NONHYPERBOLIC STEP SKEW-PRODUCTS: ENTROPY SPECTRUM OF LYAPUNOV EXPONENTS

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Abstract. We study the fiber Lyapunov exponents of step skew-product maps over a complete shift of N, $N \geq 2$, symbols and with C^1 diffeomorphisms of the circle as fiber maps. The systems we study are transitive and genuinely nonhyperbolic, exhibiting simultaneously ergodic measures with positive, negative, and zero exponents. We derive a multifractal analysis for the topological entropy of the level sets of Lyapunov exponent. The results are formulated in terms of Legendre-Fenchel transforms of restricted variational pressures, considering hyperbolic ergodic measures only, as well as in terms of restricted variational principles of entropies of ergodic measures with given exponent. We show that the entropy of the level sets is a continuous function of the Lyapunov exponent. The level set of zero exponent has positive, but not maximal, topological entropy. Under the additional assumption of proximality, there exist two unique ergodic measures of maximal entropy, one with negative and one with positive fiber Lyapunov exponent.

Contents

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1. Introduction

We will study the entropy spectrum of Lyapunov exponents, that is, the topological entropy of level sets of points with a common given Lyapunov exponent. This subject forms part of the multifractal analysis which, in general, studies thermodynamical quantities and objects (such as, for example, equilibrium states, entropies, Lyapunov exponents, Birkhoff averages, and recurrence rates) and their relations with geometrical properties (for example, fractal dimensions). Those properties are often encoded by the topological pressure.

In the uniformly hyperbolic context multifractal analysis is understood in great depth and has found already far reaching applications. There is a huge literature on this subject. To highlight a collection of results in the field at different stages of development, we refer, for example, to [R] (analyticity of pressure and its consequences), [O, PW] (multifractal analysis for conformal expanding maps and Smale's horsehoes), and [BS] (mixed spectra and restricted variational principles). In many of those references, particular attention is drawn to so-called geometric potentials because of their close relation to Lyapunov exponents, entropy, and SRB measures. Two key properties of uniformly hyperbolic systems, under which the classical context of multifractal analysis was developed so far, are the specification property (studied for example in [TV, PS, FLP]) and the existence and uniqueness of equilibrium states.

The multifractal analysis theory extends also to "one-sided" nonuniformly hyperbolic systems, that is, for example to nonuniformly expanding maps where the presence of a nonpositive Lyapunov exponent is the only obstruction to hyperbolicity, that is, the spectrum of Lyapunov exponents covers a range of hyperbolicity and the zero exponent bounds this range from one side, see for example [GPR] (expansive Markov maps of the interval) and [PR, IT] (multimodal interval maps). So far, there is not much understanding of a multifractal analysis for more complicated types of nonhyperbolic systems. It is difficult to describe all the situations that can happen in general; one natural class of systems to focus on could be the systems with a designated line field (associated with the Oseledets decomposition) for which the Lyapunov exponent takes both positive and negative values arbitrarily close to zero. Naturally, we assume topological transitivity, hence the system in question cannot split into "one-sided" nonuniformly hyperbolic parts.

Probably, the simplest setting of such a "two-sided" nonhyperbolic dynamical system is a step skew-product with a hyperbolic horseshoe map in its base and circle diffeomorphisms in its fibres. The nonuniform hyperbolicity arises from the coexistence of contracting and expanding regions which are blended by the dynamics.

The system exhibits ergodic measures which positive and negative fiber Lyapunov exponents. An important feature is the occurrence of ergodic *nonhyperbolic measures* (i.e., with zero Lyapunov exponent) with positive entropy. The considered dynamics is topologically transitive and simultaneously has "horseshoes" which are contracting and "horseshoes" which are expanding in the fiber direction. Moreover, these horseshoes are intermingled and there coexist dense sets of periodic points with negative and positive fiber Lyapunov exponents. The precise setting is discussed in Section 3.

The present paper is a continuation of [DGR_2] where properties of the space of invariant measures were investigated. Here we will concentrate on the multifractal analysis of the entropy of the level sets of fiber Lyapunov exponents. We follow a thermodynamic approach based on a restricted variational principle. The philosophy is that to obtain relevant multifractal information about the respective class of exponents one should not consider the whole variational-topological pressure, but instead its restrictions to ergodic measures with corresponding exponents, so-called *restricted pressures*. The use of restricted (sometimes also called *hidden*) pressures was initiated in [MS] (for rational maps of the Riemann sphere) and subsequently used, for example, in [GPR] (for non-exceptional rational maps) and [PR] (for multimodal interval maps). As the difficulty in our setting comes from the coexistence of negative, zero, and positive fiber Lyapunov exponents and as zero exponent measure are notoriously difficult to analyze, a natural solution is to consider the restricted pressures defined on the ergodic measures with negative and positive exponents, respectively (it turns out that zero exponent ergodic measures do not play a significant role). This approach is made possible by the fact that the uniformly hyperbolic subsystems with negative/positive Lyapunov exponents (but not the system as the whole) satisfy the specification property.

Summarizing our results: the spectrum of Lyapunov exponents is a closed interval $[\alpha_{\min}, \alpha_{\max}]$ that contains zero in its interior. For each $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ the level set $\mathcal{L}(\alpha)$ of points with fiber exponent equal to α is nonempty and its topological entropy changes continuously with α (see Figure 1). The entropy spectrum of fiber

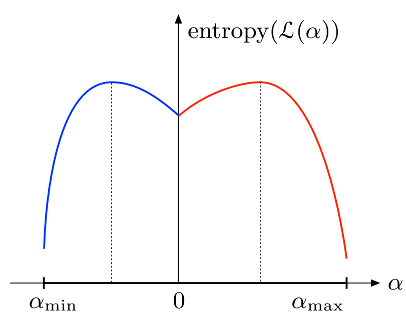

FIGURE 1. The entropy spectrum

Lyapunov exponents is described in terms of those restricted pressure functions and their respective Legendre-Fenchel transforms.

To obtain our results, we combine a thermodynamical and an orbitwise approach. On the one hand, we study the restricted variational pressure functions and extract properties from its shape. Here, one of the big selling points of this approach is that it gives us convexity for free, which turns out to be a surprisingly useful property. On the one hand, in our approach we put our hands on the orbits of the level sets (the amount of their entropy provides explicit information about them), using natural recurrence properties of the systems (which is guided by the concept of so-called blending intervals in [DGR$_2$]), and follow the "orbit-gluing approach" (which lead us to the notion of skeletons of the dynamics in [DGR$_2$]). We point out that we always work in the lowest possible regularity and consider C^1 circle diffeomorphisms as fiber maps.

Let us return to the discussion of the zero Lyapunov exponent. The greatest obstacle in our investigation, as usual in the study of nonhyperbolic systems, are points and measures with zero exponent. While for nonzero exponents we can give the full description of the Lyapunov exponent level sets, including the restricted variational principle and the exact formula for their entropy, we have very restricted tools for studying the zero exponent level set $\mathcal{L}(0)$. We are able to describe the entropy of this set, but the restricted variational principle cannot be obtained by our methods. Let us observe that the fact that $\mathcal{L}(0)$ has positive topological entropy was obtained in a similar context in [BBD] by proving the existence of ergodic measures with positive entropy and zero exponents. In this paper this property is obtained as a surprising consequence of the shape of the pressure map. Though positive, we also show that the topological entropy of $\mathcal{L}(0)$ is strictly smaller than the maximal, that is, the topological entropy of the system.

The systems we study always have (at least) two hyperbolic ergodic measure of maximal entropy, one with negative and one with positive fiber Lyapunov exponent. Indeed, this is an immediate consequence of [C], obtained from a different point of view of our system as a random dynamical system, that is, as a product of independent and identically distributed circle diffeomorphisms, also observing the fundamental fact that our hypotheses exclude the case that our system is a rotation extension of a Bernoulli shift. It is a particular case of a result in a more general setting [RH^{2}TU], stated for accessible partially hyperbolic diffeomorphisms having compact center leaves, see also [TY]. Under the additional assumption of proximality, with [Ml] we even can conclude uniqueness of ergodic measure of maximal entropy with negative and positive exponents.

Finally, we point out that the systems that we study are models for robustly transitive and nonhyperbolic diffeomorphisms and sets with compact central leaves [BDU, RH^{2}TU]. From another point of view, which in fact provides some of our tools, the systems can be also considered as actions of a group of diffeomorphisms on the circle or as random dynamical systems.

2. Statement of the results

Let $\sigma\colon \Sigma_N \to \Sigma_N$, $N \geq 2$, be the usual shift map on the space $\Sigma_N = \{0, \ldots, N-1\}^{\mathbb{Z}}$ of two-sided sequences. We equip the shift space Σ_N with the standard metric $d_1(\xi, \eta) = 2^{-n(\xi,\eta)}$, where $n(\xi, \eta) = \sup\{|\ell|\colon \xi_i = \eta_i \text{ for } i = -\ell, \ldots, \ell\}$. We equip $\Sigma_N \times \mathbb{S}^1$ with the metric $d((\xi, x), (\eta, y)) = \sup\{d_1(\xi, \eta), |x - y|\}$, where $|\cdot|$ is the usual metric on $\mathbb{S}^1$.

Consider a finite family $f_i\colon \mathbb{S}^1 \to \mathbb{S}^1$, $i = 0, \ldots, N-1$, of C^1 diffeomorphisms and the associated step skew-product

$$F\colon \Sigma_N \times \mathbb{S}^1 \to \Sigma_N \times \mathbb{S}^1, \quad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)). \tag{2.1}$$

We will consider a class of maps which are topologically transitive and "nonhyperbolic in a nontrivial sense that there are some "expanding region" and some "contracting region" (relative to the fiber direction) and that any of those can be reached from anywhere in the ambient space under forward/backward iterations. More precisely, we will require F to satisfy Axioms CEC$\pm$ and Acc$\pm$ (see Section 3).

Let $\mathcal{M}$ be the space of F-invariant probability measures supported in $\Sigma_N \times \mathbb{S}^1$, equip $\mathcal{M}$ with the weak$*$ topology, and denote by $\mathcal{M}_{\rm erg} \subset \mathcal{M}$ the subset of ergodic measures. To characterize nonhyperbolicity, given $\mu \in \mathcal{M}$ denote by $\chi(\mu)$ its *(fiber) Lyapunov exponent* which is given by

$$\chi(\mu) \stackrel{\rm def}{=} \int \log |(f_{\xi_0})'(x)|\, d\mu(\xi, x).$$

An ergodic measure μ is *nonhyperbolic* if $\chi(\mu) = 0$. Otherwise the measure is *hyperbolic*. In our setting any hyperbolic ergodic measure has either a negative or a positive exponent. Accordingly, we split the set of all *ergodic* measures and consider the decomposition

$$\mathcal{M}_{\rm erg} = \mathcal{M}_{{\rm erg},<0} \cup \mathcal{M}_{{\rm erg},0} \cup \mathcal{M}_{{\rm erg},>0} \tag{2.2}$$

into measures with negative, zero, and positive fiber Lyapunov exponent, respectively. In our setting, each component is nonempty. In general, it is very difficult to determine which type of hyperbolicity "prevails". For that we will study the spectrum of possible exponents and will perform a multifractal analysis of the topological entropy of level sets of equal (fiber) Lyapunov exponent.

To be more precise, a sequence $\xi = (\ldots \xi_{-1}.\xi_0\xi_1 \ldots) \in \Sigma_N$ can be written as $\xi = \xi^-.\xi^+$, where $\xi^+ \in \Sigma_N^+ \stackrel{\rm def}{=} \{0, \ldots, N-1\}^{\mathbb{N}_0}$ and $\xi^- \in \Sigma_N^- \stackrel{\rm def}{=} \{0, \ldots, N-1\}^{-\mathbb{N}}$. Given *finite* sequences $(\xi_0 \ldots \xi_n)$ and $(\xi_{-m} \ldots \xi_{-1})$, we let

$$f_{[\xi_0 \ldots\, \xi_n]} \stackrel{\rm def}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_0} \quad \text{and} \quad f_{[\xi_{-m} \ldots\, \xi_{-1}.]} \stackrel{\rm def}{=} (f_{[\xi_{-m} \ldots\, \xi_{-1}]})^{-1}.$$

For $n \geq 0$ denote also

$$f_\xi^n \stackrel{\rm def}{=} f_{[\xi_0 \ldots\, \xi_{n-1}]} \quad \text{and} \quad f_\xi^{-n} \stackrel{\rm def}{=} f_{[\xi_{-n} \ldots\, \xi_{-1}.]}.$$

Given $X = (\xi, x) \in \Sigma_N \times \mathbb{S}^1$ consider the *(fiber) Lyapunov exponent* of X

$$\chi(X) \stackrel{\rm def}{=} \lim_{n \to \pm\infty} \frac{1}{n} \log |(f_\xi^n)'(x)|,$$

where we assume that both limits $n \to \pm\infty$ exist and coincide. Note that in our context the exponent is nothing but the Birkhoff average of the continuous function (also called *potential*) $\varphi\colon \Sigma_N \times \mathbb{S}^1 \to \mathbb{R}$ defined for $X = (\xi, x)$ by

$$\varphi(X) \stackrel{\rm def}{=} \log |(f_{\xi_0})'(x)|. \tag{2.3}$$

We will analyze the topological entropy of the following *level sets of Lyapunov exponents*: given $\alpha \in \mathbb{R}$ let

$$\mathcal{L}(\alpha) \stackrel{\rm def}{=} \big\{X \in \Sigma_N \times \mathbb{S}^1 \colon \chi(X) = \alpha\big\}$$

assuming that the Lyapunov exponent at X is well defined and equal to α. Note that each level set is invariant but, in general, noncompact. Hence we will rely on the general concept of topological entropy $h_{\rm top}$ introduced by Bowen [B$_1$] (see Appendix). Denoting by $\mathcal{L}_{\rm irr}$ the set of points where the fiber Lyapunov exponent is not well-defined (either one of the limits does not exist or both limits exist but they do not coincide), we obtain the following *multifractal decomposition* of $\Sigma_N \times \mathbb{S}^1$

$$\Sigma_N \times \mathbb{S}^1 = \bigcup_{\alpha\in\mathbb{R}} \mathcal{L}(\alpha) \cup \mathcal{L}_{\rm irr}.$$

Note that $\mathcal{L}(\alpha)$ will be nonempty in some range of α, only. Under our axioms this range decomposes into three natural nonempty parts

$$\{\alpha\colon \mathcal{L}(\alpha)\neq\varnothing\} = [\alpha_{\min},0)\cup\{0\}\cup(0,\alpha_{\max}],$$

where

$$\alpha_{\max} \stackrel{\rm def}{=} \max\big\{\alpha\colon \mathcal{L}(\alpha)\neq\varnothing\big\},\quad \alpha_{\min} \stackrel{\rm def}{=} \min\big\{\alpha\colon \mathcal{L}(\alpha)\neq\varnothing\big\}.$$

It is easy to verify that max and min are indeed attained.

To state our main results, we need the following thermodynamical quantities. Denote by $h(\mu)$ the *entropy* of a measure μ and consider the pressures and their convex conjugates (see Section 4 for details)

$$P_*(q\varphi) \stackrel{\rm def}{=} \sup_{\mu\in\mathcal{M}_{{\rm erg},*}} \big(h(\mu)-q\chi(\mu)\big),\quad \mathcal{E}_*(\alpha) \stackrel{\rm def}{=} \inf_{q\in\mathbb{R}} \big(P_*(q\varphi)-q\alpha\big), \tag{2.4}$$

where $*$ should be replaced by <0 and >0, respectively. In the terminology of [PRS], this would be called (*positive/negative*) *variational hyperbolic pressure*, we call it simply *pressure*. For simplicity we will use the notation

$$\mathcal{P}_*(q) \stackrel{\rm def}{=} P_*(q\varphi),$$

as this is the only family of potentials whose pressure we are going to consider. Similarly, we define

$$\mathcal{P}_0(q) \stackrel{\rm def}{=} \sup_{\mu\in\mathcal{M}_{{\rm erg},0}} h(\mu).$$

Clearly,

$$\max\{\mathcal{P}_{<0}(q),\mathcal{P}_0(q),\mathcal{P}_{>0}(q)\} = P_{\rm top}(q\varphi)$$

is the classical *topological pressure* of $q\varphi$ with respect to F (see [Wa, Chapter 7]). We will also write $\mathcal{E}$ for both $\mathcal{E}_{>0}$ and $\mathcal{E}_{<0}$, because the domains of those two functions are disjoint.

Theorem 1. *Consider a transitive step skew-product map F as in* (2.1) *whose fiber maps are C^1. Assume that F satisfies Axioms CEC± and Acc±.*

Then for every $\alpha\in[\alpha_{\min},\alpha_{\max}]$ we have $\mathcal{L}(\alpha)\neq\varnothing$. Moreover,

- *for every $\alpha\in(\alpha_{\min},0)$ we have*
$$h_{\rm top}(\mathcal{L}(\alpha)) = \sup\big\{h(\mu)\colon \mu\in\mathcal{M}_{\rm erg},\chi(\mu)=\alpha\big\} = \mathcal{E}_{<0}(\alpha),$$
- *for every $\alpha\in(0,\alpha_{\max})$ we have*
$$h_{\rm top}(\mathcal{L}(\alpha)) = \sup\big\{h(\mu)\colon \mu\in\mathcal{M}_{\rm erg},\chi(\mu)=\alpha\big\} = \mathcal{E}_{>0}(\alpha),$$
- *for every $\alpha\in\{\alpha_{\min},0,\alpha_{\max}\}$ we have*
$$\lim_{\beta\to\alpha} h_{\rm top}(\mathcal{L}(\beta)) = h_{\rm top}(\mathcal{L}(\alpha)),$$
- $h_{\rm top}(\mathcal{L}(0))>0$.

Moreover, there exist (finitely many) ergodic measures μ_+,μ_- of maximal entropy $h(\mu_\pm)=\log N$ and with $\chi(\mu_-)<0<\chi(\mu_+)$.

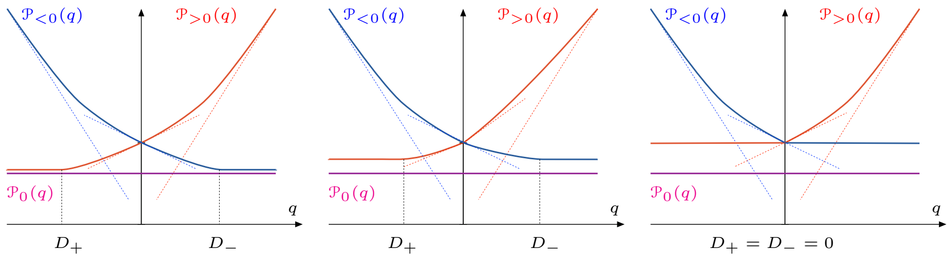

FIGURE 2. Pressures. Left figure: Under the hypothesis of Theorem 2

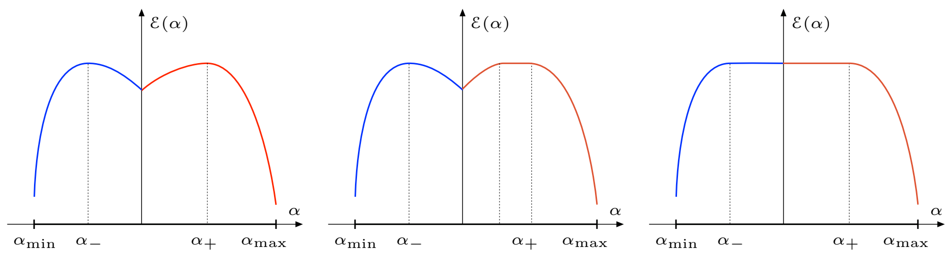

FIGURE 3. Entropy. Left figure: Under the hypothesis of Theorem 2

To prove uniqueness of the measures $\mu_\pm$ of maximal entropy, we require an additional assumption (see Section 8.1 for discussion). We say that the IFS is *proximal* if for every $x, y \in \mathbb{S}^1$ there exists at least one sequence $\xi \in \Sigma_N$ such that $|f_\xi^n(x) - f_\xi^n(y)| \to 0$ as $|n| \to \infty$.

It is easy to see that the IFS is proximal if, for example, it contains a map with one attracting and one contracting fixed point (contraction-expansion map; North pole-South pole map) and the IFS satisfies our other axioms CEC$\pm$ and Acc$\pm$.

Theorem 2. *Assume the hypothesis of Theorem 1. Assume also that the IFS is proximal. Then there exist unique ergodic F-invariant probability measures μ_- and μ_+ of maximal entropy $h(\mu_\pm) = \log N$, respectively, and satisfying*

$$\alpha_- \stackrel{\text{def}}{=} \chi(\mu_-) < 0 < \alpha_+ \stackrel{\text{def}}{=} \chi(\mu_+).$$

We have

$$h_{\text{top}}(\mathcal{L}(\alpha_-)) = h_{\text{top}}(\mathcal{L}(\alpha_+)) = \log N$$

and

$$h_{\text{top}}(\mathcal{L}(\alpha)) < \log N$$

for all $\alpha \neq \alpha_-, \alpha_+$.

Similar phenomenon as in Theorem 2 (the entropy achieving its maximum away from zero exponent) in a slightly different setting (for ergodic measures on C^2 systems) was observed in [TY]. The weak$*$ and entropy convergence means that the sequence of measures converges in the weak$*$ topology and their entropies also converge to the entropy of the limit measure.

Corollary 3. *Under the hypothesis of Theorem 2 no measure which is a nontrivial convex combination of the two ergodic measures of maximal entropy is a weak$*$ and in entropy limit of ergodic measures.*

The results in [TY] and our results suggests the following conjecture (which is indeed true for maximal entropy measures, by Corollary 3).

Conjecture 4. *For every pair of hyperbolic ergodic measures μ_1 and μ_2 with $\chi(\mu_1) < 0 < \chi(\mu_2)$ every nontrivial convex combination of μ_1 and μ_2 cannot be approximated (weak$*$ and in entropy) by ergodic measures.*

Under the hypothesis of Theorem 2, the graph of the Lyapunov spectrum $\alpha \to h_{\rm top}(\mathcal{L}(\alpha)) = \mathcal{E}(\alpha)$ is as on Figure 3 (left figure). Under the hypothesis of Theorem 1, possible shapes of the graph of the corresponding Lyapunov spectrum are as on Figure 3 (middle and right figures).

We summarise the properties of (restricted) pressure functions, its Legendre-Fenchel transform, and of the Lyapunov spectrum in the following theorem (compare Figures 2 and 3).

Theorem 5. *Under the assumptions of Theorem 1,*

a) *$\mathcal{P}_{<0}$ and $\mathcal{P}_{>0}$ are nonincreasing and nondecreasing convex functions, respectively,*
b) *(Plateaus) There are numbers $D_\pm$ and $h_\pm$ such that*
$$\mathcal{P}_{<0}(q) = h_- \text{ for all } q \geq D_- \quad \text{and} \quad \mathcal{P}_{>0}(q) = h_+ \text{ for all } q \leq D_+.$$
c) *$h_- = h_+ = h_{\rm top}(\mathcal{L}(0))$,*
d) *$D_+ \leq 0 \leq D_-$,*
e) *$\mathcal{P}_{>0}(0) = \mathcal{P}_{<0}(0) = \log N = h_{\rm top}(F)$,*
f) *$\alpha \mapsto h_{\rm top}(\mathcal{L}(\alpha))$ achieves its maximum value $\log N$ at some points*
$$\alpha_- < 0 \quad \text{and} \quad \alpha_+ > 0,$$
g) *For $\alpha < 0$ the function $\alpha \mapsto \mathcal{E}(\alpha)$ is a Legendre-Fenchel transform of $q \mapsto \mathcal{P}_{<0}(q)$. Similarly, for $\alpha > 0$ the function $\alpha \mapsto \mathcal{E}(\alpha)$ is a Legendre-Fenchel transform of $q \mapsto \mathcal{P}_{>0}(q)$. In particular, $\alpha \mapsto \mathcal{E}(\alpha)$ is a concave function on the domains $\alpha < 0$ and $\alpha > 0$, respectively,*
h) *$h_{\rm top}(\mathcal{L}(\alpha))$ is a continuous function on $[\alpha_{\min}, \alpha_{\max}]$,*
i) *$0 \leq D_R\mathcal{E}(0) < \infty$ and $0 \leq -D_L\mathcal{E}(0) < \infty$,*
j) *$h_{\rm top}(\mathcal{L}(0)) > 0$.*

Moreover, under the assumptions of Theorem 2 we have additional properties

k) *$\mathcal{P}_{>0}(q)$ and $\mathcal{P}_{<0}(q)$ are differentiable at $q = 0$,*

in items d) and i) we have strict inequalities:
$$D_+ < 0 < D_- \quad \text{and} \quad D_L\mathcal{E}(0) < 0 < D_R\mathcal{E}(0),$$
and the points α_-, α_+ in item f) are the unique numbers α for which $h_{\rm top}(\mathcal{L}(\alpha)) = \log N$.

Remark 2.1. The following questions remain open. The restricted pressures can be differentiable or nondifferentiable at the beginning of the plateaus in Theorem 5 item b). The nondifferentiability of, for example, $\mathcal{P}_{>0}$ at a_- would mean that $\mathcal{E}(\alpha)$ is linear on some interval $[0, q]$. Further regularity properties (smoothness, analyticity) of the restricted pressure functions (excluding the ends of plateaus) and of the spectrum are unknown.

The asymptote of $\mathcal{P}_{>0}$ at $q \to \infty$ is some line $\{P = \alpha_{\max}q + h_{\max}\}$, similarly $\mathcal{P}_{<0}$ is asymptotic to $\{P = \alpha_{\min}q + h_{\min}\}$, and we do not know whether $h_{\max}$ and $h_{\min}$

are equal to zero (which would mean that $h_{\rm top}(\mathcal{L}(\alpha_{\max}))=h_{\rm top}(\mathcal{L}(\alpha_{\min}))=0$; this phenomenon is sometimes referred to as *ergodic optimization*, see for example [J]).

Finally, we do not know if there exist ergodic measures with Lyapunov exponent zero and with entropy arbitrarily close to $h_{\rm top}(\mathcal{L}(0))$ (Variational principle for exponent zero).

Our approach is to treat positive, negative, and zero spectra separately. First, we recall the restricted variational principle for entropy which provides a lower bound for $h_{\rm top}(\mathcal{L}(\alpha))$, see Section 4.1. Then we show that these values can be expressed via the Legendre-Fenchel transform of the restricted pressure function in (2.4) (treating negative and positive values separately). For that we will strongly use that for any pair of uniformly hyperbolic sets with negative (positive) fiber exponents we can find a larger one containing them both and hence we can gradually approximate from below the restricted pressure $\mathcal{P}_{<0}$ ($\mathcal{P}_{>0}$), see Section 5. Finally, using the existence of so-called skeletons established in [DGR$_2$], for any α with a level set of given entropy h we can construct hyperbolic sets with entropy close to h with almost homogeneous exponents close to α. This will show that $h_{\rm top}(\mathcal{L}(\alpha))$ is limited from above by entropies of ergodic measures with exponents close to α. Concavity of the Legendre-Fenchel transform implies its continuity, which concludes the main argument.

The structure of this paper is as follows. In Section 3 we recall the most important properties of the systems we investigate, obtained in [DGR$_2$]. In Section 4 we give some basic information about the thermodynamical formalism. In Section 5 we introduce (in an abstract setting) the restricted pressures and exhausting families, then in Section 6 we construct them in the setting of our paper. Finally, in the last three sections we prove our three theorems.

3. Setting

We recall the precise setting of our axioms CEC$\pm$ and Acc$\pm$ and their main consequences, established in [DGR$_2$]. The step skew-product structure of F allows us to reduce the study of its dynamics to the study of the iterated function system (IFS) generated by the fiber maps $\{f_i\}_{i=0}^{N-1}$. In what follows we always assume that F is transitive.

3.1. **Axioms.** Given a point $x\in\mathbb{S}^1$, consider and define its *forward* and *backward orbits* by

$$\mathcal{O}^+(x)\stackrel{\rm def}{=}\bigcup_{n\ge0}\bigcup_{(\theta_0\ldots\theta_{n-1})}f_{[\theta_0\ldots\,\theta_{n-1}]}(x)\text{ and }\mathcal{O}^-(x)\stackrel{\rm def}{=}\bigcup_{m\le1}\bigcup_{(\theta_{-m}\ldots\theta_{-1})}f_{[\theta_{-m}\ldots\,\theta_{-1}.]}(x),$$

respectively. Consider also the *full orbit* of x

$$\mathcal{O}(x)\stackrel{\rm def}{=}\mathcal{O}^+(x)\cup\mathcal{O}^-(x).$$

Similarly, we define the orbits $\mathcal{O}^+(J),\mathcal{O}^-(J)$, and $\mathcal{O}(J)$ for any subset $J\subset\mathbb{S}^1$.

In requiring that the underlying IFS $\{f_i\}$ of the map F *satisfies the axioms CEC$\pm$ and Acc$\pm$* we mean that there are so-called (closed) *forward and backward blending intervals* $J^+,J^-\subset\mathbb{S}^1$ such that the following properties hold.

CEC+(J^+) (Controlled Expanding forward Covering relative to J^+). There exist positive constants $K_1,\ldots,K_5$ such that for every interval $H\subset\mathbb{S}^1$ intersecting J^+ and satisfying $|H|<K_1$ we have

- (controlled covering) there exists a finite sequence $(\eta_0 \dots \eta_{\ell-1})$ for some positive integer $\ell \le K_2 \left|\log |H|\right| + K_3$ such that
$$f_{[\eta_0 \dots \eta_{\ell-1}]}(H) \supset B(J^+, K_4),$$
where $B(J^+,\delta)$ is the δ-neighbourhood of the set J^+.
- (controlled expansion) for every $x \in H$ we have
$$\log |(f_{[\eta_0 \dots \eta_{\ell-1}]})'(x)| \ge \ell K_5.$$

CEC$-(J^-)$ (Controlled Expanding backward Covering relative to J^-). The IFS $\{f_i^{-1}\}$ satisfies the Axiom CEC+(J^+).

Acc$+(J^+)$ (forward Accessibility relative to J^+). $\mathcal{O}^+(\operatorname{int} J^+) = \mathbb{S}^1$.

Acc$-(J^-)$ (backward Accessibility relative to J^-). $\mathcal{O}^-(\operatorname{int} J^-) = \mathbb{S}^1$.

When the step skew-product F is transitive then there is a common interval $J \subset \mathbb{S}^1$ satisfying CEC$\pm(J)$ and Acc$\pm(J)$ (see Lemma 3.5 and detailed discussion in [DGR$_2$, Section 2.2]).

In what follows we recall some properties of the IFS $\{f_i\}$ and the skew-product map F satisfying the axioms above that will be used in this paper.

3.2. **Previous results from** [DGR$_2$]**.** A technical result that we extract from [DGR$_2$] claims that given an ergodic measure μ with exponent $\chi(\mu) = \alpha > 0$ and entropy $h(\mu) > 0$, for every small $\beta < 0$ there are ergodic measures with exponents close to β and positive entropy, but in this construction some entropy is lost. [DGR$_2$, Theorem 5] bounds the amount of lost entropy that is related to the size of $\alpha + |\beta|$. A specially interesting case occurs when the exponent β is taken arbitrarily close to 0^-. The estimates in [DGR$_2$, Theorem 5] are summarized in the next lemma.

Lemma 3.1 (Rephrasing partially [DGR$_2$, Theorem 5])**.** *There exists $c > 0$ such that for every ergodic measure μ with nonzero Lyapunov exponent $\chi(\mu) = \alpha \ne 0$ there is a sequence of ergodic measures ν_i with Lyapunov exponents $\chi(\nu_i) = \beta_i$, $\operatorname{sgn}\alpha \ne \operatorname{sgn}\beta_i$, such that $\beta_i \to 0$ and*
$$\lim_{i\to\infty} h(\nu_i) \ge \frac{h(\mu)}{1 + c|\alpha|}.$$

This result also implies the following.

Corollary 3.2. *There exist ergodic measures with exponents arbitrarily close to 0^+ and 0^-.*

The systems considered in this paper satisfy the so-called *skeleton property* which implies the existence of orbit pieces that allow to approximate entropy and Lyapunov exponent, see [DGR$_2$, Section 4] for details. The skeleton property is referred to some blending interval and to quantifiers corresponding to the entropy and a level set for the Lyapunov exponent. An important property is that if $\mathcal{L}(\alpha) \ne 0$ then the skeleton property holds relative to $h = h_{\rm top}(\mathcal{L}(\alpha))$ and α. Based on the skeleton property, we have the following.

Given a compact F-invariant set $\Gamma \subset \Sigma_N \times \mathbb{S}^1$, we say that Γ has *uniform fiber expansion* (*contraction*) if every ergodic measure $\mu \in \mathcal{M}(\Gamma)$ has positive (negative) Lyapunov exponent. It is *hyperbolic* if it either has uniform fiber expansion or uniform fiber contraction. We say that a set is *basic* (with respect to F) if it is compact, F-invariant, locally maximal, topologically transitive, and hyperbolic[1].

Proposition 3.3 ([DGR$_2$, Theorems 4.3 and 4.4]). *Given $\alpha \neq 0$ such that $\mathcal{L}(\alpha) \neq \varnothing$ and $h = h_{\mathrm{top}}(\mathcal{L}(\alpha)) > 0$, for every $\gamma \in (0,h)$ and every small $\lambda > 0$ there is a basic set $\Gamma \subset \Sigma_N \times \mathbb{S}^1$ such that*

1. *$h_{\mathrm{top}}(\Gamma) \in [h-\gamma, h+\gamma]$ and*
2. *every $\nu \in \mathcal{M}_{\mathrm{erg}}(\Gamma)$ satisfies $\chi(\nu) \in (\alpha-\lambda, \alpha+\lambda) \cap \mathbb{R}_-$.*

The analogous result holds for negative Lyapunov exponents exponents.

Proposition 3.4 ([DGR$_2$, Proposition 4.8 and Theorem 4.4]). *Suppose that $\mathcal{L}(0) \neq \varnothing$ and that $h = h_{\mathrm{top}}(\mathcal{L}(0)) > 0$. Then for every $\gamma \in (0,h)$ and every $\lambda > 0$ there is a basic set $\Gamma \subset \Sigma_N \times \mathbb{S}^1$ such that*

1. *$h_{\mathrm{top}}(\Gamma) \geq h - \gamma$ and*
2. *every $\nu \in \mathcal{M}_{\mathrm{erg}}(\Gamma)$ satisfies $\chi(\nu) \in (-\lambda, 0)$.*

A further consequence of the axioms CEC$\pm$ and Acc$\pm$ is that the IFS $\{f_i\}$ is forward and backward minimal. Lemma 2.2 in [DGR$_2$] states a quantitative version of this minimality. We also will use the following results which are simple consequences.

Lemma 3.5 ([DGR$_2$, Lemmas 2.2 and 2.3]). *Every nontrivial interval $I \subset \mathbb{S}^1$ contains a subinterval $J \subset I$ such that the IFS $\{f_i\}$ satisfies axioms CEC$\pm(J)$ and Acc$\pm(J)$. Moreover, there is a number $M = M(I) \geq 1$ such that for every point $x \in \mathbb{S}^1$ there are finite sequences $(\theta_1 \ldots \theta_r)$ and $(\beta_1 \ldots \beta_s)$ with $r, s \leq M$ such that*

$$f_{[\beta_1 \ldots \beta_s]}(x) \in I \quad \textit{and} \quad f_{[\theta_1 \ldots \theta_r .]}(x) \in I$$

Lemma 3.6 ([DGR$_2$, Lemma 2.4]). *For every interval $I \subset \mathbb{S}^1$ there exist $\delta = \delta(I) > 0$ and $M = M(I) \geq 1$ such that for any interval $J \subset \mathbb{S}^1$, $|J| < \delta$, there exists a finite sequence $(\tau_1 \ldots \tau_m)$, $m \leq M$, such that $f_{[\tau_1 \ldots \tau_m]}(J) \subset I$.*

We finish this section with one further conclusion which we will use in Sections 7.1 and 8.1.

Lemma 3.7. *There does not exist a Borel probability measure m on $\mathbb{S}^1$ which is f_i-invariant for every $i = 0, \ldots, N-1$.*

Proof. By contradiction, assume that there is a Borel probability measure m on $\mathbb{S}^1$ which is simultaneously f_i-invariant for all i. Let $J \subset \mathbb{S}^1$ be a blending interval and consider two closed disjoint small sub-intervals $J_1, J_2 \subset J$. By Axiom CEC$+(J)$, there is some sequence $(\eta_0 \ldots \eta_{\ell-1})$ such that $f_{[\eta_0 \ldots \eta_{\ell-1}]}(J_1) \supset J$. From this we can conclude that $m(J \setminus J_1) = 0$. Similarly, $m(J \setminus J_2) = 0$. This implies $m(J) = 0$. Hence, by Acc$\pm(J)$ we have that $m(\mathbb{S}^1) = 0$. But this is a contradiction. $\square$

4. Entropy, pressures, and variational principles

In this section we consider a general setting of a compact metric space $(\mathbf{X}, d)$, a continuous map $F \colon \mathbf{X} \to \mathbf{X}$, and a continuous function $\varphi \colon \mathbf{X} \to \mathbb{R}$. We collect some general facts about entropy and pressure.

[1]This definition mimics the usual definition of a basic set in a differentiable setting.

4.1. **Entropy: restricted variational principles.** Given $\alpha \in \mathbb{R}$ consider the level sets

$$\mathcal{L}(\alpha) \stackrel{\text{def}}{=} \big\{x \in \mathbf{X} \colon \overline{\varphi}(x) = \alpha\big\}, \qquad \text{where} \qquad \overline{\varphi}(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(F^k(x)),$$

whenever this limit exists. We study the *topological entropy of* F on the set $\mathcal{L}(\alpha)$ and consider the function

$$\alpha \mapsto h_{\rm top}(\mathcal{L}(\alpha)).$$

We will now recall some results which are known for such general setting. An upper bound for the entropy $h_{\rm top}(\mathcal{L}(\alpha))$ (which, in fact, is sharp in many cases) is easily derived applying a general result by Bowen [B_1]. Denote by $\mathcal{M}(\mathbf{X})$ the set of F-invariant probability measures and by $\mathcal{M}_{\rm erg}(\mathbf{X}) \subset \mathcal{M}(\mathbf{X})$ the subset of ergodic measures. We equip this space with the weak$*$ topology. Given $x \in \mathbf{X}$, let $V_F(x) \subset \mathcal{M}(\mathbf{X})$ be the set of (F-invariant) measures which are weak$*$ limit points as $n \to \infty$ of the empirical measures $\mu_{x,n}$

$$\mu_{x,n} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F^k(x)},$$

where δ_x is the Dirac measure supported on the point x. Given $\mu \in \mathcal{M}(\mathbf{X})$, denote by $G(\mu)$ the set of μ-generic points

$$G(\mu) \stackrel{\text{def}}{=} \Big\{x \colon \lim_{n\to\infty} \mu_{x,n} = \{\mu\}\Big\}\,.$$

Given $c \geq 0$, define the set of its "quasi regular" points by

$$QR(c) \stackrel{\text{def}}{=} \big\{y \in \mathbf{X} \colon \text{ there exists } \mu \in V_F(y) \text{ with } h(\mu) \leq c\big\}.$$

Proposition 4.1.

i) $h_{\rm top}(QR(c)) \leq c$ *(*[B_1, Theorem 2]*).*
ii) *For* μ *ergodic we have* $h(\mu) = h_{\rm top}(G(\mu))$ *(*[B_1, Theorem 3]*).*
iii) *If* F *satisfies the specification property, then for every* $\mu \in \mathcal{M}(\mathbf{X})$ *we have* $h(\mu) = h_{\rm top}(G(\mu))$ *(*[PS, Theorem 1.2] *or* [FLP, Theorem 1.1]*).*[2]

We have the following simple consequence. Let

$$\varphi(\mu) \stackrel{\text{def}}{=} \int \varphi\, d\mu.$$

Lemma 4.2. *For every* α *such that* $\mathcal{L}(\alpha) \neq \varnothing$ *we have*

$$\begin{aligned}\sup\big\{h(\mu) \colon \mu \in \mathcal{M}_{\rm erg}(\mathbf{X}), \varphi(\mu) = \alpha\big\} &\leq h_{\rm top}(\mathcal{L}(\alpha)) \\ &\leq \sup\big\{h(\mu) \colon \mu \in \mathcal{M}(\mathbf{X}), \varphi(\mu) = \alpha\big\}.\end{aligned}$$

Proof. To prove the first inequality, observe that for μ ergodic with $\varphi(\mu) = \alpha$ we have $G(\mu) \subset \mathcal{L}(\alpha)$ and by Proposition 4.1 ii) and monotonicity of topological entropy with respect to inclusion we obtain $h(\mu) = h_{\rm top}(G(\mu)) \leq h_{\rm top}(\mathcal{L}(\alpha))$.

To prove the second inequality, denote

$$H(\alpha) \stackrel{\text{def}}{=} \sup\{h(\mu) \colon \mu \in \mathcal{M}(\mathbf{X}), \varphi(\mu) = \alpha\}.$$

[2] Note that, in fact, this result holds true for any map which has the so called *g-almost product property* which is implied by the specification property (see [PS, Proposition 2.1]). The specification property is satisfied for example for every basic set (see [S]). We emphasize that the skew-product systems we study in this paper do not satisfy the specification property.

Note that for every $x \in \mathcal{L}(\alpha)$ we have $\overline{\varphi}(x) = \alpha$ and hence for every $\mu \in V_F(x)$ we have $\varphi(\mu) = \alpha$ and thus $h(\mu) \le H(\alpha)$. Hence, $\mathcal{L}(\alpha) \subset QR(H(\alpha))$ and again by monotonicity and Proposition 4.1 i) we obtain

$$h_{\rm top}(\mathcal{L}(\alpha)) \le h_{\rm top}(QR(H(\alpha))) \le H(\alpha),$$

proving the lemma. □

We recall the following classical restricted variational principle strengthening the above lemma which will play a central role in our arguments. We point out that it requires φ to be continuous, only.

Proposition 4.3 ([PS, Theorem 6.1 and Proposition 7.1] or [FLP, Theorem 1.3] and [S])**.** *If $F\colon \mathbf{X} \to \mathbf{X}$ satisfies the specification property then for every α such that $\mathcal{L}(\alpha) \neq \varnothing$ we have*

$$h_{\rm top}(\mathcal{L}(\alpha)) = \sup\big\{h(\mu)\colon \mu \in \mathcal{M}(\mathbf{X}), \varphi(\mu) = \alpha\big\}.$$

Moreover, $\big\{\varphi(\mu)\colon \mu \in \mathcal{M}_{\rm erg}(\mathbf{X})\big\}$ is an interval.

4.2. **Pressure functions.** For a measure $\mu \in \mathcal{M}(\mathbf{X})$ we define the affine functional $P(\cdot, \mu)$ on the space of continuous functions by

$$P(\varphi, \mu) \stackrel{\rm def}{=} h(\mu) + \int \varphi \, d\mu.$$

Given an F-invariant compact subset $Y \subset \mathbf{X}$, we define the *topological pressure of* φ with respect to $F|_Y$ by

$$P_{F|Y}(\varphi) \stackrel{\rm def}{=} \sup_{\mu \in \mathcal{M}(Y)} P(\varphi, \mu) = \sup_{\mu \in \mathcal{M}_{\rm erg}(Y)} P(\varphi, \mu) \tag{4.1}$$

and we simply write $P(\varphi) = P_{F|\mathbf{X}}(\varphi)$ if $Y = \mathbf{X}$ and $F|_{\mathbf{X}}$ is clear from the context. Note that definition (4.1) is nothing but the *variational principle* of the topological pressure (see [Wa, Chapter 9] for a proof and a purely topological and equivalent definition of pressure). A measure $\mu \in \mathcal{M}(Y)$ is an *equilibrium state* for φ (with respect to $F|_Y$) if it realizes the supremum in (4.1).[3] Recall that $h_{\rm top}(Y) = P_{F|Y}(0)$ is the *topological entropy* of F on Y.

We now continue by considering a decomposition of the set of ergodic measures and studying corresponding pressure functions. Given a subset $\mathcal{N} \subset \mathcal{M}(\mathbf{X})$, define

$$P(\varphi, \mathcal{N}) \stackrel{\rm def}{=} \sup_{\mu \in \mathcal{N}} P(\varphi, \mu).$$

Given $\mathcal{N} \subset \mathcal{M}(\mathbf{X})$, consider its *closed convex hull* $\overline{\rm conv}\, \mathcal{N}$, defined as the smallest closed convex set containing $\mathcal{N}$. It is an immediate consequence of the affinity of $\mu \mapsto P(\varphi, \mu)$ that

$$P(\varphi, \mathcal{N}) = P\big(\varphi, \overline{\rm conv}(\mathcal{N})\big).$$

A particular consequence of this equality and ergodic decomposition of non-ergodic measures is the fact that for $\mathcal{N} = \mathcal{M}_{\rm erg}(\mathbf{X})$ and hence $\overline{\rm conv}(\mathcal{N}) = \mathcal{M}(\mathbf{X})$ in (4.1) it is irrelevant if we take the supremum over all measures in $\mathcal{M}(\mathbf{X})$ or over the *ergodic* measures only (used to show the equality in (4.1)). The case of a general subset $\mathcal{N}$ of $\mathcal{M}(\mathbf{X})$, however, will be quite different and is precisely our focus of interest.

[3] Note that in context of the rest of the paper, skew-product maps with one-dimensional fibers, such equilibrium states indeed exist by [DF, Corollary 1.5] (see also [CY]). However, they are not unique in general, see for instance examples in [LOR, DGR_1].

We now analyze the pressure function for a subset of *ergodic* measures $\mathcal{N}\subset\mathcal{M}_{\rm erg}(\mathbf{X})$.[4] Let $q\in\mathbb{R}$ and consider the parametrized family $q\varphi\colon\mathbf{X}\to\mathbb{R}$ and the function

$$\mathcal{P}_{\mathcal{N}}(q)\stackrel{\scriptscriptstyle\rm def}{=}P(q\varphi,\mathcal{N}).$$

For each $\mu\in\mathcal{N}$ we simply write $\mathcal{P}_\mu(q)=\mathcal{P}(q,\{\mu\})$. We call $\mu\in\mathcal{M}(\mathbf{X})$ an *equilibrium state* for $q\varphi$, $q\in\mathbb{R}$, (with respect to $\mathcal{N}$) if $\mathcal{P}_{\mathcal{N}}(q)=\mathcal{P}_\mu(q)$. Let also

$$\varphi(\mathcal{N})\stackrel{\scriptscriptstyle\rm def}{=}\Big\{\int\varphi\,d\mu\colon\mu\in\mathcal{N}\Big\},\quad \underline{\varphi}_{\mathcal{N}}\stackrel{\scriptscriptstyle\rm def}{=}\inf\varphi(\mathcal{N}),\quad \overline{\varphi}_{\mathcal{N}}\stackrel{\scriptscriptstyle\rm def}{=}\sup\varphi(\mathcal{N}). \tag{4.2}$$

We list the following general properties which are easy to verify (most of these properties and the ideas behind their proofs can be found in [Wa, Chapter 9]).

(P1) The function $\mathcal{P}_\mu$ is affine and satisfies $\mathcal{P}_\mu\le\mathcal{P}_{\mathcal{N}}$ and $\mathcal{P}_\mu(0)=h(\mu)$.
(P2) Given a subset $\mathcal{N}'\subset\mathcal{N}$, then $\mathcal{P}_{\mathcal{N}'}\le\mathcal{P}_{\mathcal{N}}$.
(P3) $\mathcal{P}_{\mathcal{N}}(0)=\sup\{h(\mu)\colon\mu\in\mathcal{N}\}$.
(P4) The function $\varphi\mapsto P(\varphi,\mathcal{N})$ is continuous and $q\mapsto P(q\varphi,\mathcal{N})$ is uniformly Lipschitz continuous.
(P5) The function $\mathcal{P}_{\mathcal{N}}$ is convex. Consequently, $\mathcal{P}_{\mathcal{N}}$ is differentiable at all but at most countably many q and the left and right derivatives $D_L\mathcal{P}_{\mathcal{N}}(q)$ and $D_R\mathcal{P}_{\mathcal{N}}(q)$ are defined for all $q\in\mathbb{R}$.
(P6) We have

$$\underline{\varphi}_{\mathcal{N}}=\lim_{q\to\infty}\frac{\mathcal{P}_{\mathcal{N}}(q)}{q}=\lim_{q\to\infty}D_L\mathcal{P}_{\mathcal{N}}(q)=\lim_{q\to\infty}D_R\mathcal{P}_{\mathcal{N}}(q),$$
$$\overline{\varphi}_{\mathcal{N}}=\lim_{q\to-\infty}\frac{\mathcal{P}_{\mathcal{N}}(q)}{q}=\lim_{q\to-\infty}D_L\mathcal{P}_{\mathcal{N}}(q)=\lim_{q\to-\infty}D_R\mathcal{P}_{\mathcal{N}}(q).$$

(P7) The graph of $\mathcal{P}_{\mathcal{N}}$ has a supporting straight line of slope $\varphi(\mu)$ for every $\mu\in\mathcal{N}$. Thus, for any $\alpha\in(\underline{\varphi}_{\mathcal{N}},\overline{\varphi}_{\mathcal{N}})$ it has a supporting straight line of slope α.
(P8) If the entropy map $\mu\mapsto h(\mu)$ is upper semi-continuous on $\mathcal{M}(\mathbf{X})$ then for any number $\alpha\in(\underline{\varphi}_{\mathcal{N}},\overline{\varphi}_{\mathcal{N}})$ there is a measure $\mu_\alpha\in\mathcal{M}(\mathbf{X})$ (not necessarily ergodic and not necessarily in $\mathcal{N}$) such that $\varphi(\mu_\alpha)=\alpha$ and $q\mapsto\mathcal{P}_{\mu_\alpha}(q)$ is a supporting straight line for $\mathcal{P}_{\mathcal{N}}$.
(P9) If $\mu\in\mathcal{M}(\mathbf{X})$ is an equilibrium state for $q\varphi$ for some $q\in\mathbb{R}$ (with respect to $\mathcal{N}$), then $D_L\mathcal{P}_{\mathcal{N}}(q)\le\varphi(\mu)\le D_R\mathcal{P}_{\mathcal{N}}(q)$. Moreover, the graph of $\mathcal{P}_\mu$ is a supporting straight line for the graph of $\mathcal{P}_{\mathcal{N}}$ at $(q,\mathcal{P}_{\mathcal{N}}(q))$.
(P10) If the entropy map $\mu\mapsto h(\mu)$ is upper semi-continuous, then for any q there are equilibrium states $\mu_{L,q}$ and $\mu_{R,q}$ for $q\varphi$ (with respect to $\mathcal{N}$) such that $\varphi(\mu_{L,q})=D_L\mathcal{P}_{\mathcal{N}}(q)$ and $\varphi(\mu_{R,q})=D_R\mathcal{P}_{\mathcal{N}}(q)$. Moreover, $\mu_{L,q}$ and $\mu_{R,q}$ can be chosen to be ergodic (but not necessarily in $\mathcal{N}$).
(P11) $\mathcal{P}_{\mathcal{N}}$ is differentiable at q if and only if all equilibrium states for $q\varphi$ (with respect to $\mathcal{N}$) have the same exponent and this exponent is $\mathcal{P}_{\mathcal{N}}'(q)$. In particular, if there is a unique equilibrium state for $q\varphi$ (with respect to $\mathcal{N}$) then $\mathcal{P}_{\mathcal{N}}$ is differentiable at q.

[4]In the rest of this paper we will study the decomposition (2.2) and have in mind the particular subset of measures

$$\mathcal{N}_-=\mathcal{M}_{{\rm erg},<0},\quad\mathcal{N}_0=\mathcal{M}_{{\rm erg},0},\quad\mathcal{N}_+=\mathcal{M}_{{\rm erg},>0}.$$

(P12) If $\mu \in \overline{\mathrm{conv}}(\mathcal{N})$ is not ergodic and $\mathcal{P}_\mu(q) = \mathcal{P}_\mathcal{N}(q)$ for some q, then almost all of the measures in the ergodic decomposition of μ are equilibrium states for $q\varphi$ (with respect to $\mathcal{N}$).

4.3. **The convex conjugate of the pressure function.** One of our goals is to express the topological entropy $h_{\mathrm{top}}(\mathcal{L}(\alpha))$ of each level sets $\mathcal{L}(\alpha)$ in terms of restricted variational principles and Legendre-Fenchel transforms of appropriate pressure functions. Let us hence recall some simple facts about such transforms.

Given a subset of ergodic measures $\mathcal{N} \subset \mathcal{M}_{\mathrm{erg}}(\mathbf{X})$, we define

$$\mathcal{E}_\mathcal{N}(\alpha) \stackrel{\mathrm{def}}{=} \inf_{q\in\mathbb{R}} \big(\mathcal{P}_\mathcal{N}(q) - q\alpha\big)$$

on its domain

$$D(\mathcal{E}_\mathcal{N}) \stackrel{\mathrm{def}}{=} \Big\{\alpha \in \mathbb{R}\colon \inf_{q\in\mathbb{R}}(\mathcal{P}_\mathcal{N}(q) - q\alpha) > -\infty\Big\}.$$

Observe that $(\mathcal{P}_\mathcal{N}, \mathcal{E}_\mathcal{N})$ forms a *Legendre-Fenchel pair.*[5] We list the following general properties.

(E1) The function $\mathcal{E}_\mathcal{N}$ is concave (and hence continuous). Consequently, it is differentiable at all but at most countable many α, and the left and right derivatives are defined for all $\alpha \in D(\mathcal{E}_\mathcal{N})$.

(E2) We have

$$D(\mathcal{E}_\mathcal{N}) \supset (\underline{\varphi}_\mathcal{N}, \overline{\varphi}_\mathcal{N}).$$

(E3) If μ is an equilibrium state for $q\varphi$ for some $q \in \mathbb{R}$ (with respect to $\mathcal{N}$) and $\alpha = \varphi(\mu)$, then $h(\mu) = \mathcal{E}_\mathcal{N}(\alpha)$.

(E4) We have

$$\max_{\alpha\in D(\mathcal{E}_\mathcal{N})} \mathcal{E}_\mathcal{N}(\alpha) = \mathcal{P}_\mathcal{N}(0).$$

Moreover, this maximum is attained at exactly one value of α if, and only if, $\mathcal{P}_\mathcal{N}$ is differentiable at 0.

(E5) For every $\alpha \in D(\mathcal{E}_\mathcal{N})$ we have

$$\mathcal{E}_\mathcal{N}(\alpha) \ge \sup\big\{h(\mu)\colon \mu \in \mathcal{N}, \chi(\mu) = \alpha\big\}.$$

Lemma 4.4. *Assume that $F\colon M \to M$ is a diffeomorphism and $\mathbf{X} \subset M$ is a basic set. Let $\varphi\colon \mathbf{X} \to \mathbb{R}$ be a continuous potential. Then for $\mathcal{N} = \mathcal{M}_{\mathrm{erg}}(\mathbf{X})$ and every $\alpha \in \operatorname{int} D(\mathcal{E}_\mathcal{N})$ we have*

$$\sup\big\{h(\mu)\colon \mu \in \mathcal{N}, \varphi(\mu) = \alpha\big\} = \mathcal{E}_\mathcal{N}(\alpha).$$

Note that to show the inequality $\le$ in the lemma we, in fact, do not need hyperbolic-like properties.

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[5]The *Legendre-Fenchel transform* of a convex function $\beta\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\beta^\star(\alpha) \stackrel{\mathrm{def}}{=} \sup_{q\in\mathbb{R}} \big(\alpha q - \beta(q)\big),$$

and is convex on its domain $D(\beta^\star) = \{\alpha \in \mathbb{R}\colon \beta^\star(\alpha) < \infty\}$. In particular, the convex function β is differentiable at all but at most countably many points and

$$\beta^\star(\alpha) = \beta'(q)q - \beta(q) \quad \text{for} \quad \alpha = \beta'(q).$$

On the set of strictly convex functions the transform is involutive $\beta^{\star\star} = \beta$. Formally, it is the function $\alpha \mapsto -\mathcal{E}_\mathcal{N}(-\alpha)$ which is the Legendre-Fenchel transform of $\mathcal{P}_\mathcal{N}(q)$, but it is common practice in the context of this paper (that we will also follow) to address $\mathcal{E}_\mathcal{N}$ by this name.

Proof. Let $\alpha \in \operatorname{int} D(\mathcal{E}_\mathcal{N})$. Fix any $q \in \mathbb{R}$. Observe that

$$\begin{aligned}\sup\big\{h(\mu)\colon \mu\in\mathcal{N},\varphi(\mu)=\alpha\big\} &= \sup\big\{h(\mu)+q\varphi(\mu)\colon \mu\in\mathcal{N},\varphi(\mu)=\alpha\big\}-q\alpha\\ &\le \sup\big\{h(\mu)+q\varphi(\mu)\colon \mu\in\mathcal{N}\big\}-q\alpha\\ &= \mathcal{P}_\mathcal{N}(q)-q\alpha.\end{aligned}$$

Since q was arbitrary, we can conclude

$$\sup\big\{h(\mu)\colon \mu\in\mathcal{N},\varphi(\mu)=\alpha\big\}\le \inf_{q\in\mathbb{R}}\big(\mathcal{P}_\mathcal{N}(q)-q\alpha\big)=\mathcal{E}_\mathcal{N}(\alpha).$$

To prove the other inequality, first recall [B$_2$] that for any Hölder continuous potential $\tilde\varphi\colon \mathbf{X}\to\mathbb{R}$ and $\tilde q\in\mathbb{R}$ there is a unique equilibrium state for $\tilde q\tilde\varphi$. By property (P8) applied to $\mathbf{X}$ and $\mathcal{N}$, there is a measure $\mu_\alpha\in\mathcal{M}(\mathbf{X})$ (not necessarily ergodic) such that $\varphi(\mu_\alpha)=\alpha$ and $q\mapsto\mathcal{P}_{\mu_\alpha}(q)$ is a supporting straight line for $\mathcal{P}_\mathcal{N}$. Hence, there is $q=q(\alpha)$ such that $\mathcal{P}_\mathcal{N}(q)=h(\mu_\alpha)+qh(\mu_\alpha)$. If μ_α was already ergodic then we are done. Otherwise, note that we can find $\tilde\varphi\colon\mathbf{X}\to\mathbb{R}$ Hölder continuous and arbitrarily close to the continuous potential $\varphi\colon\mathbf{X}\to\mathbb{R}$ and $\tilde q$ arbitrarily close to q and an ergodic equilibrium state $\tilde\nu\in\mathcal{N}$ for $\tilde q\tilde\varphi$ such that $\varphi(\tilde\nu)=\alpha$. By (P4) we have that $P(\tilde q\tilde\varphi,\mathcal{N})$ is arbitrarily close to $P(q\varphi,\mathcal{N})$. Hence, for such $\tilde\nu$ we have

$$h(\tilde\nu)=P(\tilde q\tilde\varphi,\mathcal{N})-\tilde q\alpha=\big(P(q\varphi,\mathcal{N})-q\alpha\big)+\big(P(\tilde q\tilde\varphi,\mathcal{N})-P(q\varphi,\mathcal{N})\big)+\big(q\alpha-\tilde q\alpha\big).$$

Thus, we can conclude

$$\sup\big\{h(\nu)\colon \nu\in\mathcal{N},\varphi(\nu)=\alpha\big\}\ge\big(P(q\varphi,\mathcal{N})-q\alpha\big).$$

Taking the infimum over all $q\in\mathbb{R}$ we obtain

$$\sup\big\{h(\nu)\colon \nu\in\mathcal{N},\varphi(\nu)=\alpha\big\}\ge \inf_{q\in\mathbb{R}}\big(\mathcal{P}_\mathcal{N}(q)-q\alpha\big)=\mathcal{E}_\mathcal{N}(\alpha).$$

This finishes the proof of the lemma. □

5. Exhausting families

In this section we present a general principle to perform a multifractal analysis which was used in several contexts having some hyperbolicity (see, for example, [GR] for Markov maps on the interval, [GPR] for non-exceptional rational maps of the Riemann sphere, or [BG] for geodesic flows of rank one surfaces). Note that for general dynamical systems – and, in particular, in the setting of the present paper – we cannot expect the specification property to be satisfied on the whole space. For this reason, we will consider in the following sections certain families of subsets (basic sets, see Section 6.1) on which we do have specification. In this section we are going to present the general theory of *restricted pressures* which allows us to obtain dynamical properties of the full system knowing the properties of those subsets.

Let $(\mathbf{X},d)$ be a compact metric space, $F\colon\mathbf{X}\to\mathbf{X}$ a continuous map, and $\varphi\colon\mathbf{X}\to\mathbb{R}$ a continuous potential. Fix a set of ergodic measures $\mathcal{N}\subset\mathcal{M}_{\rm erg}(\mathbf{X})$. Recall that we defined for $\alpha\in D(\mathcal{E}_\mathcal{N})$

$$\mathcal{E}_\mathcal{N}(\alpha)=\inf_{q\in\mathbb{R}}\big(\mathcal{P}_\mathcal{N}(q)-q\alpha\big).$$

A sequence of compact F-invariant sets $\mathbf{X}_1,\mathbf{X}_2,\ldots\subset\mathbf{X}$ is said to be $(\mathbf{X},\varphi,\mathcal{N})$-*exhausting* if the following holds: for every $i\ge1$ we have

(exh1) $\mathcal{M}_{\rm erg}(\mathbf{X}_i)\subset\mathcal{N}$, (exh2) $F|_{\mathbf{X}_i}$ has the specification property,

(exh3) Given $\mathcal{M}_i = \mathcal{M}_{\rm erg}(\mathbf{X}_i)$ let $\mathcal{P}_i = \mathcal{P}_{\mathcal{M}_i}$ and

$$\mathcal{E}_i(\alpha) \stackrel{\rm def}{=} \inf_{q\in\mathbb{R}} \big(\mathcal{P}_i(q) - q\alpha\big).$$

Then for every $\alpha \in \operatorname{int} D(\mathcal{E}_i)$ the restricted variational principle holds

$$\mathcal{E}_i(\alpha) = \sup \big\{h(\mu) \colon \mu \in \mathcal{M}_i, \varphi(\mu) = \alpha\big\}.$$

(exh4) for every $q \in \mathbb{R}$ we have

$$\lim_{i\to\infty} P_{F|\mathbf{X}_i}(q\varphi) = \mathcal{P}_{\mathcal{N}}(q).$$

(exh5) Let $\underline{\varphi}_{\mathcal{N}}$ and $\overline{\varphi}_{\mathcal{N}}$ be as in (4.2), then

$$\underline{\varphi}_{\mathcal{N}} = \lim_{i\to\infty} \underline{\varphi}_{\mathcal{M}_i}, \quad \overline{\varphi}_{\mathcal{N}} = \lim_{i\to\infty} \overline{\varphi}_{\mathcal{M}_i}.$$

Note that $(\mathcal{P}_i, \mathcal{E}_i)$ forms a Legendre-Fenchel pair for every $i \geq 1$.

Lemma 5.1. *It holds* $\lim_{i\to\infty} \mathcal{E}_i(\alpha) = \mathcal{E}_{\mathcal{N}}(\alpha)$. *In particular,* $\operatorname{int} D(\mathcal{E}_{\mathcal{N}}) = (\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}})$.

Proof. Note that property (exh4) of pointwise convergence of convex functions of pressures $\mathcal{P}_i$ to the convex function of pressure $\mathcal{P}_{\mathcal{N}}$ and the fact that $\mathcal{E}_i$ and $\mathcal{E}_{\mathcal{N}}$ are their Legendre-Fenchel transforms imply the claim, see for instance [Wi]. □

The following result will be the main step in establishing the lower bounds for entropy in Theorem 1. We derive it in the general setting of this subsection.

Proposition 5.2. *Assume that there exists an increasing family of sets* $(\mathbf{X}_i)_i \subset \mathbf{X}$ *which is* $(\mathbf{X}, \varphi, \mathcal{N})$*-exhausting. Then*

- *we have*

$$(\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}}) \subset \varphi(\mathcal{N}) \subset [\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}}].$$

  In particular, $\varphi(\mathcal{N})$ *is an interval.*
- *For every* $\alpha \in (\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}})$ *we have* $\mathcal{L}(\alpha) \neq \varnothing$ *and*

$$h_{\rm top}(\mathcal{L}(\alpha)) \geq \mathcal{E}_{\mathcal{N}}(\alpha) = \lim_{i\to\infty} \sup \big\{h(\mu) \colon \mu \in \mathcal{M}(\mathbf{X}_i), \varphi(\mu) = \alpha\big\}.$$

Proof. By condition (exh4) and the property of pointwise convergence of convex functions to a convex function (see (P5)), we can conclude that for every i

$$P_{F|\mathbf{X}_{n(i)}}(q\varphi) \geq \mathcal{P}_{\mathcal{N}}(q) - \frac{1}{i}$$

for all $q \in [-i, i]$ and some sequence $(n(i))_i$. For simplicity, allowing a change of indices, we will assume that $n(i) = i$.

A particular consequence of specification of $F|_{\mathbf{X}_i}$ is that by Proposition 4.3 the set $\varphi(\mathcal{M}_i)$ is an interval. Together with (exh5) this implies that $\varphi(\mathcal{N})$ is an interval and we have

$$(\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}}) \subset \varphi(\mathcal{N}) = \bigcup_{i\geq 1} \varphi(\mathcal{M}_i) \subset [\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}}], \tag{5.1}$$

proving the first item.

Let $\alpha \in (\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}})$. For every index i, by Proposition 4.3, we have

$$h_{\rm top}(\mathcal{L}(\alpha) \cap \mathbf{X}_i) = \sup \big\{h(\mu) \colon \mathcal{M}_i, \varphi(\mu) = \alpha\big\} \leq h_{\rm top}(\mathcal{L}(\alpha)),$$

where for the inequality we use monotonicity of entropy. By (5.1), there is $i = i(\alpha) \ge 1$ such that $\alpha \in \varphi(\mathcal{M}_i)$ and, in particular, we have $\mathcal{L}(\alpha) \neq \varnothing$. By (exh3), for every $\alpha \in (\underline{\varphi}_{\mathcal{N}}, \overline{\varphi}_{\mathcal{N}})$ and i sufficiently big, we have

$$\mathcal{E}_i(\alpha) = \sup\big\{h(\mu)\colon \mathcal{M}_i, \varphi(\mu) = \alpha\big\}.$$

By Lemma 5.1 we have $\lim_{i\to\infty} \mathcal{E}_i(\alpha) = \mathcal{E}_{\mathcal{N}}(\alpha)$, concluding the proof of the proposition. □

Remark 5.3. The exhausting property for appropriate $\mathcal{N}$ is the essential step to relate the lower bound in the restricted variational principle (4.2) to the Legendre-Fenchel transform of the restricted pressure function $\mathcal{P}_{\mathcal{N}}$. This is the requirement (exh3).

6. Homoclinic relations and construction of exhausting families

In this section we return to consider a transitive step skew-product map F as in (2.1) whose fiber maps are C^1 and satisfies Axioms CEC± and Acc±. Recall that the map F has ergodic measures with exponents arbitrarily close to 0^+ and 0^-, see Corollary 3.2. The goal of this section is to prove the following proposition.

Proposition 6.1. *Consider the set of ergodic measures $\mathcal{N} = \mathcal{M}_{\mathrm{erg},<0}$ and the potential $\varphi\colon \Sigma_N \times \mathbb{S}^1 \to \mathbb{R}$ in (2.3).*

Then there is a $(\Sigma_N \times \mathbb{S}^1, \varphi, \mathcal{N})$-exhausting family $\{\mathbf{X}_i\}$ consisting of nested basic sets and $\varphi(\mathcal{N}) = [\alpha_{\min}, 0)$. The analogous statement is true for $\mathcal{N} = \mathcal{M}_{\mathrm{erg},>0}$ with $\varphi(\mathcal{N}) = (0, \alpha_{\max}]$.

6.1. Homoclinic relations.

We say that a periodic point is *hyperbolic* or a *saddle* of F if its (fiber) Lyapunov exponent is nonzero. In our partially hyperbolic setting with one-dimensional central bundle there are only two possibilities: a saddle has either a positive or negative (fiber) Lyapunov exponent. We say that two saddles are of *the same type* if either both have negative exponents or both have positive exponents. Note that all saddles in a basic set are of the same (expanding/contracting) type. We say that two basic sets are of the *same type* if their saddles are of the same type.

Given a saddle P we define the *stable* and *unstable sets* of its orbit $\mathcal{O}(P)$ by

$$W^{\mathrm{s}}(\mathcal{O}(P)) \stackrel{\mathrm{def}}{=} \{X\colon \lim_{n\to\infty} d(F^n(X), \mathcal{O}(P)) = 0\},$$

and

$$W^{\mathrm{u}}(\mathcal{O}(P)) \stackrel{\mathrm{def}}{=} \{X\colon \lim_{n\to\infty} d(F^{-n}(X), \mathcal{O}(P)) = 0\},$$

respectively.

We say that a point X is a *homoclinic point* of P if $X \in W^{\mathrm{s}}(\mathcal{O}(P)) \cap W^{\mathrm{u}}(\mathcal{O}(P))$. Two saddles P and Q of the same index are *homoclinically related* if the stable and unstable sets of their orbits intersect cyclically, that is, if

$$W^{\mathrm{s}}(\mathcal{O}(P)) \cap W^{\mathrm{u}}(\mathcal{O}(Q)) \neq \varnothing \neq W^{\mathrm{s}}(\mathcal{O}(Q)) \cap W^{\mathrm{u}}(\mathcal{O}(P)).$$

In our setting, homoclinic intersections behave the same as transverse homoclinic intersections. As in the differentiable case, to be homoclinically related defines an equivalence relation on the set of saddles of F. The *homoclinic class* of a saddle P, denoted by $H(P, F)$, is the closure of the set of saddles which are homoclinically related to P. A homoclinic class can be also defined as the closure of the homoclinic points of P. As in the differentiable case, a homoclinic class is a F-invariant and transitive set.[6]

Lemma 6.2. *Any pair of saddles $P, Q \in \Sigma_N \times \mathbb{S}^1$ of the same type are homoclinically related.*

Proof. Let us assume that P and Q both have negative exponents. The proof of the other case is analogous and omitted. Let $P = (\xi, p)$ and $Q = (\eta, q)$, where $\xi = (\xi_0 \ldots \xi_{n-1})^{\mathbb{Z}}$ and $\eta = (\eta_0 \ldots \eta_{m-1})^{\mathbb{Z}}$. By hyperbolicity, there is $\delta > 0$ such that

$$f_\xi^n\big([p-\delta, p+\delta]\big) \subset (p-\delta, p+\delta) \quad \text{and} \quad f_\eta^m\big([q-\delta, q+\delta]\big) \subset (q-\delta, q+\delta)$$

and such that those maps are uniformly contracting on those intervals. This immediately implies that

$$[.(\xi_0 \ldots \xi_{n-1})^{\mathbb{N}}] \times [p-\delta, p+\delta] \subset W^{\mathrm{s}}(\mathcal{O}(P)),$$
$$[.(\eta_0 \ldots \eta_{m-1})^{\mathbb{N}}] \times [q-\delta, q+\delta] \subset W^{\mathrm{s}}(\mathcal{O}(Q)).$$

Similarly we get

$$[(\xi_0 \ldots \xi_{n-1})^{-\mathbb{N}}.] \times \{p\} \subset W^{\mathrm{u}}(\mathcal{O}(P)), \quad [(\eta_0 \ldots \eta_{m-1})^{-\mathbb{N}}.] \times \{q\} \subset W^{\mathrm{u}}(\mathcal{O}(Q)).$$

By Lemma 3.5 there are $(\beta_0 \ldots \beta_s)$ and $(\gamma_0 \ldots \gamma_r)$ such that

$$f_{[\beta_0 \ldots \beta_s]}(q) \in (p-\delta, p+\delta) \quad \text{and} \quad f_{[\gamma_0 \ldots \gamma_r]}(p) \in (q-\delta, q+\delta).$$

By construction, this implies that

$$\big((\eta_0 \ldots \eta_{m-1})^{-\mathbb{N}}.\beta_0 \ldots \beta_s(\xi_0 \ldots \xi_{n-1})^{\mathbb{N}}, q\big) \in W^{\mathrm{u}}(\mathcal{O}(Q)) \cap W^{\mathrm{s}}(\mathcal{O}(P)),$$
$$\big((\xi_0 \ldots \xi_{n-1})^{-\mathbb{N}}.\gamma_0 \ldots \gamma_r(\eta_0 \ldots \eta_{m-1})^{\mathbb{N}}, p\big) \in W^{\mathrm{s}}(\mathcal{O}(P)) \cap W^{\mathrm{u}}(\mathcal{O}(P)).$$

This proves that P and Q are homoclinically related. $\square$

6.2. **Existence of exhausting families.** Let us start by recalling the following well-known fact about homoclinically related basic sets. For a proof we refer to [R, Section 7.4.2].

Lemma 6.3 (Bridging)**.** *Consider two basic sets Λ_1, Λ_2 of a diffeomorphism Φ which are homoclinically related. Then there is a basic set Λ of Φ containing $\Lambda_1 \cup \Lambda_2$. In particular, for every continuous potential φ, we have*

$$\max\big\{P_{\Phi|\Lambda_1}(\varphi), P_{\Phi|\Lambda_2}(\varphi)\big\} \le P_{\Phi|\Lambda}(\varphi).$$

We will base our arguments also on the following result that translates results of from Pesin-Katok theory to our setting.

[6]These assertions are folklore ones, details can be found, for instance, in [DER, Section 3]. Note that in our skew-product context the standard transverse intersection condition between the invariant sets of the saddles in the definition of a homoclinic relation is not required and does not make sense. However, since the dynamics in the central direction is non-critical (the fiber maps are diffeomorphisms and hence have no critical points) the intersections between invariant sets of saddles of the same type behave as "transverse" ones and the arguments can be translated to the skew-product setting (here the fact that the fiber direction is one-dimensional is essential).

Lemma 6.4. *Let $\mu \in \mathcal{M}_{\mathrm{erg},<0}$ with $h = h(\mu) > 0$ and $\alpha = \chi(\mu) < 0$.*

Then for every $\gamma \in (0, h)$ and every $\lambda \in (0, \alpha)$ there exists a basic set $\Gamma = \Gamma(\gamma, \lambda) \subset \Sigma_N \times \mathbb{S}^1$ such that for all $q \in \mathbb{R}$ we have

$$P_{F|\Gamma}(q\varphi) \ge h(\mu) + q \int \varphi \, d\mu - \gamma - q\lambda.$$

The analogous statement is true for $\mathcal{M}_{\mathrm{erg},>0}$.

Proof. By Proposition 3.3, there exists a basic set Γ such that $h_{\mathrm{top}}(\Gamma) \ge h - \gamma$ and that for every $\nu \in \mathcal{M}_{\mathrm{erg}}(\Gamma)$ we have $\chi(\nu) \in (\alpha - \lambda, \alpha + \lambda)$. The variational principle (4.1) immediately implies the lemma. □

We are now prepared to prove Proposition 6.1.

Proof of Proposition 6.1. We first construct an exhausting family. Given $i \ge 1$, let us first construct a basic set X_i of contracting type such that

$$P_{F|X_i}(q\varphi) \ge \mathcal{P}_{\mathcal{N}}(q) - \frac{1}{i} \tag{6.1}$$

for all $q \in [-i, i]$. By Lipschitz continuity property (P4) of pressure, there is a Lipschitz constant Lip and a finite subset $q_1, \ldots, q_\ell$ of $[-i, i]$ such that for every $q \in [-i, i]$ there is q_k with

$$\mathrm{Lip}\, |q_k - q| \|\varphi\| < \frac{1}{4i}.$$

To prove (6.1), given q_k, by Lemma 6.4 there is a basic set $X_{i,k}$ such that

$$P_{F|X_{i,k}}(q_k\varphi) \ge \mathcal{P}_{\mathcal{N}}(q_k) - \frac{1}{4i}.$$

Applying Lemma 6.3 consecutively to the finitely many basic sets $X_{i,1}, \ldots, X_{i,\ell}$, we obtain a basic set X_i containing all these sets and satisfying (6.1). This shows (exh4) and (exh5).

By construction, all basic sets are of contracting type and hence all ergodic measures have negative Lyapunov exponent and we have (exh1). Each of them clearly satisfies (exh2) (basic sets have the specification property [S]). By Lemma 4.4 we have the restricted variational principle (exh3).

What remains to prove is that $\varphi(\mathcal{N}) = [\alpha_{\min}, 0)$. By Corollary 3.2), the Lyapunov exponents of ergodic measures extend all the way to 0, that is, $\overline{\varphi}_{\mathcal{N}} = 0$. On the other hand, note that by (P5) we can choose an increasing sequence $(q_j)_j$ tending to $-\infty$ such that $\mathcal{P}_{\mathcal{N}}$ is differentiable at all such q_j. By (P11) and (P12) for every j there is an ergodic equilibrium state μ_j for $q_j\varphi$ and $\varphi(\mu_j) \to \underline{\varphi}_{\mathcal{N}}$. Taking μ' which is a weak$*$ limit of $(\mu_j)_j$ as $j \to \infty$, then there is an ergodic measure μ'' in its ergodic decomposition such that $\varphi(\mu'') = \underline{\varphi}_{\mathcal{N}}$. In particular, we can conclude $\mathcal{L}(\underline{\varphi}_{\mathcal{N}}) \neq \varnothing$ and $\alpha_{\min} = \underline{\varphi}_{\mathcal{N}}$. This concludes the proof that $\varphi(\mathcal{N}) = [\alpha_{\min}, 0)$.

The statement for $\mathcal{N} = \mathcal{M}_{\mathrm{erg},>0}$ is proved analogously.

The proof of the proposition is now complete. □

7. Proof of Theorem 1. Entropy spectrum

In the first section we deal with maximal entropy measures. In the remaining sections, we first study the non-zero part of the spectrum and thereafter analyze the zero level set.

7.1. **Measure(s) of maximal entropy.** To prove the last statement of Theorem 1 about the existence of measure(s) of maximal entropy note that all those measures project to the $(1/N,\dots,1/N)$-Bernoulli measure in the base. Hence, we can use the known results about the behaviour of Bernoulli measures for random dynamical systems. By [C, Theorem 8.6] (stated for products of independently and identically distributed (i.i.d.) diffeomorphisms on a compact manifold) for every Bernoulli measure $\mathfrak{b}$ in $\mathcal{M}(\Sigma_N)$ there exists a (at least one) F-ergodic measure $\mu_+^{\mathfrak{b}}$ with positive exponent and a (at least one) F-ergodic measure $\mu_-^{\mathfrak{b}}$ with negative exponent, both projecting to $\mathfrak{b}=\pi_*\mu_\pm^{\mathfrak{b}}$. Indeed, note that our axioms exclude the possibility of a measure being simultaneously preserved by all the fiber maps, see Lemma 3.7. For $\mathfrak{b}$ being the $(1/N,\dots,1/N)$-Bernoulli measure we simply write $\mu_\pm$.

There are various ways to prove that there are only finitely many hyperbolic ergodic F-invariant measures projecting to the same Bernoulli measure. For example, in our setting it is a consequence of [RH^{2}TU, Theorem 1].

7.2. **The level sets with negative/positive exponents.** By Proposition 6.1 there is a $(\Sigma_N\times\mathbb{S}^1,\varphi,\mathcal{M}_{\mathrm{erg},<0})$-exhausting family (analogously for $\mathcal{M}_{\mathrm{erg},>0}$). Hence, in particular, for every $\alpha\in(\alpha_{\min},0)\cup(0,\alpha_{\max})$ we have $\mathcal{L}(\alpha)\neq\varnothing$ and together with Proposition 5.2 we have

$$h_{\mathrm{top}}(\mathcal{L}(\alpha))\ge\mathcal{E}_{\mathcal{N}}(\alpha)=\lim_{i\to\infty}\sup\big\{h(\mu)\colon\mu\in\mathcal{M}(X_i),\varphi(\mu)=\alpha\big\}.$$

By Lemma 4.2, for every $\alpha\in(\alpha_{\min},\alpha_{\max})$ we have

$$\sup\big\{h(\mu)\colon\mu\in\mathcal{M}_{\mathrm{erg}}(X),\chi(\mu)=\alpha\big\}\le h_{\mathrm{top}}(\mathcal{L}(\alpha)).$$

Together with Lemma 7.1, which is a consequence of general facts on the Legendre-Fenchel transform and facts from [DGR$_2$] (see Section 4), the statements of Theorem 1 about the negative/positive part of the spectrum will follow.

We will analyze the negative part of the spectrum, the analysis of the positive part is analogous and it will be omitted.

Lemma 7.1. *For every $\alpha\in(\alpha_{\min},0)$ we have $h_{\mathrm{top}}(\mathcal{L}(\alpha))\le\mathcal{E}_{<0}(\alpha)$.*

Proof. First, recall that by (E5) for every $\alpha<0$ we have

$$\mathcal{E}_{<0}(\alpha)\ge\sup\big\{h(\mu)\colon\mu\in\mathcal{M}_{\mathrm{erg},<0},\varphi(\mu)=\alpha\big\}.\tag{7.1}$$

Arguing by contradiction, let us assume that there are $\alpha\in(\alpha_{\min},0)$ and $\delta>0$ so that

$$h_{\mathrm{top}}(\mathcal{L}(\alpha))\ge\mathcal{E}_{<0}(\alpha)+2\delta.$$

Then, by continuity of $\mathcal{E}_{<0}(\cdot)$, property (E1), there exists $\varepsilon>0$ such that for every $\alpha'\in(\alpha-2\varepsilon,\alpha+2\varepsilon)$ we have

$$h_{\mathrm{top}}(\mathcal{L}(\alpha))\ge\mathcal{E}_{<0}(\alpha')+\delta.$$

By Proposition 3.3, there exists a basic set $\Gamma\subset\Sigma_N\times\mathbb{S}^1$ such that

$$h_{\mathrm{top}}(F,\Gamma)>h_{\mathrm{top}}(\mathcal{L}(\alpha))-\delta,$$

and that for every $\nu\in\mathcal{M}_{\mathrm{erg}}(\Gamma)$ we have $\chi(\nu)\in(\alpha-\varepsilon,\alpha+\varepsilon)$. Taking the measure of maximal entropy $\nu\in\mathcal{M}_{\mathrm{erg}}(\Gamma)$, with the above for every $\alpha'\in(\alpha-2\varepsilon,\alpha+2\varepsilon)$ we have

$$h(\nu)=h_{\mathrm{top}}(F,\Gamma)>\mathcal{E}_{>0}(\alpha').$$

However, $\alpha'=\chi(\nu)\in(\alpha-\varepsilon,\alpha+\varepsilon)$ would then contradict (7.1). This proves the lemma. $\square$

7.3. **The level sets with zero and extremal exponents.**

Lemma 7.2. $h_0 \stackrel{\text{def}}{=} \lim_{\alpha\searrow 0} h_{\rm top}(\mathcal{L}(0)) = \lim_{\alpha\nearrow 0} h_{\rm top}(\mathcal{L}(0))$.

Proof. By the first part of Theorem 1 proved in Section 7.2 for $\alpha \in (0,\alpha_{\max})$ we have $h_{\rm top}(\mathcal{L}(\alpha)) = \mathcal{E}_{>0}(\alpha)$, hence by (E1) this is a concave function in α. Similarly for $\alpha \in (\alpha_{\min}, 0)$. So we can define the numbers $h_0^\pm = \lim_{\alpha\to 0^\pm} h_{\rm top}(\mathcal{L}(\alpha))$.

By the restricted variational principle in the first part of Theorem 1, for every sequence $\alpha_k \nearrow 0$ there is a sequence of ergodic measures $(\mu_k)_{k\ge 0}$ such that $\chi(\mu_k) = \alpha_k$ and $h(\mu_k) \to h_0^+$. As a consequence of Lemma 3.1 there is a corresponding sequence $(\mu_{-k})_{k\ge1}$ with $\chi(\mu_{-k}) \searrow 0$ and $h(\mu_{-k}) \to h_0^+$. This implies that $h_0^- \ge h_0^+$. Reversing the roles of the negative and positive exponents we get $h_0^+ \ge h_0^-$ and hence $h_0^+ = h_0^-$, proving the lemma. $\square$

Lemma 7.3. $h_{\rm top}(\mathcal{L}(0)) \le h_0$.

Proof. As a consequence of Proposition 3.4 together with Lemma 7.2, for every $\gamma > 0$ and $\lambda > 0$ there exists $\alpha \in (-\lambda, 0)$ such that $h_{\rm top}(\mathcal{L}(\alpha)) \ge h_{\rm top}(\mathcal{L}(0)) - \gamma$. The assertion then follows. $\square$

Lemma 7.4. *For $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$ we have $h_{\rm top}(\mathcal{L}(\alpha)) \le \lim_{\beta\to\alpha} h_{\rm top}(\mathcal{L}(\beta))$.*

Proof. We consider $\alpha = \alpha_{\max}$, the other case is analogous. By the part of Theorem 1 proved already in Section 7.2, for every $\beta \in (0, \alpha_{\max})$ we have already $h_{\rm top}(\mathcal{L}(\beta)) = \mathcal{E}(\beta)$. By Lemma 4.2 and the fact that the ergodic decomposition of a measure with extremal exponent has almost surely only ergodic measures with extremal exponents, we have

$$h_{\rm top}(\mathcal{L}(\alpha_{\max})) \le \sup\big\{h(\mu)\colon \mu \in \mathcal{M}_{\rm erg}, \varphi(\mu) = \alpha_{\max}\big\} \stackrel{\text{def}}{=} h_{\alpha_{\max}}.$$

Hence, for every $q \in \mathbb{R}$ we have

$$\mathcal{P}_{>0}(q) \ge h_{\alpha_{\max}} + q\alpha_{\max},$$

which implies

$$\inf_{q\in\mathbb{R}} \big(\mathcal{P}_{>0}(q) - q\alpha_{\max}\big) \ge h_{\alpha_{\max}}.$$

By [Wi], the left hand side is not larger than $\lim_{\beta\to\alpha_{\max}} \mathcal{E}(\beta)$. $\square$

What remains to prove is the following result. Together with the results in Section 7.2, it will complete the proof of Theorem 1.

Proposition 7.5. *For every $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ we have $\limsup_{\beta\to\alpha} \mathcal{E}(\beta) \le h_{\rm top}(\mathcal{L}(\alpha))$.*

For $\alpha \in (\alpha_{\min}, \alpha_{\max})\setminus\{0\}$ the result of the above proposition follows already from Section 7.2. Moreover, for $\alpha \in \{\alpha_{\min}, \alpha_{\max}\}$ this result can be easily obtained by the following arguments: Take the weak$*$ limit μ of a sequence of measures $(\mu_k)_k$ converging in exponent to α and in entropy to $h = \limsup_{\beta\to\alpha} \mathcal{E}(\beta)$. Indeed, such sequences exist by the already obtained description in the interior of the spectrum. The ergodic decomposition of μ contains an ergodic measure μ' with exponent α and entropy at least h. The set of μ'-generic points is contained in $\mathcal{L}(\alpha)$ which will imply the assertion.

So, we only need to prove the proposition for $\alpha = 0$. However, the proof is completely general. We start by showing some preliminary results.

Lemma 7.6. *Given $\mu \in \mathcal{M}_{\mathrm{erg},<0}$, for every small $\varepsilon > 0$ there exist an interval $I = I(\mu, \varepsilon) \subset \mathbb{S}^1$ and a constant $K = K(\mu, \varepsilon) > 1$ such that for every $n \geq 1$ there exists a set $\Xi(n)$ consisting of finite sequences of length n with cardinality*

$$\operatorname{card}(\Xi(n)) \geq K^{-1} e^{n(h(\mu)-\varepsilon)}$$

such that for each $(\rho_1 \ldots \rho_n) \in \Xi(n)$ and each point $x \in I$ for every $k \in \{1, \ldots, n\}$ the derivative satisfies

$$K^{-1} e^{k(\chi(\mu)-\varepsilon)} \leq |(f_{[\rho_1 \ldots \rho_k]})'(x)| \leq K e^{k(\chi(\mu)+\varepsilon)}.$$

Proof. The first fact is a consequence of ergodicity, the definition of a Lyapunov exponent, the Brin-Katok, the Birkhoff ergodic, and the Egorov theorems (details are given in [DGR$_2$, Proposition 3.1]). Recall the definition of separated points, see [Wa, Chapter 7]. Given $\varepsilon \in (0, \chi(\mu)/2)$, there are a constant $K > 1$ and for $n \geq 1$ a set of $(n, 1)$-separated points $X_i = (\rho^i, x_i) \in \Sigma_N \times \mathbb{S}^1$ of cardinality at least $K^{-1} e^{n(h(\mu)-\varepsilon)}$ such that for every i for every $\ell = 0, \ldots, n-1$ we have

$$K^{-1} e^{\ell(\chi(\mu)-\varepsilon/2)} \leq |(f_{[\rho^i_0 \ldots \rho^i_{\ell-1}]})'(x_i)| \leq K e^{\ell(\chi(\mu)+\varepsilon/2)}. \tag{7.2}$$

As X_i are points with uniform contraction, we obtain control of distortion on some small neighborhood whose size depends on the constant K.

Claim 7.7 ([DGR$_2$, Proposition 3.4])**.** *Given $\varepsilon_D > 0$, let $\delta_0 > 0$ be such that*

$$\max_{i=0,\ldots,N-1} \max_{x,y \in \mathbb{S}^1, |y-x| \leq 2\delta_0} \Big| \log \frac{|f_i'(y)|}{|f_i'(x)|} \Big| \leq \varepsilon_D.$$

If $(\xi, x) \in \mathbb{S}^1$ and $r > 0$ and $n \geq 1$ are such that for every $\ell = 0, \ldots, n-1$ we have

$$|(f_\xi^\ell)'(x)| < \frac{1}{r} \delta_0 e^{-\ell \varepsilon_D},$$

then for every $\ell = 0, \ldots, n-1$ we have

$$\sup_{x,y \colon |y-x| \leq r} \frac{|(f_\xi^\ell)'(y)|}{|(f_\xi^\ell)'(x)|} \leq e^{\ell \varepsilon_D}.$$

Fixing some $\varepsilon < \frac{1}{2} \min\{\varepsilon, |\chi(\mu) - \varepsilon|\}$, let $\delta_0 > 0$ as in the claim and choose also $r > 0$ such that $K < \delta_0 / r$. Thus, for every i and every $y \in (x_i - r, x_i + r)$ for every $\ell = 0, \ldots, n-1$ with (7.2) we obtain

$$K e^{\ell(\chi(\mu)-\varepsilon/2)} e^{-\ell \varepsilon_D} \leq |(f_{[\rho^i_0 \ldots \rho^i_{\ell-1}]})'(y)| \leq K e^{\ell(\chi(\mu)+\varepsilon/2)} e^{\ell \varepsilon_D}.$$

Dividing now $\mathbb{S}^1$ into intervals of length r, at least one interval I of them must contain at least $K^{-1} r^{-1} e^{n(h(\mu)-\varepsilon)}$ starting points x_i of $(n, 1)$-separated trajectories. Exchanging now Kr for K we are done. □

Proof of Proposition 7.5. Given $h = \limsup_{\beta \to \alpha} \mathcal{E}(\alpha)$, there is a sequence of ergodic measures $(\mu_k)_{k \geq 0}$ with Lyapunov exponents converging to α and with the upper limit of entropies equal to h. We aim to prove that $h_{\mathrm{top}}(\mathcal{L}(\alpha)) \geq h$. Without weakening of assumptions, by passing to subsequence we can assume that all the measures μ_k have exponents of the same sign (for example, negative) and that their entropies converge to h.

The proof has two steps. First, we construct a large subset of $\mathcal{L}(\alpha)$. Second, we estimate its entropy.

Step 1: A large subset $\Xi\subset\Sigma_N$ in the projection of $\mathcal{L}(\alpha)$. We consider forward orbits first, that is, we construct forward orbits on which the Lyapunov exponent is α. Fix a sequence $\varepsilon_k\searrow 0$ and apply Lemma 7.6 to all measures μ_k: we get intervals $I_k=I(\mu_k,\varepsilon_k)$, constants K_k, and for every $n\ge 1$ a set $\Xi_k(n)$ of finite sequences of length n. As, by our choice of sequences, $h(\mu_k)\to h$, we can assume that our constants K_k and ε_k are such that for every $k\ge 0$ and for every $n\ge 1$ there are at least

$$K_k^{-1}e^{n(h-\varepsilon_k)}\le \operatorname{card}(\Xi_k(n))\le K_ke^{n(h+\varepsilon_k)}$$

finite sequences $(\rho_1\ldots\rho_n)$ of length n such that for every $x\in I_k$ we have

$$K_k^{-1}e^{n(\chi(\mu_k)-\varepsilon_k)}\le |(f_{[\rho_1\ldots\,\rho_n]})'(x)|\le K_ke^{n(\chi(\mu_k)+\varepsilon_k)}.$$

To each interval I_k we associate numbers $\delta_k>0$ and $M_k>0$ provided by Lemma 3.6. As the chosen orbit pieces are uniformly contracting, we can fix a sequence of sufficiently fast increasing natural numbers $(n_k)_k$ such that for each k for every $(\rho_1\ldots\rho_{n_k})\in\Xi_k(n_k)$ we have that

$$|f_{[\rho_1\ldots\,\rho_{n_k}]}(I_k)|<\delta_{k+1}.$$

Note that n_k can be chosen arbitrarily large. We will specify the choice of this sequence below. Hence, by Lemma 3.6, we can associate to it a finite sequence $(\tau_1\ldots\tau_m)$, $m\le M_{k+1}$, which depends on the initial sequence, such that

$$(f_{[\tau_1\ldots\,\tau_m]}\circ f_{[\rho_1\ldots\,\rho_{n_k}]})(I_k)\subset I_{k+1}.$$

We consider now the set of all such concatenated finite sequences defined by

$$\Xi_k'\overset{\text{def}}{=}\{(\rho_1\ldots\,\rho_{n_k}\tau_1\ldots\,\tau_m)\colon(\rho_1\ldots\,\rho_{n_k})\in\Xi_k(n_k)\},$$

note that here $(\tau_1\ldots\,\tau_m)$ depends on $(\rho_1\ldots\,\rho_{n_k})$. We write $(\rho_1\ldots\,\rho_{n_k})=\varrho$ and $(\tau_1\ldots\,\tau_m)=\vartheta$. We say that ϱ is a *main sequence* and that ϑ is a *connecting sequence*. Finally, we consider the set Ξ^+ of all one-sided infinite sequences

$$\Xi^+\overset{\text{def}}{=}\{\varrho_1\vartheta_1\varrho_2\vartheta_2\ldots\varrho_k\vartheta_k\ldots\colon\varrho_k\vartheta_k\in\Xi_k'\}.$$

Now given any point $x\in I_1$, for every $k\ge 1$ we have already obtained orbit pieces of the form

$$(f_{[\varrho_\ell\vartheta_\ell]}\circ\ldots\circ f_{[\varrho_1\vartheta_1]})(x)\in I_{\ell+1},\quad \ell=1,\ldots,k,$$

where $\varrho_\ell\in\Xi_\ell'$. The cardinality of those pieces is at least $\operatorname{card}\Xi_1'\cdots\operatorname{card}\Xi_k'$. By our choice of quantifiers for every $x\in I_1$ and every $\xi\in\Xi^+$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log|(f_\xi^n)'(x)|=\alpha. \tag{7.3}$$

Until now we proved, that for some interval I_1 there exists a large set of forward-infinite symbolic sequences ξ^+ such that for *every* point $x\in I_1$ the forward orbit of (x,ξ^+) satisfies (7.3). Now we can do the construction in the other (time) direction. It is completely analogous: we take the sequence of measures $\mu_{-1}=\mu_1,\mu_{-2}=\mu_2,\ldots$, take for each of them a set of sequences provided by Lemma 7.6 and connect them using Lemma 3.6. Note that constructed backward itineraries are expanding, thus for a backward itinerary we just get a point following it instead an interval of points as in the forward itinerary. Note also that for each finite concatenation we get a closed interval of starting points and that these intervals form a nested sequence, the point is given by the intersection of these intervals. We obtain a set Ξ^- of backward-infinite symbolic sequences such that for each of them there exist a corresponding backward orbit (y,ξ^-) satisfying

$$\lim_{n\to-\infty}\frac{1}{n}\log|(f_\xi^n)'(y)|=\alpha.$$

and ending at some point in $y\in\mathbb{S}^1$. By Lemma 3.6 we can make each of those backward orbits end at some point in I_1. Hence, each of those trajectories can be prolonged into the future by any $\xi\in\Xi^+$. We obtain a large set of two-sided infinite sequences

$$\Xi\stackrel{\text{def}}{=}\Xi^-.\Xi^+=\{\xi^-.\xi^+\colon\xi^\pm\in\Xi^\pm\}$$

such that $\pi(\mathcal{L}(\alpha))\supset\Xi$, where $\pi\colon\Sigma_N\times\mathbb{S}^1\to\Sigma_N$ denotes the natural projection.

We formalize the meaning of the "large set".

Lemma 7.8. *There are an appropriate choice of the sequence $(n_k)_k$, a constant $K>1$, and a function $\varepsilon=\varepsilon(n)\searrow 0$ such that for every $n\ge 0$, for every $j\in\{0,\dots,n\}$ such that $n-j\le j$ there exist at least $K^{-1}\cdot e^{n(h-\varepsilon(n))}$ different finite sequences $(\rho_1\dots\rho_j)$ such that each of them has at least $K^{-1}\cdot e^{(n-j)(h-\varepsilon(j))}e^{-j\varepsilon(j)}$ and at most $K\cdot e^{(n-j)(h+\varepsilon(j))}e^{j\varepsilon(j)}$ different continuations to sequences in Ξ^+. The same statement holds for Ξ^-, modulo time reversal.*

Proof. We will present the proof for Ξ^+, the other case is analogous. We take the sequence $(n_k)_k$ such that

$$\sum_{i=1}^{k-1}n_i+M_i<n_k,\quad n_k,\tag{7.4}$$

$$\log(K_kK_{k+1})\le n_{k-1}\varepsilon_k\tag{7.5}$$

and

$$\frac{M_k(h+1)}{n_{k-1}}\le\varepsilon_k.\tag{7.6}$$

By the construction above, any sequence in $\xi\in\Xi^+$ can be written as a concatenation of main and connecting sequences $\varrho_1\vartheta_1\varrho_2\vartheta_2\dots\varrho_k\vartheta_k\dots$ such that $|\varrho_k|=n_k$ and $|\vartheta_k|\le M_k$.

First, if j and n are both within a block ϱ_k then by our choice of the constants it follows indeed that at j we have a number of continuations which is in between

$$K_k^{-1}e^{(n-j)(h-\varepsilon_k)}\quad\text{ and }\quad K_ke^{(n-j)(h+\varepsilon_k)}.$$

Suppose now that $j\in\varrho_k$ and $n\notin\varrho_k$. Hence,

$$n_{k-1}\le j\le\sum_{i=1}^{k}n_i+M_i.\tag{7.7}$$

We claim that either $n\in\vartheta_k$ or $n\in\varrho_{k+1}$. Otherwise, we would have $n-j>n_{k+1}$ and by (7.4) we hence would obtain

$$n_{k+1}>\sum_{i=1}^{k}n_i+M_i>j,$$

a contradiction with our hypothesis.

Suppose that $j\in\varrho_k$ and $n\in\varrho_{k+1}$ (the other cases are simpler and thus omitted). Consider a finite sequence $\hat\xi\hat\varrho_k$ of length j such that there is some continuation of

$\hat{\varrho}_k$ to $\varrho_k = \hat{\varrho}_k\tilde{\varrho}_k \in \Xi_k(n_k)$. By Lemma 7.6, for each such finite sequence $\hat{\xi}\hat{\varrho}_k$ the number of such continuations of length $n_k - j$ is between

$$K_k^{-1}e^{(n_k-j)(h-\varepsilon_k)} \quad \text{and} \quad K_k e^{(n_k-j)(h+\varepsilon_k)}.$$

Each of those sequences then can then continued by a connecting sequence whose length m is between 0 and M_k. Finally, this finite sequence of length $n_k + m$ can be prolonged to length n and the number of such continuations is between

$$K_{k+1}^{-1}e^{(n-n_k-M_k)(h-\varepsilon_{k+1})} \quad \text{and} \quad K_{k+1}e^{(n-n_k)(h+\varepsilon_{k+1})}.$$

Summarizing, the number of continuations is between

$$K_k^{-1}K_{k+1}^{-1}e^{(n-j-M_k)(h-\varepsilon_k)} \quad \text{and} \quad K_k K_{k+1}e^{(n-j)(h+\varepsilon_k)}.$$

Note that

$$K_{k+1}K_k e^{(n-j)h}e^{(n-j)\varepsilon_k} \le e^{(n-j)(h+\varepsilon_k)}e^{n_{k-1}\varepsilon_k} \le e^{(n-j)(h+\varepsilon_k)}e^{j\varepsilon_k},$$

where we use (7.5) and that $j \ge n_{k-1}$ by (7.7). This provides the inequality $\le$ in the lemma, with $K = 1$. On the other hand, using equation (7.5) and (7.6) we get

$$K_{k+1}^{-1}K_k^{-1}e^{(n-j)(h-\varepsilon_k)}e^{-M_k(h-\varepsilon_k)} \ge e^{(n-j)(h-\varepsilon_k)}e^{-2j\varepsilon_k}.$$

To complete the proof note that the index k depended on j and hence we can find the sequence $\varepsilon(j)$ as claimed in the lemma. The constant K in the lemma takes care of the remaining cases. □

We demand that the sequence $(n_k)_k$ in Lemma 7.8 was chosen such that the there defined function $\varepsilon(\cdot)$ is such that $\varepsilon(n_k) \le \varepsilon_k$ for every $k \ge 1$. Observe that all the conditions on $(n_k)_k$ are that this sequence grows sufficiently fast. Hence, we can satisfy all those conditions simultaneously.

Note that Ξ depends on the choice of chosen quantifiers $\Xi = \Xi((\varepsilon_k)_k, (n_k)_k)$.

Step 2: Entropy of $\Xi = \Xi((\varepsilon_k)_k, (n_k)_k)$. To estimate the entropy of the set Ξ we use the following classical Frostman's lemma (see [Mt]):

Lemma 7.9 (Mass distribution principle). *Consider a compact metric space* (X, d) *and subset* $\Xi \subset X$. *Let* ν *be a finite Borel measure such that* $\nu(\Xi) > 0$.

Suppose that there exists $D > 0$ *such that for every* $x \in \Xi$ *it holds*

$$\liminf_{\varepsilon\to 0} \frac{\log \nu(B(x,\varepsilon))}{\log \varepsilon} \ge D.$$

Then $\mathrm{HD}(\Xi) \ge D$, *where* HD *denotes the Hausdorff dimension.*

Remark 7.10. Note that for the standard metric for every subset $\Xi \subset \Sigma_N$ its Hausdorff dimension is equal to the topological entropy of σ on Ξ, recall the definition in Appendix.

We apply the above to Ξ as defined above. In view of Lemma 7.9 and Remark 7.10, the following lemma will imply the proposition.

Lemma 7.11. *For appropriate choices of the sequences* $(\varepsilon_k)_k$ *and* $(n_k)_k$ *we have* $h_{\rm top}(\sigma, \Xi) \ge h$, *where* $\Xi = \Xi\big((\varepsilon_k)_k, (n_k)_k\big)$.

Proof. Consider the sequence $(m_k)_{k\ge 0}$ given by $m_k = 2^k$. We define a probability measure ν as being the product of a measure ν^+ depending only on the (forward) one-sided sequences Σ_N^+ and a measure ν^- depending only on the (backward) one-sided sequences Σ_N^-, $\nu(\Delta) = \nu^+(\Delta^+)\cdot\nu^-(\Delta^-)$, where Δ^- and Δ^+ denote its projection to Σ_N^- and Σ_N^+, respectively. The measure ν^+ is constructed as follows (the measure ν^- is analogous): for every $k \ge 1$

- ν^+ is uniformly distributed on all cylinders of level m_1 intersecting Ξ,
- for every cylinder of level m_k intersecting Ξ, ν^+ is uniformly subdistributed on its subcylinders of level m_{k+1} intersecting Ξ.

Given $n \ge 1$, consider some cylinder $\Delta = \Delta^-.\Delta^- = \Delta_{\xi_{-m}\cdots\xi_{-1}.\xi_0\cdots\xi_{n-1}}$ which has nonempty intersection with Ξ. We are going to estimate $\nu(\Delta) = \nu^+(\Delta^+)\cdot\nu^-(\Delta^-)$. Let us consider Δ^+ (the other term is analogous). There exists a unique index $k \ge 0$ such that $m_k < n \le m_{k+1}$.

Note that the sequence $(n_k)_k$ grows much faster than the sequence $(m_k)_k$. We have

$$\nu^+(\Delta^+) = \nu^+(\Delta^+_{m_0})\cdot\left(\frac{\nu^+(\Delta^+_{m_1})}{\nu^+(\Delta^+_{m_0})}\cdots\frac{\nu^+(\Delta^+_{m_k})}{\nu^+(\Delta^+_{m_{k-1}})}\right)\cdot\frac{\nu^+(\Delta^+)}{\nu^+(\Delta^+_{m_k})} = T_1\cdot T_2\cdot T_3,$$

where Δ_i^+ is the corresponding parent m_i-cylinder of Δ^+, $i = 1,\ldots,k$.

Let us now estimate the above three terms T_1, T_2, and T_3. By Lemma 7.8 applied to $n = m_0$ and $j = 0$ we have

$$T_1 \le Ke^{m_0(h-\varepsilon(m_0))}.$$

For every $i = 1,\ldots,k$ by Lemma 7.8 applied to $n = m_i$ and $j = m_{i-1}+1$, we obtain that each cylinder $\Delta^+_{m_i}$ contains at most

$$Ke^{-(m_i-m_{i-1})(h+\varepsilon(m_{i-1}))}e^{m_{i-1}\varepsilon(m_{i-1})}$$

cylinders of length m_{i+1}. Hence we can estimate

$$T_2 \le \prod_{i=1}^{k} Ke^{-(m_i-m_{i-1})(h+\varepsilon(m_{i-1}))}e^{m_{i-1}\varepsilon(m_{i-1})} = e^{-(m_k-m_0)h}S(k),$$

where

$$\lim_{k\to\infty}\frac{1}{m_k}\log S(k) = 0.$$

Finally, by Lemma 7.8 applied to $n = m_{k+1}$ and $j = m_k+1$ as well as to $n = m_{k+1}$ and $j = n+1$ we get

$$T_3 \le \frac{Ke^{-(m_{k+1}-m_k)(h-\varepsilon(m_{k+1}))}e^{m_k\varepsilon(m_k)}}{K^{-1}e^{-(m_{k+1}-n)(h+\varepsilon(n))}e^{-n\varepsilon(n)}} = e^{-(n-m_k)h}R(k),$$

where

$$\lim_{k\to\infty}\frac{1}{m_k}\log R_k = 0.$$

Putting together the previous estimates, we get

$$\nu^+(\Delta^+) \le e^{-nh}Q(k), \quad \text{where} \quad \lim_{k\to\infty}\frac{1}{m_k}\log Q(k) = 0.$$

Given $n \ge 1$ and $\xi \in \Xi$, denote by $\Delta_n^+(\xi)$ the cylinder of length n containing ξ. Hence, as $n \ge 1$ was arbitrary for every $\xi \in \Xi$ we obtain

$$-\frac{1}{n+1}\log\nu^+(\Delta_n^+(\xi)) \ge h + P(n),$$

where $\lim_{n\to\infty} P(n) = 0$. Analogous arguments apply to ν^-.

Denote by $\Delta_{m,n}(\xi)$ the cylinder $\Delta_{\xi_{-m}\dots\xi_{-1}.\xi_0\dots\xi_{n-1}} = \Delta^-_m(\xi).\Delta^+_n(\xi)$. Hence, considering the product measure ν, for every $\xi\in\Xi$ and every $m,n\ge 1$ we obtain

$$-\frac{1}{m+n+1}\log\nu(\Delta_{m,n}(\xi)) = -\frac{\log\left(\nu(\Delta^+_n(\xi))\nu(\Delta^-_m(\xi))\right)}{m+n+1}\ge h+P(m)+P(n).$$

Note that, by construction, we have $\nu(\Xi)=1$. By Lemma 7.9 applied to the probability measure ν on the space Σ_N we obtain $h_{\rm top}(\sigma,\Xi)\ge h$. □

As the projection π does not increase entropy, we conclude $h_{\rm top}(F,\mathcal{L}(\alpha))\ge h$. The proof of Proposition 7.5 is now complete. □

8. Proof of Theorem 2. Measures of maximal entropy

8.1. **Synchronization.** As explained in Section 7.1, any measure(s) of maximal entropy (with respect to F) project(s) to the $(1/N,\dots,1/N)$-Bernoulli measure $\mathfrak{b}$.

We call a Bernoulli measure $\mathfrak{b}=(b_1,\dots,b_N)$ *nondegenerate* if all weights b_i, $i=1,\dots,N$, are positive. By [C, Theorem 8.6] together with Lemma 3.7, given any nondegenerate Bernoulli measure $\mathfrak{b}$, there exists a (at least one) F-ergodic measure $\mu^{\mathfrak{b}}_+$ with positive exponent and a (at least one) F-ergodic measure $\mu^{\mathfrak{b}}_-$ with negative exponent, both projecting to $\mathfrak{b}=\pi_*\mu^{\mathfrak{b}}_\pm$.

Following for example [Ml], given a Bernoulli measure $\mathfrak{b}$, we say that an IFS $\{f_i\}_{i=0}^{N-1}$ with probabilities $\mathfrak{b}$ is *forward synchronizing* if for every $x,y\in\mathbb{S}^1$ for $\mathfrak{b}$-almost every one-sided sequence $\xi\in\Sigma_N^+$ we have

$$|f^n_\xi(x)-f^n_\xi(y)|\to 0. \tag{8.1}$$

Backward synchronization is defined the same way, but for the IFS $\{f_i^{-1}\}_{i=0}^{N-1}$.

The IFS $\{f_i\}_{i=0}^{N-1}$ is (*forward*) *proximal* if for every $x,y\in\mathbb{S}^1$ there exists at least one sequence $\xi\in\Sigma_N^+$ such that (8.1) holds, *backward proximality* we define analogously. By [Ml, Theorem E], proximality of the IFS implies that every nondegenerate Bernoulli measure satisfies forward synchronization. Similarly, backward proximality implies backward synchronization.[7]

Lemma 8.1. *For every Bernoulli measure $\mathfrak{b}$ satisfying synchronization, the measures $\mu^{\mathfrak{b}}_\pm$ provided by* [C, Theorem 8.6] *are unique.*

Proof. Consider the set

$$B=\left\{(x,y,\xi)\in\mathbb{S}^1\times\mathbb{S}^1\times\Sigma_N^+\colon \limsup_{n\to\infty}|f^n_\xi(x)-f^n_\xi(y)|>0\right\}$$

and write

$$B_{(x,y)}=\left(\{(x,y)\}\times\Sigma_N^+\right)\cap B.$$

Note that forward synchronization means that for every $(x,y)\in\mathbb{S}^1\times\mathbb{S}^1$ it holds that $\mathfrak{b}(B_{(x,y)})=0$.

Given $\xi\in\Sigma_N^+$, divide $\mathbb{S}^1$ into equivalence classes by the relation

$$x\sim_\xi y \iff \lim_{n\to\infty}|f^n_\xi(x)-f^n_\xi(y)|=0.$$

One can easily check that it is indeed an equivalence relation. As the fiber maps are homeomorphisms, those equivalence classes are simply connected, i.e. intervals. Note that in principle in a given fiber ξ there may exist uncountably many classes.

[7] In fact, [Ml, Theorem E] (stated for groups of circle homeomorphisms) shows even that we have exponential synchronization, that is, for a given Bernoulli measure convergence in (8.1) is exponential. However, we will not make use of this fact.

However there can exist only countably many classes which are nontrivial intervals. Let us denote these classes by $C_i(\xi)$ with $i \in I(\xi)$. We will see that in almost every fiber there is exactly one nontrivial class, and that it is of Lebesgue measure 1 and hence the whole circle minus one point.

Note that if $(x,y,\xi) \notin B$, $x \neq y$, then there is an index $i = i(x,y,\xi)$ such that $x, y \in C_i(\xi)$. As the diagonal $\{(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1; x = y\}$ has $\mathrm{Leb} \times \mathrm{Leb}$ measure zero, we have

$$(\mathrm{Leb} \times \mathrm{Leb} \times \mathfrak{b})(B^c) = \int_{\Sigma_N^+} \left(\sum_{i \in I(\xi)} \mathrm{Leb}(C_i(\xi)))^2 \right) d\mathfrak{b}(\xi). \tag{8.2}$$

By the comments above, synchronization, and Fubini's Theorem we have that $(\mathrm{Leb} \times \mathrm{Leb} \times \mathfrak{b})(B^c) = 1$, hence the integrand in (8.2) is 1 almost everywhere. As $\sum \mathrm{Leb}\, C_i(\xi) \leq 1$, $\sum (\mathrm{Leb}\, C_i(\xi))^2 = 1$ can happen if, and only if, $I(\xi) = \{1\}$ and $\mathrm{Leb}\, C_1(\xi) = 1$. Therefore, for $\mathfrak{b}$-almost every ξ the index set $I(\xi)$ consists of exactly one element and there is exactly one class $C(\xi)$ with full Lebesgue measure. This class is the whole circle except one point. Let us denote by $z(\xi)$ this missing point.

Clearly, the disintegration of the positive exponent invariant measure $\mu_+^{\mathfrak{b}}$ is supported on $z(\xi)$ for almost all ξ. In particular, $\mu_+^{\mathfrak{b}}$ is unique.

The same arguments applied to the IFS $\{f_i^{-1}\}$ prove uniqueness of $\mu_-^{\mathfrak{b}}$. □

8.2. **End of the proof of Theorem 2.** We can now conclude the proof of Theorem 2. Assume that the second conclusion in the theorem is not true, that is, that there exists $\alpha \neq \alpha_\pm = \chi(\mu_\pm)$ such that $h_{\mathrm{top}}(\mathcal{L}(\alpha)) = \log N$. Let us assume that $\alpha \geq 0$, the proof of the other case is analogous. By (E4) we have $\log N = \mathcal{E}_{\mathcal{N}}(\alpha')$ with $\mathcal{N} = \mathcal{M}_{\mathrm{erg},>0}$ for all α' between α and α_+. Hence, by (E4) the function $\mathcal{P}_{\mathcal{N}}$ is not differentiable at 0. By (P10), there exist two ergodic measures of maximal entropy (ergodic equilibrium states for 0) (with respect to $\mathcal{N}$) with exponents given by the (different) left and right derivatives $D_{L/R}\mathcal{P}_{\mathcal{N}}(0)$, these derivatives being nonnegative by the choice of $\mathcal{N}$. Hence, there would exist two ergodic measures of maximal entropy with two distinct nonnegative Lyapunov exponents, contradicting Lemma 8.1. □

8.3. **Proof of Corollary 3.** By Lemma 8.1 applied to the (Bernoulli) measure of maximal entropy we have that there are exactly two measures of maximal entropy. Arguing by contradiction, suppose that there is another measure μ of maximal entropy which is the weak$*$ and in entropy limit of a sequence of ergodic measures. If this measure is ergodic we are done. Otherwise almost every measure in its (nontrivial) ergodic decomposition has maximal entropy. Hence this measure is a (nontrivial) linear combination of μ_+ and μ_- and, in particular, $\alpha = \chi(\mu) \in (\alpha_-, \alpha_+)$. Without weakening of assumptions let $\alpha \geq 0$. Recall that the function $\mathcal{E}_{>0}$ is continuous and has a unique global maximum at α_+. Hence for δ small $\mathcal{E}_{>0}(\alpha') < \log N - \delta$ for all α' in a small neighbourhood of α. If there would exist such a sequence of ergodic measures weak$*$ (and hence in Lyapunov exponent) and in entropy converging to μ then eventually the Lyapunov exponents of the measures would be arbitrarily close to α and their entropies arbitrarily close to $\log N$. This provides a contradiction with the above inequality. □

9. Proof of Theorem 5. Shapes of pressure and Lyapunov spectrum

In this section we will prove Theorem 5. Recall the properties of pressure, Legendre-Fenchel transform and convex functions given in Section 4.

The most important property, property g), (which we are going to constantly apply below) is formulated in Theorem 1.

Property a): convexity follows from basic properties of pressure, the derivative of the pressure function is equal to the Lyapunov exponent of the corresponding equilibrium state by the definition of Legendre-Fenchel transform. For the pressure $\mathcal{P}_{>0}$ all the equilibrium states have nonnegative Lyapunov exponent, for the pressure $\mathcal{P}_{<0}$ all the equilibrium states have nonpositive Lyapunov exponent.

Property b) follows from the fact that the sets $\mathcal{M}_{\mathrm{erg},>0}$ and $\mathcal{M}_{\mathrm{erg},<0}$ contain measures with arbitrarily small Lyapunov exponent (Corollary 3.2) we know that the limit derivative of both $\mathcal{P}_{>0}$ (as $q \to -\infty$) and $\mathcal{P}_{<0}$ (as $q \to \infty$) is zero. The fact that those are indeed plateaus, not asymptotic behaviour, follows from property i) proved below. Indeed, by the definition of Legendre-Fenchel transform, $D_- = D_R\mathcal{E}(0)$ and $D_+ = D_L\mathcal{E}(0)$.

Property c) follows from Theorem 1. Indeed, by the definition of Legendre-Fenchel transform, $h_+ = \lim_{\alpha\searrow 0}\mathcal{E}(\alpha)$ and $h_- = \lim_{\alpha\nearrow 0}\mathcal{E}(\alpha)$.

Property d) follows from Theorem 1 and property a). Indeed, a concave function with maximum in the interior of the domain is nonincreasing to the right of the maximum and nondecreasing to the left of the maximum.

Property e) follows immediately from Theorem 1, because by the basic properties of entropy $\mathcal{P}_{>0}(0)$ is the supremum of entropies of ergodic measures with positive Lyapunov exponents (and similarly $\mathcal{P}_{<0}(0)$ - negative Lyapunov exponents) and those classes of measures both contain a measure of maximal entropy.

Properties f) and h) are formulated in Theorem 1.

Property i) is the only one that needs some more extended proof. An immediate consequence of Lemma 3.1 and property h) is that there exists $c > 0$ such that

$$\mathcal{E}(0) \geq \frac{\mathcal{E}(\alpha)}{1 + c|\alpha|}$$

for all $\alpha \neq 0$. Hence,

$$\frac{\mathcal{E}(\alpha) - \mathcal{E}(0)}{|\alpha|} \leq c\mathcal{E}(0).$$

Passing with α to zero, we get

$$\max(D_R\mathcal{E}(0), -D_L\mathcal{E}(0)) \leq c\mathcal{E}(0), \tag{9.1}$$

which proves the finiteness of derivatives $D_L\mathcal{E}(0), D_R\mathcal{E}(0)$. The other inequality follows from convexity of $\mathcal{E}_{<0}$ and $\mathcal{E}_{>0}$ and property f).

Property j) is proved in the course of proof of property i). Indeed, equation (9.1) implies that if $h_{\mathrm{top}}(\mathcal{L}(0)) = 0$ then $D_L\mathcal{E}(0), D_R\mathcal{E}(0) = 0$. However, $D_L\mathcal{E}(0), D_R\mathcal{E}(0) = 0$ implies that $\mathcal{E}$ has maximum at zero, that is, $h_{\mathrm{top}}(\mathcal{L}(0)) = \log N$ - a contradiction.

Assume now that we are under assumptions of Theorem 2, that is that we have exactly two maxima of the spectrum $\mathcal{E}(\alpha)$, achieved at points $\alpha_- < 0$ and $\alpha_+ > 0$. As the concave function with a unique maximum in the interior of the domain has negative derivative (or one-sided derivatives if the derivative is not defined) to the right of the maximum and positive derivative to the left of the maximum, the required changes in items d) and i) follow immediately. The property k) follows from (local) uniqueness of the maximum by (P11). □

Appendix. Entropy

Let X be a compact metric space. Consider a continuous map $f\colon X\to X$, a set $Y\subset X$, and a finite open cover $\mathscr{A}=\{A_1,A_2,\ldots,A_n\}$ of X. Given $U\subset X$ we write $U\prec\mathscr{A}$ if there is an index j so that $U\subset A_j$, and $U\not\prec\mathscr{A}$ otherwise. Taking $U\subset X$ we define

$$n_{f,\mathscr{A}}(U):=\begin{cases}0 & \text{if } U\not\prec\mathscr{A},\\ \infty & \text{if } f^k(U)\prec\mathscr{A}\ \forall k\in\mathbb{N},\\ \ell & \text{if } f^k(U)\prec\mathscr{A}\ \forall k\in\{0,\ldots,\ell-1\}, f^\ell(U)\not\prec\mathcal{A}.\end{cases}$$

If $\mathcal{U}$ is a countable collection of open sets, given $d>0$ let

$$m(\mathscr{A},d,\mathcal{U}):=\sum_{U\in\mathcal{U}}e^{-d\,n_{f,\mathscr{A}}(U)}.$$

Given a set $Y\subset X$, let

$$m_{\mathscr{A},d}(Y):=\lim_{\rho\to0}\inf\Big\{m(\mathscr{A},d,\mathcal{U})\colon Y\subset\bigcup_{U\in\mathcal{U}}U, e^{-n_{f,\mathcal{A}}(U)}<\rho \text{ for every } U\in\mathcal{U}\Big\}.$$

Analogously to the Hausdorff measure, $d\mapsto m_{\mathcal{A},d}(Y)$ jumps from ∞ to 0 at a unique critical point and we define

$$h_{\mathscr{A}}(f,Y):=\inf\{d\colon m_{\mathscr{A},d}(Y)=0\}=\sup\{d\colon m_{\mathscr{A},d}(Y)=\infty\}.$$

The *topological entropy* of f on the set Y is defined by

$$h_{\rm top}(f,Y):=\sup_{\mathscr{A}}h_{\mathscr{A}}(f,Y),$$

When $Y=X$, we simply write $h_{\rm top}(X)=h_{\rm top}(f,X)$.

By [B$_1$, Proposition 1], in the case of Y compact this definition is equivalent to the canonical definition of topological entropy (see, for example, [Wa, Chapter 7]).

References

[BS] L. Barreira and B. Saussol, *Variational principles and mixed multifractal spectra,* Trans. Amer. Math. Soc. **353** (2001), 3919–3944.

[BBD] Ch. Bonatti, J. Bochi, and L. J. Díaz, *Robust Criterion for the Existence of Nonhyperbolic Ergodic Measures*, Comm. Math. Phys. **344** (2016), 751–795.

[BDU] Ch. Bonatti, L. J. Díaz, and R. Ures, *Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms*, J. Inst. Math. Jussieu **1** (2002), 513–541.

[B$_1$] R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125–136.

[B$_2$] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Second revised edition*, Lecture Notes in Mathematics, 470, Springer-Verlag, Berlin, 2008.

[BG] K. Burns and K. Gelfert, *Lyapunov spectrum for geodesic flows of rank 1 surfaces*, Discrete Contin. Dyn. Syst. **34** (2014), 1841–1872.

[CY] W. Cowieson and L.-S. Young, *SRB measures as zero-noise limits*, Ergodic Theory Dynam. Systems **25** (2005), 1115–1138.

[C] H. Crauel, *Extremal exponents of random dynamical systems do not vanish*, J. Dyn. Diff. Equations **2** (1990), 245–291.

[DER] L. J. Díaz, S. Esteves, and J. Rocha, *Skew product cycles with rich dynamics: from totally non-hyperbolic dynamics to fully prevalent hyperbolicity*, Dyn. Syst. **31** (2016),1–40.

[DF] L. J. Díaz and T. Fisher, *Symbolic extensions and partially hyperbolic diffeomorphisms*, Discrete Contin. Dyn. Sys. **29** (2011),1419–1441.

[DGR$_1$] L. J. Díaz and K. Gelfert, *Procupine-like horseshoes: Transitivity, Lyapunov spectrum, and phase transitions*, Fund. Math. **216** (2012), 55–100.

[DGR$_2$] L. J. Díaz, K. Gelfert, and M. Rams, *Nonhyperbolic step skew-products: Ergodic approximation*, To appear in: Ann. Inst. H. Poincaré Anal. Non Linéaire.

[FLP] A. Fan, L. Liao, and J. Peyriére, *Generic points in systems of specification and Banach valued Birkhoff ergodic averages*, Discrete Contin. Dyn. Sys. **21** (2008), 1103–1128.

[GPR] K. Gelfert, F. Przytycki, and M. Rams, *On the Lyapunov spectrum for rational maps*, Math. Annalen **348** (2010), 965–1004.

[GR] K. Gelfert and M. Rams, *The Lyapunov spectrum of some parabolic systems*, Ergodic Theory Dynam. Systems **29** (2009), 919–940.

[IT] G. Iommi and M. Todd, *Dimension theory for multimodal maps*, Ann. Henri Poincaré **12** (2011), no. 3, 591–620.

[J] O. Jenkinson, *Ergodic optimization*, Discrete Contin. Dyn. Syst. **15** (2006), 197–224.

[LOR] R. Leplaideur, K. Oliveira, and I. Rios, *Equilibrium states for partially hyperbolic horseshoes*, Ergodic Theory Dynam. Systems **31** (2011), 179–195.

[MS] N. Makarov and S. Smirnov, *On "thermodynamics" of rational maps. I. Negative spectrum*, Comm. Math. Phys. **211** (2000), 705–743.

[Ml] D. Malicet, *Random walks on* $\mathrm{Homeo}(\mathbb{S}^1)$, Preprint `arXiv:1412.8618`.

[Mt] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability*, Cambridge Studies in Advanced Mathematics, 44, Cambridge University Press, Cambridge, 1995.

[O] L. Olsen, *A multifractal formalism*, Adv. Math. **116** (1995), 82–196.

[PW] Y. Pesin and H. Weiss, *The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples*, Chaos **7** (1997), 89–106.

[PS] C.-E. Pfister and W.G. Sullivan, *On the topological entropy of saturated sets*, Ergodic Theory Dynam. Systems **27** (2007), 929–956.

[PR] F. Przytycki, J. Rivera-Letelier, *Geometric pressure for multimodal maps of the interval*, Preprint `arXiv:1405.2443v1`.

[PRS] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Syst. **24** (2004), 891–914.

[R] C. Robinson, *Dynamical Systems. Stability, Symbolic Dynamics, and Chaos*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL (1995).

[RH^{2}TU] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, and R. Ures, *Maximizing measures for partially hyperbolic systems with compact center leaves*, Ergodic Theory Dynam. Systems **32** (2012), 825–839.

[R] D. Ruelle, *Thermodynamic Formalism: the Mathematical Structures of Classical Equilibrium Statistical Mechanics*, Encyclopedia of Mathematics and its Applications **5**, Reading, MA, Addison-Wesley (1978).

[S] K. Sigmund, *On dynamical systems with the specification property*, Trans. Amer. Math. Soc. **190** (1974), 285–299.

[TY] A. Tahzibi and J. Yang, *Strong hyperbolicity of ergodic measures with large entropy*, Preprint `arXiv:1606.09429`.

[TV] F. Takens and E. Verbitskiy, *On the variational principle for the topological entropy of certain non-compact sets*, Ergodic Theory Dynam. Systems **23** (2003), 317–348.

[Wa] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

[Wi] R. Wijsman, *Convergence of sequence of convex sets, cones, and functions. II*, Trans. Amer. Math. Soc. **123** (1966), 32–45.

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