \leq K_G \left(\sum_{k=n+1}^{m} |x_k|^2\right)^{1/2} \leq K_G \left(\sum_{k=n+1}^{\infty} |x_k|^2\right)^{1/2},$$

which converges to zero as $n \rightarrow \infty$ by the order-continuity of $X$. \(\square\)

**Corollary 2.2.** If two order-continuous Banach lattices $X$ and $Y$ are isomorphic as Banach spaces, then $X(\ell_2)$ and $Y(\ell_2)$ are isomorphic as Banach spaces.

If $X$ is a separable order-continuous nonatomic Banach lattice then $X$ can be represented as (i.e. is linearly and order isomorphic with) a Köthe function space on $0, 1$ in such a way that $L_{\infty} 0, 1 \subset X \subset L_1 0, 1$ and inclusions are continuous. It will then follow that $L_{\infty}$ is dense in $X$, and the dual of $X$ can be represented as a space of functions, namely $X^* = \{f \in L_1 : \int|fg|dt < \infty \text{ for every } g \in X\}$.

Now we are ready to state our main result. Let us observe that for re-arrangement invariant function spaces on $0, 1$ this result was proved in [9] (cf. also [16] 2.d) by a quite different technique.

**Theorem 2.3.** Let $X$ be an order continuous, nonatomic Banach lattice with an unconditional basis. Then $X$ is isomorphic as a Banach space to $X(\ell_2)$.

**Proof.** We will represent $X$ as a Köthe function space on $0, 1$ as described above. Suppose $(\phi_n)_{n=1}^{\infty}$ is a normalized unconditional basis of $X$. Then there is an order-continuous atomic Banach lattice $Y$ which we identify as a sequence space and
operators $U : X \to Y$ and $V : Y \to X$ such that $UV = I_Y$, $VU = I_X$ and $U(e_n) = e_n$ for $n = 1, 2, \ldots$, where $e_n$ denotes the canonical basis vectors in $Y$. We can regard $Y^*$ as a space of sequences and further suppose that $e_n Y^* = e_n$. We will identify $Y(\ell_2)$ as a space of double sequences with canonical basis $(e_{mn})_{m,n=1}^\infty$; thus for any finitely nonzero sequence we have $\sum a_{mn} e_{mn}(\ell_2) = \sum_m (\sum_n |a_{mn}|^2)^{1/2} e_{mY}$.

Let $r_n$ denote the Rademacher functions and for each fixed $f \in X$ note that $(r_n f)$ converges weakly to zero, since for $g \in X^*$ we have $\lim_{n \to \infty} \int r_n f g \, dt = 0$. In particular we have for each $m \in \mathbb{N}$ that $(r_n \phi_m)$ converges weakly to zero. It follows by a standard gliding hump technique that if $\eta = (2\mathcal{A})^{-1}$ then we can find for each $(m, n) \in \mathbb{N}^2$ an integer $k(m, n)$ and disjoint subsets $(A_{mn})$ of $\mathbb{N}$ so that $U(\phi_{r_k(m, n)}) \chi_{A_{mn}} - U(\phi_{r_k(m, n)}) = \eta$.

Identifying $Y^*$ as a sequence space, we let $\psi_m = U^*(e_m)$ and then define $v_{m,n} = \chi_{A_{mn}} U(\phi_{r_k(m, n)}) \in Y$ and $v_{n,m} = \chi_{A_{mn}} V^* (\psi_{r_k(m, n)}) \in Y^*$. Now suppose $(a_{mn})$ is a finitely nonzero double sequence. Then

$$\sum_{m,n} a_{mn} v_{mn} \leq \left( \sum_{m,n} |a_{mn}|^2 U(\phi_{r_k(m, n)})^2 \right)^{1/2} Y$$

$$\leq K_U \sum_{m,n} |a_{mn}|^2 |\phi_{r_k(m, n)}|^2 \right)^{1/2} X$$

$$= K_U \sum_{m,n} \left( \sum_n |a_{mn}|^2 \right) \phi_{r_k(m, n)}^2 \right)^{1/2} X$$

$$= K_U \sum_{m,n} \left( \sum_n |a_{mn}|^2 \right) \phi_{r_k(m, n)}^2 \right)^{1/2} X$$

$$\leq K_U \sum_{m,n} \left( \sum_n |a_{mn}|^2 \right)^{1/2} |e_{m,Y}|^2$$

$$= K_U \sum_{m,n} a_{mn} e_{mn} Y(\ell_2).$$

Here we have used Krivine's theorem twice. It follows that we can define a linear operator $S : Y(\ell_2) \to Y$ by $S e_{mn} = v_{mn}$ and then $S \leq K_U^2 Y$.

Similar calculations yield that for any finitely nonzero double sequence $(b_{mn})$ we have:

$$\sum_{m,n} b_{mn} v_{mn} \leq K_U^2 Y \sum_{m,n} \left( \sum_n |b_{mn}|^2 \right)^{1/2} e_{mY}.$$
that $0/0 = 0$,
\[
\sum_{(m,n) \in F} a_{mn}e_{mn}Y(l_2) = \sum_{m} \beta_m \alpha_m
\]
\[
= \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} a_{mn} l_2^2
\]
\[
= \langle y_l, \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} a_{mn} v_{mn}^* \rangle
\]
\[
\leq y_l^* \sum_{(m,n) \in F} \beta_m \alpha_m^{-1} a_{mn} v_{mn}^* v_{mn}^*
\]
\[
\leq K_2^2 K_4 y_l.
\]
Thus for each $F$ the map $T_F : Y \to Y(l_2)$ given by $T_F y = \sum_{(m,n) \in F} \langle y_l, v_{mn}^* \rangle e_{mn}$ has norm at most $K_2^2 K_4$. More generally, we have $T_F y \leq K_2^2 K_4 x_k y$ where $A_F = \cup_{(m,n) \in F} A_{mn}$.

It follows that for each $y \in Y$ the series $\sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn}$ converges (unconditionally) in $Y(l_2)$. We can thus define an operator $T : Y \to Y(l_2)$ by $T y = \sum_{m,n} \langle y, v_{mn}^* \rangle e_{mn}$ and $T \leq K_2^2$. Now notice that $T S(e_{mn}) = c_{mn} e_{mn}$ where $c_{mn} = \langle v_{mn}, v_{mn}^* \rangle$. But:
\[
\langle v_{mn}, v_{mn}^* \rangle = \langle v_{mn}, Y^* \psi_{m} r_k(m,n) \rangle
\]
\[
\geq \langle U(\psi_{m} r_k(m,n)), Y^* (\psi_{m} r_k(m,n)) \rangle - \eta \psi_{m} \psi_{m}^*
\]
\[
= \langle \phi_{m}, \psi_{m} \rangle - \eta \psi_{m} \psi_{m}^*
\]
\[
\geq 1 - \eta U \geq 1/2.
\]
Thus $T S$ is invertible and so it follows that $Y(l_2)$ is isomorphic to a complemented subspace of $Y$. It then follows from the Pelczynski decomposition technique that $Y \sim Y(l_2)$; more precisely $Y \sim Y(l_2) \oplus W$ for some $W$ and so $Y \sim Y(l_2) \oplus (Y(l_2) \oplus W) \sim Y(l_2) \oplus Y \sim Y(l_2)$.

**Remark.** The order continuity of the Banach lattice $X$ is essential. In [14] a nonatomic Banach lattice $X$ (actually an M-space) was constructed which is isomorphic to $C_0$. In particular $X$ has an unconditional basis but is not isomorphic to $X(l_2)$.

**Theorem 2.4.** Let $Y$ be a Banach space with an unconditional basis. Then $Y$ is isomorphic to an order-continuous nonatomic Banach lattice if and only if $Y \sim Y(l_2)$.

**Remark.** Here again we regard $Y$ as an order-continuous Banach lattice.

**Proof.** One direction follows immediately from Theorem 2.3 and Corollary 2.2. For the other direction, it is only necessary to show that if $Y \sim Y(l_2)$ then $Y$ is isomorphic to order-continuous nonatomic Banach lattice. To this we introduce the space $Y(L_2)$; this is the space of sequences of functions $(f_n)$ in $L_2$ such that $\sum f_n e_n$ converges in $Y$. We set $(f_n)_{Y(L_2)} = \sum f_n e_n$. It is clear that $Y(L_2)$ is an order-continuous Banach lattice. Now if $(g_n)$ is an orthonormal basis of $L_2$ we define $W : Y(l_2) \to Y(L_2)$ by $W(\sum_{m,n} a_{mn} r_k(m,n)) = (\sum_n a_{mn} r_k(m,n))_{m=1}^\infty$ and it is easy to see that $W$ is an isometric isomorphism.
Proposition 2.5. If $X$ is a non-atomic order continuous Banach lattice with unconditional basis, then $X \sim X \oplus X$ and $X \sim X \oplus \mathbb{R}$.

Proof. Both facts follow from Theorem 2.3. \qed

Note that for spaces with unconditional basis both properties do not hold in general (see [5] and [6])

Proposition 2.6. Let $X$ be an order continuous non-atomic Banach lattice with an unconditional basis and let $Y$ be a complemented subspace of $X$. Assume that $Y$ contains a complemented subspace isomorphic to $X$. Then $X \sim Y$.

Proof. The proof is a repetition of the proof of Proposition 2.3.5. of [16]. \qed

3. Hardy spaces

We recall that $H_1(T^n)$ is defined to be the space of boundary values of functions $f$ holomorphic in the unit disk $D$ and such that

$$\sup_{0 < r < 1} \int_{T^n} \left| f(re^{i \theta_1}, re^{i \theta_2}, \ldots, re^{i \theta_n}) \right| \, dt_1 \, dt_2 \ldots \, dt_n < \infty.$$ 

The basic theory of such spaces is explained in [18].

Let us consider first the case $n = 1$. Then $\Re H_1$ is defined to be the space of real functions $f \in L_1(T)$ such that for some $F \in H_1(T)$ we have $\Re F = f$. $\Re H_1$ is normed by $f_1 + \min \{ F \in H_1 : \Re F = f \}$. Then $H_1$ is isomorphic to the complexification of $\Re H_1$, and further when considered as a real space is isomorphic to $\Re H_1$. Further it was shown in [17] that $\Re H_1$ has an unconditional basis and is isomorphic a space of martingales $H_1(\delta)$. To define the space $H_1(\delta)$ let $(h_n)_{n \geq 1}$ be the usual enumeration of the Haar functions on $I = [0, 1)$ normalized so that $h_{n+1} = 1$. Then suppose $f \in L_1$ is of the form $f = \sum a_n h_n$. We define $f_{H_1(\delta)} = f(\sum_n |a_n|^2 h_n^2)^{1/2} dt$ and $H_1(\delta) = \{ f : f_{H_1(\delta)} < \infty \}$.

These considerations can be extended to the case $n > 1$. In a similar way, $H_1(T^n)$ is isomorphic to the complexification of, and is also real-isomorphic to, a martingale space $H_1(\delta^n)$. Here we define for $\alpha \in M = \mathbb{N}^n$ the function $h_\alpha \in L_1(T^n)$ by $h_\alpha(t_1, \ldots, t_n) = \prod h_{\alpha_k}(t_k)$. Then $H_1(\delta^n)$ consists of all $f = \sum_{\alpha \in M} a_\alpha h_\alpha$ such that $f_{H_1(\delta^n)} = f(\sum_n |a_n|^2 h_n^2)^{1/2} dt < \infty$.

It is clear from the definition that the system $(h_\alpha)_{\alpha \in M}$ is an unconditional basis of $H_1(\delta^n)$. We can thus define a space $H_1(\delta^n, \ell_2) = H(\delta^n)(\ell_2)$ as in Section 1; since $H_1(\delta^n)$ has cotype two, this space is isomorphic to $\text{Rad} H_1(\delta^n)$. The following theorem is due to Bourgain [2]:

Theorem 3.1. $H_1(\delta, \ell_2)$ is not isomorphic to a complemented subspace of $H_1(\delta)$.

In a subsequent paper [3] Bourgain implicitly extended this result to higher dimensions.

Theorem 3.2. For every $n = 1, 2, \ldots$ the space $H_1(\delta^n, \ell_2)$ is not isomorphic to any complemented subspace of $H_1(\delta^n)$.

Sketch of proof. For $n = 1$ this Theorem is proved in detail in [2]. The subsequent paper [3] states only the weaker fact that $H_1(\delta^n)$ is not isomorphic to $H_1(\delta^{n+1})$. His proof however gives the above Theorem as well. All that is needed is to change in Section 3 of [3] condition $(m+1)$ and Lemma 4. Before we formulate the appropriate
condition we need some further notation. By $BMO(\delta^n)$ we will denote the dual of $H_1(\delta^n)$ and by $BMO(\delta^n, \ell_2)$ we will denote the dual of $H_1(\delta^n, \ell_2)$. The space $H_1(\delta^n, \ell_2)$ has an unconditional basis given by $(h_\alpha \otimes e_k)_{\alpha \in \mathcal{M}, k \in \mathbb{N}}$. In our notation from Section 2 $h_\alpha \otimes e_k$ is a sequence of $H_1(\delta^n)$-functions which consists of zero functions except at the $k$-th place where there is $h_\alpha$. The same element can be treated as an element of the dual space. Note that the natural duality gives

$$\langle h_\alpha \otimes e_k, h_{\alpha'} \otimes e_{k'} \rangle = \begin{cases} \int_{\mathbb{R}^n} y_{h_\alpha}^1, & \text{when } \alpha = \alpha' \text{ and } k = k' \\ 0, & \text{otherwise.} \end{cases}$$

Now we are ready to state the new condition (m+1): Let $\Phi : H_1(\delta^n, \ell_2) \rightarrow H_1(\delta^n)$ and $\Phi^\times : BMO(\delta^n, \ell_2) \rightarrow BMO(\delta^n)$ be bounded linear operators (note that $\Phi^\times$ is not the adjoint of $\Phi$). Then for every $\varepsilon > 0$ there exists a set $A \subseteq \mathcal{M}$ such that $\sum_{\alpha \in A} \|h_\alpha\| = 1$ and integers $k_\alpha$ for $\alpha \in A$ such that

$$\sum_{\alpha \in A} \int_{\mathbb{R}^n} \|\Phi(h_\alpha \otimes e_{k_\alpha})\| \cdot \|\Phi^\times(h_\alpha \otimes e_{k_\alpha})\| < \varepsilon.$$ 

With this condition one can repeat the proof from [3] and obtain the Theorem. \(\square\)

**Corollary 3.3.** We have the following

$$\ell_2 \subset H_1(\delta) \subset H_1(\delta, \ell_2) \subset H_1(\delta^n) \subset H_1(\delta^n, \ell_2) \subset \ldots$$

where $X \subset Y$ means that $X$ is isomorphic to a complemented subspace of $Y$ but $Y$ is not isomorphic to a complemented subspace of $X$.

**Proof.** It is well known and easy to check that the map $h_\alpha \otimes e_k \mapsto h_\alpha(t_1, \ldots, t_n) \cdot r_k(t_{n+1})$ where $r_k$ is the $k$-th Rademacher function gives the desired complemented embedding. That no smaller space is isomorphic to a complemented subspace of a bigger one is the above theorem of Bourgain. \(\square\)

**Corollary 3.4.** The spaces $H_1(\delta^n)$ is not isomorphic to a nonatomic Banach lattice for $n = 1, 2, \ldots$. The spaces $H_1(\delta^n, \ell_2)$ are each isomorphic to a nonatomic Banach lattice.

**Proof.** The first claim follows directly from Theorem 3.1, 3.2 and Theorem 2.3. We only have to observe that (since $H_1(\delta^n)$ does not contain any subspace isomorphic to $c_0$ and indeed has cotype two) any Banach lattice isomorphic as a Banach space to $H_1(\delta^n)$ is order continuous (see Theorem 1.c.4 of [16]). The second claim follows from Corollary 2.4. \(\square\)

**Remark.** For $H_p(\mathbb{T}^n)$ with $0 < p < \infty$ we have the following situation. When $1 < p < \infty$ the orthogonal projection from $L_p(\mathbb{T}^n)$ onto $H_p(\mathbb{T}^n)$ is bounded so then $H_p(\mathbb{T}^n)$ is isomorphic to $L_p(\mathbb{T}^n)$. This implies in particular that these spaces are isomorphic to nonatomic lattices. When $0 < p < 1$ then $H_p(\mathbb{T}^n)$ admit only purely atomic orders as $p$-Banach lattices. To see this observe that if $X$ is not a purely atomic $p$-Banach lattice then its Banach envelope (for definition and properties see [11]) is a Banach lattice which is not purely atomic. On the other hand it is known
that the Banach envelope of $H_p(T^n)$ is isomorphic to $\ell_1$. For $n = 1$ this can be found in [11] Theorem 3.9, for $n > 1$ the proof uses Theorem 2′ of [19] but otherwise is the same; alternatively see [11] Theorem 3.5, for a proof using bases. When we compare it with the observation from [1] mentioned in the Introduction, that $\ell_1$ is not isomorphic to any nonatomic Banach lattice, we conclude that the spaces $H_p(T^n)$ cannot be isomorphic to any nonatomic $p$-Banach lattice.

**Remark.** For the dual spaces $H_1(T^n)^* = BMO(T^n)$ the situation is rather different. We first observe the following proposition:

**Proposition 3.5.** For any Banach space $X$ the spaces $\ell_1(X)^∗(= \ell_\infty(X^*))$ and $L_1(0,1;X)^*$ are isomorphic.

**Proof.** Clearly $\ell_1(X)^*$ is isomorphic to a 1-complemented subspace of $L_1(X)^*$. Now let $\chi_{n,k} = \chi_{[(k-1)2^\pi,k2^\pi)}$ for $1 \leq k \leq 2^n$ and $n = 0, 1, \ldots$. Let $T : \ell_1(X) \to L_1(X)$ be defined by $T((x_n)) = \sum x_n \chi_{n,k}$ where $n = 2^m + k - 1$. Let $L_1(D_N;X)$ be the subspace of all functions measurable with respect to the finite algebra generated by the sets $((k-1)2^{-N},k2^{-N})$ for $1 \leq k \leq 2^N$ and define $S_N : L_1(D_N;X) \to \ell_1(X)$ by setting $S(x \otimes \chi_{N,k})$ to be the element with $x$ in position $2^N + k - 1$ and zero elsewhere. Then applying Exercise 7 of [6] of [22] (cf. [8] Proposition 1), we obtain that $L_1(X)^*$ is isomorphic to a complemented subspace of $\ell_1(X)^*$. Then by the Pelczynski decomposition technique we obtain the proposition. □

Now from the Proposition, observe that, since $H_1(T^n)^* \sim \ell_1(H_1(T^n))$, we have $L_1(H_1(T^n))^* \sim BMO(T^n)$ and clearly this isomorphism induces a nonatomic (but not order-continuous) lattice structure on $BMO(T^n)$. (It is easy to see that a space which contains a copy of $\ell_\infty$ cannot have an order-continuous lattice structure, because it fails the separable complementation property.)

4. **Rad $H_1$ and tent spaces**

The space $H_1(\delta, \ell_2)$ is, as observed in Section 2, isomorphic to Rad $H_1$ and has a structure as a nonatomic Banach lattice. The complex space Rad $H_1$ is easily seen to be isomorphic to the vector-valued space $H_1(T,\ell_2)$ consisting of the boundary values of the space of all functions $F$ analytic in the unit disk $D$ with values in a Hilbert space $\ell_2$ and such that:

$$\sup_{0<r<1} \int_0^{2\pi} F(re^{i\theta}) d\theta = F < \infty.$$ 

To see this isomorphism just note that $H_1(T,\ell_2)$ can be identified with the space of sequences $(f_n)$ in $H_1$ such that

$$(f_n) = \int_0^{2\pi} \left( \sum_{n=1}^{\infty} |f_n(e^{i\theta})|^2 \right)^{1/2} d\theta = \infty.$$ 

This is in turn easily seen to be equivalent to the norm of $\sum r_n f_n$ in $L_2(0,1;H_1)$ (see [16] Theorem 1.d.6).

We now show that a nonatomic Banach lattice isomorphic to Rad $H_1$ arises naturally in in harmonic analysis. More precisely we will show that tent space $T^1$ which was introduced and studied by R. Coifman, Y. Meyer and E. Stein in [4] is isomorphic to Rad $H_1$. Tent spaces are useful in some questions of harmonic
analysis (cf. [7] or [21]). They can be defined over \( \mathbb{R}^n \) but for the sake of simplicity we will consider them only over \( \mathbb{R} \).

Let us fix \( \alpha > 0 \). For \( x \in \mathbb{R} \) we define

\[
\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R} \times \mathbb{R}^+ : |x - y| < \alpha t\}.
\]

Given a function \( f(y, t) \) defined on \( \mathbb{R} \times \mathbb{R}^+ \) we put

\[
f_\alpha = \int_\mathbb{R} \left( \int_{\Gamma_\alpha(x)} |f(y, t)| t^{-2} \, dy \, dt \right)^{1/2} \, dx.
\]

It was shown in [4] Proposition 4 that for different \( \alpha \)'s the norms \( f_\alpha \) are equivalent i.e. for \( 0 < \alpha < \beta < \infty \) there is a \( C = C(\alpha, \beta) \) such that for every \( f \) we have

\[(4.1) \quad f_\alpha \leq f_\beta \leq Cf_\alpha.
\]

This implies that the space \( T^1 = \{ f(y, t) : f_\alpha < \infty \} \) does not depend on \( \alpha \). Observe that \( T^1 \) is clearly a non-atomic Banach lattice.

Our main result of this Section is

**Theorem 4.1.** The space \( T^1 \) is lattice-isomorphic to \( H_1(\delta, L_2) \) and hence isomorphic to \( \text{Rad} H_1 \).

Actually for the proof of this Theorem it is natural to work with the dyadic \( H_1 \) space on \( \mathbb{R} \). This space, which we denote \( H_1(\delta, \infty) \) can be defined as follows:

Let \( I_{nk} = k \cdot 2^n, (k + 1) \cdot 2^n \) for \( n, k = 0, \pm 1, \pm 2 \ldots \) and let \( h_{nk} \) be the function which is equal to 1 on the left hand half of \( I_{nk} \), -1 on the right hand half of \( I_{nk} \) and zero outside \( I_{nk} \). In other words \( h_{nk} \) is the Haar system on \( \mathbb{R} \). The system \( \{h_{nk}\}_{n,k=0,\pm 1,\pm 2,\ldots} \) is a complete orthogonal system. For a function \( f = \sum_{n,k} a_{nk} h_{nk} \) we define its \( H_1(\delta, \infty) \)-norm by

\[(4.2) \quad f = \int_\mathbb{R} \left( \sum_{n,k} |a_{nk}|^2 |h_{nk}|^2 \right)^{1/2} \, dt.
\]

That this space is isomorphic to the space \( H_1(\delta) \) follows from the work of Sjölin and Stromberg [20]. However, slightly more is true:

**Lemma 4.2.** The atomic Banach lattices \( H_1(\delta) \) and \( H_1(\delta, \infty) \) are lattice-isomorphic (or, equivalently the natural normalized unconditional bases of these spaces are permutable equivalent).

**Proof.** For any subset \( A \) of \( \mathbb{Z}^2 \) write \( H_A \) for the closed linear span of \( \{ h_{nk} : (n, k) \in A \} \) in \( H_1(\delta, \infty) \). For \( m \in \mathbb{Z} \) let \( A_m = \{(n, k) : I_{nk} \subset 2^{-m-1}, 2^{-m}\} \) and \( B_m = \{(n, k) : I_{nk} \subset -2^{-m}, -2^{-m-1}\} \). Let \( D = \bigcup_{m \in \mathbb{Z}} (A_m \cup B_m) \) and \( D_+ = \bigcup_{m \geq 0} A_m \). Then it is clear that \( H_D \) and \( H_{D_+} \) are each lattice isomorphic to \( \ell_1(\ell_1(\delta)) \). Now \( H_1(\delta, \infty) \) is lattice isomorphic to \( H_D \oplus H_E \) where \( E = \{(m, 0), (m, -1) : m \in \mathbb{Z}\} \). It is easy to show that \( H_E \) is lattice isomorphic to \( \ell_1(\ell_1) \). Similarly \( H_1(\delta) \) is lattice-isomorphic to \( H_1(D_+) \oplus \ell_1 \) and this completes the proof of the lemma. \( \square \)

**Remark.** Note also that \( H_1(\delta) \) is lattice-isomorphic to \( \ell_1(H_1(\delta)) \).

**Proof of the Theorem.** We will prove that \( T^1 \) is lattice-isomorphic to \( H_1(\delta, \infty, L_2) \). Let us introduce squares \( A_{nk} \subset \mathbb{R} \times \mathbb{R}^+ \) defined as \( A_{nk} = I_{nk} \times 2^n, 2^{n+1} \) for
It is geometrically clear that squares \( \{A_{nk}\}_{n,k=0,\pm1,\pm2,...} \) are essentially disjoint and that they cover \( \mathbb{R} \times \mathbb{R}^+ \). For \( j = 0, 1, 2 \) we define

\[
A_{nk}^j = (k + \frac{1}{3})2^n, (k + \frac{2+j}{3})2^n \times 2^n, 2^{n+1}.
\]

Note that in this way we divide each \( A_{nk} \) into three essentially disjoint rectangles. Let \( D_j = \bigcup_{n,k} A_{nk}^j \). Let \( T^1_j \) be the subspace of \( T^1 \) consisting of all functions whose support is contained in \( D_j \). Clearly \( T^1 = T^1_0 \oplus T^1_1 \oplus T^1_2 \), so it is enough to show that \( T^1_j \) is lattice-isomorphic to \( H_1(\delta_{\infty}, L_2) \).

We write \( f \in T^1_j \) as \( f = \sum_{n,k} f_{nk}^j \) where \( f_{nk}^j = f^j \cdot \chi_{A_{nk}^j} \). We start with \( j = 1 \).

For any \( \alpha > 0 \) we have

\[
\int_{\Gamma_\alpha(x)} f^1 \text{d}x = \int_{\mathbb{R}} \left( \int_{\Gamma_{\alpha}(x)} |f^1(y, t)|^2t^{-2} \text{d}y \text{d}t \right)^{1/2} \text{d}x
\]

\[
= \int_{\mathbb{R}} \left( \int_{\Gamma_{\alpha}(x)} \sum_{n,k} |f^1_{nk}(y, t)|^2t^{-2} \text{d}y \text{d}t \right)^{1/2} \text{d}x
\]

\[
= \int_{\mathbb{R}} \left( \sum_{n,k} \int_{\Gamma_{\alpha}(x)} |f^1_{nk}(y, t)|^2t^{-2} \text{d}y \text{d}t \right)^{1/2} \text{d}x.
\]

If we now take \( \alpha = \frac{2}{3} \) we have \( \Gamma_{\alpha}(x) \supset A_{nk}^1 \) for all \( x \in I_{nk} \), so from (4.3) we get

\[
\int_{\Gamma_\alpha(x)} f^1 \text{d}x \geq \int_{\mathbb{R}} \left( \sum_{n,k} \chi_{I_{nk}}(x) \int_{A_{nk}^1} |f^1_{nk}(y, t)|^2t^{-2} \text{d}y \text{d}t \right)^{1/2} \text{d}x.
\]

On the other hand when we take \( \alpha = \frac{1}{6} \) we have \( \Gamma_{\alpha}(x) \cap A_{nk}^1 = \emptyset \) for all \( x \notin I_{nk} \), so from (4.3) we get

\[
\int_{\Gamma_\alpha(x)} f^1 \text{d}x \leq \int_{\mathbb{R}} \left( \sum_{n,k} \chi_{I_{nk}}(x) \int_{A_{nk}^1} |f^1_{nk}(y, t)|^2t^{-2} \text{d}y \text{d}t \right)^{1/2} \text{d}x.
\]

For each \( (n, k) \) the subspace of \( T^1 \) consisting of functions supported on \( A_{nk}^1 \) is easily seen to be isometric to the Hilbert space. If we fix an isometry between this space and \( \ell_2 \) we obtain from (4.2), (4.3) and (4.4) that \( T^1_j \) is lattice-isomorphic to \( H_1(\delta_{\infty}, L_2) \). In order to complete the proof of the Theorem it is enough to show that \( T^1_0 \) and \( T^1_2 \) are lattice-isomorphic to \( T^1_1 \). This isomorphism can be given by \( \sum_{nk} f_{nk}^j \mapsto \sum_{nk} f_{nk}^1 \). The fact that this map is really an isomorphism follows from:

**Lemma 4.3.** Let \( \phi(t) \) be a uniformly bounded measurable function on \( \mathbb{R}^+ \). For a function \( f \) defined on \( \mathbb{R} \times \mathbb{R}^+ \) we define

\[
A_{\phi}(f)(y, t) = f(y + t\phi(t), t).
\]

Then \( A_{\phi} : T^1 \rightarrow T^1 \) is a continuous linear operator.
Proof of the Lemma. Since

\[ \int_{\Gamma_{\alpha}(x)} |A_{\alpha}(f)(y, t)|^2 t^{-2} dt \]
\[ = \int_{\mathbb{R}^+} (t^{-2} \int_{x-\alpha t}^{x+\alpha t} |A_{\alpha}(f)(y, t)|^2 dy) dt \]
\[ = \int_{\mathbb{R}^+} (t^{-2} \int_{x-\alpha t}^{x+\alpha t} |f(y, t)|^2 dy) dt \]
\[ \leq \int_{\mathbb{R}^+} (t^{-2} \int_{x-(\phi_{-\alpha}+\alpha) t}^{x+(\phi_{-\alpha}+\alpha) t} |f(y, t)|^2 dy) dt \]
\[ = \int_{\Gamma_{\alpha+\alpha}(x)} |f(y, t)|^2 t^{-2} dy dt \]

the Lemma follows. \[ \square \]

REFERENCES


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