MULTIFRACTAL ANALYSIS OF THE BIRKHOFF SUMS OF SAINT-PETERSBURG POTENTIAL

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Abstract. Let $((0,1],T)$ be the doubling map in the unit interval and φ be the Saint-Petersburg potential, defined by $\varphi(x)=2^n$ if $x\in(2^{-n-1},2^{-n}]$ for all $n\geq 0$. We consider asymptotic properties of the Birkhoff sum $S_n(x)=\varphi(x)+\cdots+\varphi(T^{n-1}(x))$. With respect to the Lebesgue measure, the Saint-Petersburg potential is not integrable and it is known that $\frac{1}{n\log n}S_n(x)$ converges to $\frac{1}{\log 2}$ in probability. We determine the Hausdorff dimension of the level set $\{x:\lim_{n\to\infty}S_n(x)/n=\alpha\}$ $(\alpha>0)$, as well as that of the set $\{x:\lim_{n\to\infty}S_n(x)/\Psi(n)=\alpha\}$ $(\alpha>0)$, when $\Psi(n)=n\log n$, n^a or 2^{n^γ} for $a>1$, $\gamma>0$. The fast increasing Birkhoff sum of the potential function $x\mapsto 1/x$ is also studied.

1. Introduction

Let T be the doubling map on the unit interval $(0,1]$ defined by

$$Tx=2x-\lceil 2x\rceil+1,$$

where $\lceil x\rceil$ is the smallest integer larger than or equal to x. Let ϵ_1 be the function defined by $\epsilon_1(x)=\lceil 2x\rceil-1$ and $\epsilon_n(x):=\epsilon_1(T^{n-1}x)$ for $n\geq 2$. Then each real number $x\in(0,1]$ can be expanded into *an infinite series* as

$$x=\frac{\epsilon_1(x)}{2}+\cdots+\frac{\epsilon_n(x)}{2^n}+\cdots. \tag{1.1}$$

We call (1.1) the binary expansion of x and also write it as

$$x=[\epsilon_1(x)\epsilon_2(x)\ldots].$$

The Saint-Petersburg potential is a function $\varphi:(0,1]\to\mathbb{R}$ defined as

$$\varphi(x)=2^n \text{ if } x\in(2^{-n-1},2^{-n}],\ \forall n\geq 0.$$

We remark that the definition of φ is equivalent to

$$\varphi(x)=2^n \text{ where } n\geq 0 \text{ is the smallest integer such that } \epsilon_{n+1}(x)=1.$$

and is also equivalent to

$$\varphi(x)=2^n \text{ if the binary expansion of } x \text{ begins with } 0^n1,$$

where $0^n(n\geq 0)$ means a block with n consecutive zeros.

The name of Saint-Petersburg potential is motivated by the famous Saint-Petersburg game in probability theory. The Saint-Petersburg potential is of infinite expectation with respect to the Lebesgue measure. Furthermore, it increases exponentially fast near to the point 0.

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In this paper, we are concerned with the following Birkhoff sums of the Saint-Petersburg potential:

$$\forall n \geq 1, \quad S_n(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \qquad x \in (0,1].$$

Let

$$I = \{x \in (0,1] : \epsilon_1(x) = 1\}.$$

Define the hitting time of $x \in (0,1]$ to I as

$$n(x) := \inf\{n \geq 0 : T^n x \in I\}.$$

Then

$$n(x) = n \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0.$$

Using $n(x)$, we define a new dynamical system $\widehat{T} : (0,1] \to (0,1]$ by

$$\widehat{T}(x) = T^{n(x)+1}(x) = 2^{n+1}\Big(x - \frac{1}{2^{n+1}}\Big) \text{ if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0,$$

called the acceleration of T, in order that φ and $\varphi \circ \widehat{T}$ are independent. Let

$$\widehat{S}_n(x) := \varphi(x) + \varphi(\widehat{T}(x)) + \cdots + \varphi(\widehat{T}^{n-1}(x)), \qquad x \in (0,1].$$

The convergence in probability of $\widehat{S}_n(x)$ is well known (e.g. [6, p.253]) which states that for any $\epsilon > 0$, the Lebesgue measure λ of

$$\left\{x \in (0,1] \ : \ \left|\frac{\widehat{S}_n(x)}{n \log n} - \frac{1}{\log 2}\right| \geq \epsilon\right\}$$

tends to 0 as $n \to \infty$.

Let $\{\Psi_n\}_{n \geq 1}$ be an increasing sequence such that $\Psi_n \to \infty$ as $n \to \infty$. Then it was shown in [5] that almost surely either

$$\lim_{n\to\infty} \frac{\widehat{S}_n(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{\widehat{S}_n(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n \geq 1} \lambda(\{x \in (0,1] : \varphi(x) \geq \Psi_n\}) < \infty \quad \text{or} \quad = \infty.$$

Let $n_1 = n_1(x) = n(x) + 1$ and $n_k = n_k(x) = n_1(\widehat{T}^{k-1}x) = n(\widehat{T}^{k-1}x)$ for $k \geq 2$. It is direct to see that

$$\forall \ell \geq 1, \quad S_{n_1 + \cdots + n_\ell}(x) = 2\widehat{S}_\ell(x) - \ell.$$

Moreover, the ergodicity of T (of $\widehat{T}$) implies

$$\lim_{\ell\to\infty} \frac{n_1(x) + \cdots + n_\ell(x)}{\ell} = \int_0^1 (n(x)+1)\, d\lambda(x) = 2.$$

Combining these two facts together, we obtain the same convergence results as above if we replace $\widehat{S}_n$ by S_n.

In this article, we want to give a complete multifractal analysis of the Birkhoff sum S_n.

First, for any $\alpha \geq 1$, we consider the level set

$$E(\alpha) = \left\{x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} S_n(x) = \alpha\right\}.$$

For $t \in \mathbb{R}$ and $q > 0$, define

$$P(t,q) := \log \sum_{j=1}^{\infty} 2^{-tj - q(2^j - 1)}.$$

Then P is a real-analytic function. Furthermore, for each $q > 0$, there is a unique $t(q) > 0$ such that $P(t(q), q) = 0$. This function $q \mapsto t(q)$ is real-analytic, strictly decreasing and convex.

Denote by $\dim_H$ the Hausdorff dimension. The function $\alpha \mapsto \dim_H E(\alpha)$, called the Birkhoff spectrum of the Saint-Petersburg potential φ, is proved to be the Legendre transformation of the function $q \mapsto t(q)$.

Theorem 1.1. *For any $\alpha \geq 1$ we have*

$$\dim_H E(\alpha) = \inf_{q>0}\{t(q) + q\alpha\}.$$

Consequently, $\dim_H E(1) = 0$ and the function $\alpha \mapsto \dim_H E(\alpha)$ is real-analytic, strictly increasing, concave, and has limit 1 as $\alpha \to \infty$.

The Birkhoff spectrum of a continuous potential was obtained for full shifts [13], for topologically mixing subshifts of finite type [4], and for repellers of a topologically mixing $C^{1+\epsilon}$ expanding map [2]. A continuous potential in a compact space is bounded, hence these classical results are all for bounded potentials. Our Theorem 1.1 gives a Birkhoff spectrum for an unbounded function with a singular point. To prove Theorem 1.1, we will transfer our question to a Birkhoff spectrum problem of an interval map with infinitely many branches and we will apply the techniques developed in [9] for continued fraction dynamical system and in [8] for general expanding interval maps with infinitely many branches.

We also study the Birkhoff sums $S_n(x)$ of fast increasing rates. Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function. For $\beta > 0$, consider the level set

$$E_\Psi(\beta) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = \beta \right\}.$$

Theorem 1.2. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n\log n,\ \ \Psi(n) = n^a\ \ (a > 1),\ \ \Psi(n) = 2^{n^\gamma}\ \ (0 < \gamma < 1/2),$$

then for any $\beta > 0$, $\dim_H E_\Psi(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1/2$, then for any $\beta > 0$, the set $E_\Psi(\beta)$ is empty.

Our method for studying the fast increasing Birkhoff sum of Saint-Petersburg potential also works for the fast increasing Birkhoff sum of the potential $g : x \mapsto 1/x$ which is continuous with singular point 0.

Denote by $S_n g(x)$ the Birkhoff sum

$$S_n g(x) := g(x) + g(T(x)) + \cdots + g(T^{n-1}(x)), \qquad x \in (0,1].$$

For $\beta > 0$, let

$$F_\Psi(\beta) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n g(x) = \beta \right\}.$$

Theorem 1.3. *If $\Psi(n)$ is one of the following*

$$\Psi(n) = n\log n,\ \ \Psi(n) = n^a\ \ (a > 1),\ \ \Psi(n) = 2^{n^\gamma}\ \ (0 < \gamma < 1/2),$$

then for any $\beta > 0$, $\dim_H F_\Psi(\beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1/2$, then for any $\beta > 0$, the set $F_\Psi(\beta)$ is empty.

We remark that these multifractal analysis on the Birkhoff sums of fast increasing rates have been done for some special potentials in continued fraction dynamical system ([9, 11, 12]).

2. Birkhoff spectrum of the Saint-Petersburg potential

In this section, we will obtain the Birkhoff spectrum of the Saint-Petersburg potential, i.e. the Hausdorff dimension of the following level set:

$$E(\alpha) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad (\alpha \geq 1).$$

We will transfer our question to a Birkhoff spectrum problem for an interval map with infinitely many branches.

2.1. **Transference lemma.** Recall that the Saint-Petersburg potential φ is given by

$$\varphi(x) = 2^n, \text{ if } \ x = [0^n 1, \cdots]$$

where $x = [\epsilon_1 \epsilon_2, \cdots]$ denotes the digit sequence in the binary expansion of x. Recall also the definition of hitting time $n(x)$ and the acceleration $\widehat{T}$ of the doubling map T in Section 1. Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \ x \in (0,1].$$

In fact, ϕ is nothing but the function satisfying

$$\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x).$$

With the notation $n_1 = n(x) + 1 \geq 1$, and $n_k = n(\widehat{T}^{k-1}x) + 1$ for $k \geq 2$ given in Section 1, we have

$$\phi(\widehat{T}x) = \sum_{j=0}^{n(\widehat{T}x)} \varphi(T^j(\widehat{T}x)) = \sum_{j=n_1}^{n_2-1} \varphi(T^j x).$$

Hence,

$$\sum_{j=0}^{n_1+\cdots+n_\ell-1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x) = 2^{n_1} + \cdots + 2^{n_\ell} - \ell. \tag{2.1}$$

Note that the derivative of $\widehat{T}$ satisfies

$$|\widehat{T}'|(x) = 2^{n(x)+1} = 2^{n_1} = \phi(x) + 1. \tag{2.2}$$

We have

$$n_1 + \cdots + n_\ell = \sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x).$$

Recall the set in question:

$$E(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \qquad (\alpha \geq 1).$$

Define

$$\widetilde{E}(\alpha) := \left\{ x \in (0,1] : \lim_{\ell\to\infty} \frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \alpha \right\} \qquad (\alpha \geq 1).$$

The following lemma shows the two level sets are the same.

Lemma 2.1. *For all $\alpha \geq 1$, we have $E(\alpha) = \widetilde{E}(\alpha)$.*

Proof. It is evident that $E(\alpha) \subset \widetilde{E}(\alpha)$, because, as discussed above,

$$\frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1+\cdots+n_\ell-1} \varphi(T^j x). \tag{2.3}$$

Now, we show the other direction. Take an $x \in \widetilde{E}(\alpha)$, express x in its binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \cdots 0^{n_\ell-1}1 \cdots].$$

In fact, $n_\ell - 1$ is the recurrence time for $n(\widehat{T}^{\ell-1}x)$, for each $\ell \geq 1$.

By (2.1), we have, at present,

$$\lim_{\ell\to\infty} \frac{1}{n_1 + \cdots + n_\ell} \sum_{j=0}^{n_1+\cdots+n_\ell-1} \varphi(T^j x) = \alpha.$$

So, we are required to check it holds for all n.

For any $\epsilon > 0$, there exists $\ell_0 \in \mathbb{N}$ such that, for any $\ell \geq \ell_0$,

$$\alpha - \epsilon \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell} \leq \alpha + \epsilon. \tag{2.4}$$

For any $n_1 + \cdots + n_\ell < n < n_1 + \cdots + n_\ell + n_{\ell+1}$ with $\ell \geq \ell_0$, it is trivial that

$$\frac{2^{n_1} + \cdots + 2^{n_\ell} - \ell}{n_1 + \cdots + n_\ell + n_{\ell+1}} \leq \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) \leq \frac{2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1}{n_1 + \cdots + n_\ell}.$$

Thus, it suffices to show that

$$2^{n_{\ell+1}} = o(n_1 + \cdots + n_\ell), \tag{2.5}$$

which also implies

$$n_{\ell+1} = o(n_1 + \cdots + n_\ell).$$

Let M_0 be a large integer such that, for all $M \geq M_0$, $2^M \geq 4\alpha M$. So, when $n_{\ell+1} \leq M_0$, there is nothing to prove. So, we always assume $2^{n_{\ell+1}} \geq 4\alpha n_{\ell+1}$.

By (2.4), we have

$$2^{n_1} + \cdots + 2^{n_\ell} - \ell \geq (\alpha - \epsilon)(n_1 + \cdots + n_\ell),$$
$$2^{n_1} + \cdots + 2^{n_\ell} + 2^{n_{\ell+1}} - \ell - 1 \leq (\alpha + \epsilon)(n_1 + \cdots + n_\ell + n_{\ell+1}).$$

These give

$$2^{n_{\ell+1}} \leq 2\epsilon(n_1 + \cdots + n_\ell) + (\alpha + \epsilon)n_{\ell+1} + 1.$$

So, we have

$$2^{n_{\ell+1}} \leq 4\epsilon(n_1 + \cdots + n_\ell).$$ □

2.2. **Dimension of $\widetilde{E}(\alpha)$.** Now we calculate the Hausdorff dimension of the set $\widetilde{E}(\alpha)$. At first, we give a notation.

- For each finite word $w \in \bigcup_{n\geq 1}\{0,1\}^n$ of length n, a T-*dyadic* cylinder of order n is defined as

$$I_n(w) = \{x \in (0,1] : (\epsilon_1(x), \cdots, \epsilon_n(x)) = w\}.$$

- For $(n_1, \cdots, n_\ell) \in \mathbb{N}^\ell$, a $\widehat{T}$-*dyadic* cylinder of order ℓ is defined as

$$D_\ell(n_1, \cdots, n_\ell) = \{x \in (0,1] : n_k(x) = n_k,\ 1 \leq k \leq \ell\}.$$

Proof of Theorem 1.1. We consider the potential function with two parameters

$$\psi_{t,q} := -t\log|\widehat{T}'| - (\log 2)\cdot q\phi.$$

Then we can define a Ruelle operator

$$\mathcal{L}_{t,q}f(x) := \sum_{y\in\widehat{T}^{-1}x} e^{\psi_{t,q}(y)}f(y).$$

By the Ruelle-Perron-Frobenius transfer operator theory [10, Section 2], for any $q>0$ (to satisfy the condition 2.2 of [10]), we can find an eigenvalue $\lambda_{t,q}$ and an eigenfunction $h_{t,q}$ for $\mathcal{L}_{t,q}$ and an eigenfunction $\nu_{t,q}$ for the conjugate operator $\mathcal{L}^*_{t,q}$. Then the pressure function $P(t,q)=\log\lambda_{t,q}$ and the Gibbs measure $\mu_{t,q}$ is given by $h_{t,q}\cdot\nu_{t,q}$.

The pressure function can be computed by

$$P(t,q) = \lim_{\ell\to\infty}\frac{1}{\ell}\log\sum_{\widehat{T}^\ell x = x}\exp(S_\ell\psi_{t,q}(x)) = \log\sum_{j=1}^{\infty}2^{-tj-q(2^j-1)}.$$

Now we calculte the local dimension of the Gibbs measure $\mu_{t,q}$. Let $D_\ell(x)$ be the $\widehat{T}$-dyadic cylinder containing x of order ℓ. By the Gibbs property of $\mu_{t,q}$,

$$\begin{aligned}\frac{\log\mu_{t,q}(D_\ell(x))}{\log|D_\ell(x)|} &= \frac{S_\ell\psi_{t,q}(x)-\ell P(t,q)}{-S_\ell\log|\widehat{T}'|(x)} = \frac{-tS_\ell\log|\widehat{T}'|(x)-(\log 2)\cdot qS_\ell\phi(x)-\ell P(t,q)}{-S_\ell\log|\widehat{T}'|(x)}\\ &= t+q\frac{S_\ell\phi(x)}{S_\ell\log_2|\widehat{T}'|(x)}+\frac{\ell P(t,q)}{S_\ell\log|\widehat{T}'|(x)}. \end{aligned}\tag{2.6}$$

• Upper bound. For each $q>0$, let $t(q)$ be the the number such that $P(t(q),q)=0$. Then we have for all $x\in\widetilde{E}(\alpha)$, we have

$$\begin{aligned}\liminf_{r\to0}\frac{\log\mu(B(x,r))}{\log r} &\le \liminf_{\ell\to\infty}\frac{\log\mu_{t,q}(B(x,|D_\ell(x)|))}{\log|D_\ell(x)|}\\ &\le \liminf_{\ell\to\infty}\frac{\log\mu_{t,q}(D_\ell(x))}{\log|D_\ell(x)|} = t(q)+q\alpha,\end{aligned}$$

where for the second inequality the trivial inclusion $D_\ell(x)\subset B(x,|D_\ell(x)|)$ is used. By Billingsley Lemma, this gives an upper bound of the Hausdorff dimension of $\widetilde{E}(\alpha)$. Thus we have

$$\dim_H\widetilde{E}(\alpha)\le\inf_{q>0}\{t(q)+q\alpha\}.$$

• Lower bound. Let q_0 be the point such that the following infimum is attained

$$\inf_{q>0}\{t(q)+q\alpha\}.$$

Then

$$t'(q_0)+\alpha=0. \tag{2.7}$$

Claim (I): The measure $\mu_{t(q_0),q_0}$ is supported on E_α. Since $P(t(q),q)=0$,

$$\frac{\partial P}{\partial t}t'(q)+\frac{\partial P}{\partial q}=0. \tag{2.8}$$

On the other hand, by the ergodicity of the measure $\mu_{t,q}$, we have for $\mu_{t,q}$ almost all x,

$$\lim_{\ell\to\infty}\frac{S_\ell\phi(x)}{S_\ell\log_2|\widehat{T}'|(x)} = \frac{\int\phi d\mu_{t,q}}{\int\log|\widehat{T}'|d\mu_{t,q}}\cdot\log 2.$$

By Ruelle-Perron-Frobenius transfer operator theory ([10], Proposition 6.5),

$$\int(\log 2)\cdot\phi d\mu_{t,q} = -\frac{\partial P}{\partial q}\quad\text{and}\quad\int\log|\widehat{T}'|d\mu_{t,q} = -\frac{\partial P}{\partial t}.$$

Thus by (2.8) and then (2.7), for $\mu_{t(q_0),q_0}$ almost all x,

$$\lim_{\ell\to\infty}\frac{S_\ell\phi(x)}{S_\ell\log_2|\widehat{T}'|(x)}=\frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}}=-t'(q_0)=\alpha.$$

This shows Claim (I).

Claim (II). For $\mu_{t(q_0),q_0}$ almost all x,

$$\lim_{n\to\infty}\frac{\log\mu_{t(q_0),q_0}(I_n(x))}{\log 2^{-n}}=t(q_0)+q_0\alpha,$$

where $I_n(x)$ is the T-dyadic cylinder of order n containing x.

On one hand, by (2.6) and then by (2.8) and (2.7), one has for $\mu_{t(q_0),q_0}$ almost all x

$$\lim_{\ell\to\infty}\frac{\log\mu_{t(q_0),q_0}(D_\ell(x))}{\log|D_\ell(x)|}=t(q_0)+q_0\frac{\frac{\partial P}{\partial q}}{\frac{\partial P}{\partial t}}=t(q_0)+q_0\alpha. \tag{2.9}$$

On the other hand, note that for any $x\in E(\alpha)$, if the binary expansion of x is $x=[0^{n_1-1}10^{n_2-1}1\ldots]$, then for any $\delta>0$, for ℓ large enough,

$$(\alpha-\delta)\ell\le 2^{n_1}+\cdots+2^{n_\ell}-\ell=S_\ell\phi(x)\le(\alpha+\delta)\ell.$$

Hence

$$n_\ell=O(\log\ell),$$

which implies

$$\lim_{\ell\to\infty}\frac{\log|D_\ell(x)|}{\log|D_{\ell+1}(x)|}=\lim_{\ell\to\infty}\frac{n_1+\cdots+n_\ell}{n_1+\cdots+n_\ell+n_{\ell+1}}=1. \tag{2.10}$$

Thus

$$\lim_{n\to\infty}\frac{\log\mu_{t(q_0),q_0}(I_n(x))}{\log 2^{-n}}=\lim_{\ell\to\infty}\frac{\log\mu_{t(q_0),q_0}(D_\ell(x))}{\log|D_\ell(x)|}.$$

This shows Claim (II).

To conclude the desired lower bound, we apply the classical mass transference principle (see [3, Proposition 4.2]). Since the Hausdorff dimension will not be changed if we replace the δ-coverings by T-dyadic cylinder coverings (see [3, Section 2.4]), the lower bound of the Hausdorff dimension can be given by the mass transference principle on T-dyadic cylinders. By the above two claims and Egorov's theorem, for any $\eta>0$, there exists an integer N_0 such that the set

$$\Big\{x\in E_\alpha:\mu(I_n(x))\le|I_n(x)|^{t(q_0)+q_0\alpha-\eta},\ n\ge N\Big\}$$

is of $\mu_{t(q_0),q_0}$ positive measure. So, it implies that

$$\dim_H E_\alpha\ge t(q_0)+q_0\alpha-\eta.$$

Note that ([10, Lemma 7.5]) the function $q\mapsto t(q)$ is a decreasing convex function such that

$$t(0)=1,\quad \lim_{q\to\infty}(t(q)+q)=0,$$

and

$$\lim_{q\to 0^+}t'(q)=-\infty,\quad \lim_{q\to+\infty}t'(q)=-1.$$

Therefore, we have proved for any $\alpha\in(1,+\infty)$

$$\dim_H(\widetilde{E}(\alpha))=\inf_{q>0}\{t(q)+q\alpha\},$$

which is Legendre transformation. All the properties stated in Theorem 1.1 are satisfied by the function $\alpha\mapsto\dim_H(\widetilde{E}(\alpha))$ which is the same function as $\alpha\mapsto\dim_H(E(\alpha))$ by Lemma 2.1.

For the end point $\alpha = 1$, it suffices to note that the level set $E(1)$ is nothing but the set of numbers with frequency of the digit 1 in its binary expansion being 1. Thus the Hausdorff dimension of $E(1)$ is 0. Hence, the Legendre transformation formula for the Hausdorff dimension of $E(\alpha)$ $(\alpha > 1)$ holds also for $\alpha = 1$. $\square$

3. Fast increasing Birkhoff sum

At first, we give two simple observations.

Lemma 3.1. *Let W be an integer such that $2^t \le W < 2^{t+1}$ for some positive integer t. For any $0 \le n \le t$, among the integers between W and $W(1+2^{-n})$, there is one $V = V(W,n)$ whose binary expansion of V has at most $n+2$ digits 1 and ends with at least $t-n$ zeros.*

Proof. By the assumption, we have $2^{-n}W \ge 2^{t-n}$. Thus among the $2^{-n}W$ consecutive integers from W to $W(1+2^{-n})$ there is at least one integer which is divisible by 2^{t-n} which means there is an integer $\ell \ge 1$ such that

$$W \le \ell 2^{k-n} \le W(1+2^{-n}).$$

Let $V = \ell 2^{t-n}$ and note that V is an integer whose binary expansion ends with at least $t-n$ zeros. Since $\ell 2^{t-n} \le W(1+2^{-n}) < 2^{t+2}$, we conclude that $\ell 2^{t-n}$ has at most $(t+2)-(t-n) = n+2$ digits 1 in its binary expansion. $\square$

In the follows, the base of the logarithm is taken to be 2.

Lemma 3.2. *For each integer W, and any integer $n \le \log W$, we can find a word w with length*

$$|w| \le (n+2)(2+\log W)$$

and for any $x \in I_{|w|}(w)$

$$W \le \sum_{j=0}^{|w|-1} \varphi(T^j x) \le W(1+2^{-n}).$$

Proof. Let V be an integer given in Lemma 3.1. Then $W \le V \le W(1+2^{-n})$. Moreover if we write this number V in binary expansion:

$$V = 2^{t_1} + \cdots + 2^{t_p},$$

one has that $\lfloor \log W \rfloor + 1 \ge t_1 > \cdots > t_p \ge \lfloor \log W \rfloor - n$ and $p \le n+2$. Consider the word

$$w = (10^{t_1-1}1, 10^{t_2-1}1, \cdots, 10^{t_p-1}1)$$

here the word $10^{t_p-1}1$ is 1 when $t_p = 0$. Then we can check that the length of w satisfies

$$|w| = (t_1+1) + \cdots + (t_p+1) \le p(t_1+1) \le (n+2)(2+\log W),$$

and for any $x \in I_{|w|}(w)$,

$$\sum_{j=0}^{|w|-1} \varphi(T^j x) = V.$$

Hence, the proof is completed. $\square$

We also need the following lemma whose proof is left for the reader.

Lemma 3.3. *For any $m \ge 1$, define*

$$F_m = \{x \in (0,1] : \epsilon_{km}(x) = 1, \text{ for all } k \ge 1\}.$$

Then $\dim_H F_m = \frac{m-1}{m}$.

Before the proof Theorem 1.2, we show the following lemma.

Lemma 3.4. *Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$. Assume that there exists a subsequence N_k satisfying the following conditions*

$$N_k - N_{k-1} \to \infty, \ \Psi(N_k) - \Psi(N_{k-1}) \to \infty, \tag{3.1}$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \to 1, \quad \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0, \tag{3.2}$$

as $k \to \infty$. Then the set

$$E_\Psi(1) = \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = 1\Big\}$$

has Hausdorff dimension 1.

Proof. Fix a large integer m and write

$$\mathcal{U} = \Big\{u = (\epsilon_1, \cdots, \epsilon_m) : \epsilon_m = 1, \epsilon_i \in \{0,1\}, i \neq m\Big\}.$$

To avoid the abuse of notation, by the first assumption of (3.1), we assume $N_k - N_{k-1} \gg m$ for all $k \geq 1$ by setting $N_0 = 0$ and $\Psi(N_0) = 0$.

For each $k \geq 1$, we write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let $\{n_k\}$ be a sequence of integers tending to ∞ such that

$$n_k \leq \log W_k, \quad n_k \cdot \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0.$$

By the second assumptions of (3.1) and (3.2), this sequence of $n_k \geq 0$ do exist.

Now for W_k and n_k, let w_k be the word given in Lemma 3.2. Then the length a_k of w_k satisfies

$$\begin{aligned} a_k \leq& (n_k+2)(2+\log W_k) \\ =& (n_k+2)\left(2+\log(\Psi(N_k) - \Psi(N_{k-1}))\right) = o(N_k - N_{k-1}) \end{aligned} \tag{3.3}$$

and for any $x \in I_{a_k}(w_k)$,

$$W_k \leq \sum_{j=0}^{a_k-1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}). \tag{3.4}$$

Define t_k, ℓ_k to be the integers satisfying

$$N_k - N_{k-1} - a_k = t_k m + \ell_k,$$

for some $0 \leq \ell_k < m$.

Let w_k $(k \geq 1)$ be given as the above. We define a Cantor subset of $E_\Psi(1)$ as follows.

Level 1 of the Cantor subset. Define

$$E_1 = \Big\{I_{N_1}(u_1, \cdots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in \mathcal{U}, 1 \leq i \leq t_1\Big\}.$$

For simplicity, we use $I_{N_1}(U_1)$ to denote a general cylinder in E_1.

Level 2 of the Cantor subset. This level is composed by sublevels for each cylinder $I_{N_1}(U_1) \in E_1$. Fix an element $I_{N_1} = I_{N_1}(U_1) \in E_1$. Define

$$E_2(I_{N_1}(U_1)) = \Big\{I_{N_2}(U_1, u_1, \cdots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in \mathcal{U}, 1 \leq i \leq t_2\Big\}.$$

Then

$$E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1}).$$

For simplicity, we use $I_{N_2}(U_2)$ to denote a general cylinder in E_2.

From Level k to $k+1$. Fix $I_{N_k}(U_k) \in E_k$. Define

$$E_{k+1}(I_{N_k}(U_k)) = \Big\{ I_{N_{k+1}}(U_k, u_1, \cdots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in \mathcal{U}, 1 \le i \le t_{k+1} \Big\}.$$

Then

$$E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).$$

Up to now we have constructed a sequence of nested sets $\{E_k\}_{k\ge 1}$. Set

$$F = \bigcap_{k \ge 1} E_k.$$

We claim that

$$F \subset E(\Psi).$$

In fact, for all $x \in F$, by construction, for each $k \ge 1$,

$$\begin{aligned}
&\sum_{n=N_{k-1}}^{N_k-1} \varphi(T^n x) \\
=& \sum_{n=N_{k-1}}^{N_{k-1}+t_k m-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m}^{N_{k-1}+t_k m+\ell_k-1} \varphi(T^n x) + \sum_{n=N_{k-1}+t_k m+\ell_k}^{N_k-1} \varphi(T^n x) \\
=& t_k O(2^m) + \ell_k + W_k(1 + O(2^{-n_k})) \\
=& O\left(\frac{(N_k - N_{k-1})2^m}{m} \right) + (\Psi(N_k) - \Psi(N_{k-1}))(1 + O(2^{-n_k})).
\end{aligned}$$

Since $n_k \to \infty$ which implies $2^{-n_k} \to 0$ as $k \to \infty$, we have

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k)\big(1 + o(1)\big) + O\left(\frac{N_k 2^m}{m} \right).$$

By the assumption $\Psi(n)/n \to \infty$ as $n \to \infty$, we then deduce

$$\sum_{n=0}^{N_k-1} \varphi(T^n x) = \Psi(N_k) + o(\Psi(N_k)),$$

Thus

$$\lim_{k\to\infty} \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_k)} = 1. \tag{3.5}$$

While, for each $N_{k-1} < N \le N_k$

$$\frac{\sum_{n=0}^{N_{k-1}-1} \varphi(T^n x)}{\Psi(N_k)} \le \frac{\sum_{n=0}^{N-1} \varphi(T^n x)}{\Psi(N)} \le \frac{\sum_{n=0}^{N_k-1} \varphi(T^n x)}{\Psi(N_{k-1})}.$$

So by the first assumption of (3.2), we deduce from (3.5) that

$$\lim_{n\to\infty} \frac{1}{\Psi(n)} S_n(x) = 1.$$

This proves $x \in E_\Psi(1)$ and hence $F \subset E_\Psi(1)$.

In the following, we will construct a Hölder continuous function from F to F_m. Recall that

$$F_m = \big\{ x \in (0,1] : \epsilon_{km}(x) = 1, \text{ for all } k \ge 1 \big\}.$$

Define

$$\begin{aligned} f : F &\to F_m \\ x &\mapsto y \end{aligned}$$

where y is obtained by eliminating the digits $\{(\epsilon_{N_k-\ell_k-a_k+1}, \cdots, \epsilon_{N_k})\}_{k\geq 1}$ in the binary expansion of x. Now we calculate the Hölder expoent of f.

Take two points $x_1, x_2 \in F$ closed enough. Let n be the smallest integer such that $\epsilon_n(x_1) \neq \epsilon_n(x_2)$ and k be the integer such that $N_k < n \leq N_{k+1}$. Note that by the construction of F, the digits sequence

$$\{(\epsilon_{N_k-\ell_k-a_k+1}, \cdots, \epsilon_{N_k})\}_{k\geq 1} \text{ and } \{\epsilon_{N_k+tm}\}_{1\leq t\leq t_{k+1}}$$

are the same for all $x \in F$. So we must have

$$N_k < n < N_{k+1} - \ell_{k+1} - a_{k+1}. \tag{3.6}$$

Since n is strictly less than $N_{k+1} - \ell_{k+1} - a_{k+1}$ and $\epsilon_{N_k+tm}(x_1) = \epsilon_{N_k+tm}(x_2) = 1$ for all $1 \leq t \leq t_{k+1}$, thus, at most m steps after the position n, saying n', $\epsilon_{n'}(x_1) = \epsilon_{n'}(x_2) = 1$. So it follows that

$$|x_1 - x_2| \geq \frac{1}{2^{n+m}}.$$

Again by the construction and the definition of the map f, we have $y_1 = f(x_1)$ and $y_2 = f(x_2)$ have common digits up to the position $n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)$. Thus, it follows

$$|f(x_1) - f(x_2)| \leq \frac{1}{2^{n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)}}.$$

Recall that $\ell_k < m$ and $a_1 + \cdots + a_k = o(N_k)$ (see (3.3)) and also that $N_k/k \to \infty$ as $k \to \infty$ (by (3.1)). We have

$$1 \geq \frac{n-1-(\ell_1+a_1)-\cdots-(\ell_k+a_k)}{n+m} \geq \frac{n-1-km-o(N_k)}{n+m} = 1 + o(1),$$

which implies that f is $(1-\eta)$-Hölder for any $\eta > 0$. Thus

$$\dim_H F \geq (1-\eta) \dim_H F_m.$$

By Lemma 3.3, we then have

$$\dim_H F \geq (1-\eta)\frac{m-1}{m}.$$

By the arbitrariness of $\eta > 0$ and letting $m \to \infty$, we conclude that $\dim_H E(\Psi) = 1$. This finishes the proof. □

Proof of Theorem 1.2. Observe that the Hausdorff dimensions of $E_\Psi(\beta)$ are equal to that of $E_\Psi(1)$ for all $\beta > 0$. We need only replace $\Psi(n)$ by $\beta\Psi(n)$. But it will not change the order of Ψ and all calculations are the same to that for $E_\Psi(1)$.

To show $\dim_H E_\Psi(1) = 1$, we can apply Lemma 3.4 directly. If $\Psi(n) = n \log n$, we can choose $N_k = k^2$. For $\Psi(n) = n^a$ $(a > 1)$, we can also choose $N_k = k^2$. Suppose now $\Psi(n) = 2^{n^\gamma}$ with $0 < \gamma < 1/2$. Let $\delta > 0$ be small such that

$$\frac{\gamma}{1-\gamma} + \delta\gamma < 1 \tag{3.7}$$

which is possible since $\gamma < 1/2$. Take

$$N_k = \lfloor k^{\frac{1}{1-\gamma}+\delta} \rfloor. \tag{3.8}$$

Then we have

$$N_{k+1} - N_k \approx k^{\frac{\gamma}{1-\gamma}+\delta}, \tag{3.9}$$

and

$$\log(\Psi(N_{k+1}) - \Psi(N_k)) \approx \log(\Psi'(N_k)(N_{k+1} - N_k)) \\ \approx N_k^\gamma + \log(N_{k+1} - N_k) \approx N_k^\gamma.$$

Here we write $A \approx B$ when $A/B \to 1$. This shows the validity of (3.1). Moreover,

$$\frac{\log(\Psi(N_{k+1}) - \Psi(N_k))}{N_{k+1} - N_k} \approx \frac{k^{\frac{\gamma}{1-\gamma}+\gamma\delta}}{k^{\frac{\gamma}{1-\gamma}+\delta}} = k^{-\delta(1-\gamma)} \to 0 \ (k \to \infty).$$

Thus the second assumption of (3.2) is satisfied. At last, for the first assumption in (3.2), by (3.7)

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} = 2^{(k-1)^{\frac{\gamma}{1-\gamma}+\delta\gamma} - k^{\frac{\gamma}{1-\gamma}+\delta\gamma}} \to 1.$$

Hence Lemma 3.4 applies.

Now suppose that $\Psi(n) = 2^{n^\gamma}$ with $1/2 \le \gamma < 1$. Let $\beta > 0$ be given. Then, by (2.1) we have for $x \in E_\Psi(\beta)$, if x has binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \cdots 0^{n_\ell-1}1 \cdots]$$

then

$$\begin{aligned} \frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \to \beta, \\ \frac{S_{n_1+n_2+\cdots+n_\ell+1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell+1)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{(n_1+n_2+\cdots+n_\ell+1)^\gamma}} \to \beta. \end{aligned} \tag{3.10}$$

Since

$$\frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell+1)^\gamma}} \to 1,$$

by dividing the two limits of (3.10), we deduce that

$$\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell + 2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell} = 1 + \frac{2^{n_{\ell+1}-1}}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell} \to 1,$$

which implies that

$$\frac{S_{n_1+n_2+\cdots+n_{\ell+1}}(x)}{S_{n_1+n_2+\cdots+n_\ell}(x)} = 1 + \frac{2^{n_{\ell+1}} - 1}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell} \to 1.$$

Combining with (3.10), we get

$$1 \leftarrow \frac{\Psi(n_1 + \cdots + n_{\ell+1})}{\Psi(n_1 + \cdots + n_\ell)} = \frac{2^{(n_1+n_2+\cdots+n_\ell+n_{\ell+1})^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}.$$

Thus

$$\begin{aligned} &(n_1 + n_2 + \cdots + n_\ell + n_{\ell+1})^\gamma - (n_1 + n_2 + \cdots + n_\ell)^\gamma \\ &= (n_1 + n_2 + \cdots + n_\ell)^\gamma \left(\left(1 + \frac{n_{\ell+1}}{n_1 + n_2 + \cdots + n_\ell}\right)^\gamma - 1 \right) \\ &\approx \frac{\gamma n_{\ell+1}}{(n_1 + n_2 + \cdots + n_\ell)^{1-\gamma}} \to 0. \end{aligned}$$

Therefore, for any $\varepsilon > 0$, there exists $k_0 \ge 1$ such that for all $j > k_0$,

$$n_j < \varepsilon(n_1 + n_2 + \cdots + n_{j-1})^{1-\gamma}. \tag{3.11}$$

Then for any $k_0 < j \le \ell$

$$n_j < \varepsilon(n_1 + n_2 + \cdots + n_\ell)^{1-\gamma}.$$

This implies

$$\begin{aligned} S_{n_1+n_2+\cdots+n_\ell}(x) &= 2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell \\ &\le M + \ell 2^{\epsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}} - \ell, \end{aligned}$$

with $M := 2^{n_1} + \cdots + 2^{n_{k_0}}$. Thus we have

$$\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} < \frac{M + \ell 2^{\epsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}. \tag{3.12}$$

By observing $n_j \ge 1$, we deduce that the upper bound of (3.12) converges to 0 for $1/2 \le \gamma < 1$, a contradiction to (3.10). Hence $E_\Psi(\beta)$ is an empty set.

Suppose that $\gamma \ge 1$. Then by (2.1) we have for $x \in E_\Psi(\beta)$

$$\begin{aligned} \frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \to \beta, \\ \frac{S_{n_1+n_2+\cdots+n_\ell-1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell-1)} &= \frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \to \beta. \end{aligned} \tag{3.13}$$

However,

$$\frac{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell}{2^{n_1} + 2^{n_2} + \cdots + 2^{n_\ell} - \ell - 1} \to 1 \quad \text{but} \quad \frac{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell-1)^\gamma}} \ge 2,$$

which is a contradiction. $\square$

4. The potential $1/x$

In fact, the techniques in Section 3 can be applied to the continuous potential $g : x \mapsto 1/x$ on $(0, 1]$ which has a singularity at 0.

Proof of Theorem 1.3. We first note that if $x \in (0, 1]$ has binary expansion $x = [0^n 1^s \dots]$, then $\varphi(x) = 2^n$ and

$$2^n \le g(x) \le 2^n + 2^{n-s+1} = 2^n(1 + 2^{-s+1}). \tag{4.1}$$

In Lemma 3.2, for an integer W, and for any integer $n \le \log W$, we can construct instead of the words $w = (10^{t_1-1}1, 10^{t_2-1}1, \cdots, 10^{t_p-1}1)$, the following word

$$w = (10^{t_1-1}1^{s+1}, 10^{t_2-1}1^{s+1}, \cdots, 10^{t_p-1}1^{s+1}).$$

Then the length of the word satisfies

$$|w| = \sum_{i=1}^{p} (t_i + s + 1) \le p(t_1 + s + 1) \le (n + 2)(\log W + s + 2). \tag{4.2}$$

By (4.1), for any $x \in I_{|w|}(w)$,

$$W + s(n + 2) \le \sum_{j=0}^{|w|-1} g(T^j x) \le W(1 + 2^{-n}) \cdot (1 + 2^{-s}) + 2s(n + 2).$$

For each $k \ge 1$, we still write

$$W_k := \Psi(N_k) - \Psi(N_{k-1})$$

and let n_k, s_k be a sequence of integers tending to ∞ such that

$$n_k \cdot \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0, \tag{4.3}$$

$$\frac{n_k \cdot s_k}{N_k - N_{k-1}} \to 0, \tag{4.4}$$

and

$$\frac{n_k \cdot s_k}{\Psi(N_k) - \Psi(N_{k-1})} \to 0. \tag{4.5}$$

By (3.1) and (3.2), these two sequences of $n_k \geq 0$, $s_k \geq 0$ do exist.

Now for W_k and n_k, s_k, let w_k be the word given as above. Then by (4.3) and (4.4), the length a_k of w_k satisfies

$$\begin{aligned} a_k \leq & (n_k + 2)(\log W_k + s_k + 2) \\ = & (n_k + 2)\left(\log(\Psi(N_k) - \Psi(N_{k-1})) + s_k + 2\right) \\ = & o(N_k - N_{k-1}) \end{aligned} \tag{4.6}$$

and for any $x \in I_{a_k}(w_k)$,

$$\begin{aligned} W_k + s_k(n_k + 2) \leq & \sum_{j=0}^{a_k - 1} g(T^j x) \\ & \leq W_k(1 + 2^{-n_k}) \cdot (1 + 2^{-s_k}) + 2s_k(n_k + 2). \end{aligned} \tag{4.7}$$

Hence by (4.5) we still have the same estimation:

$$\sum_{n=0}^{N_k - 1} g(T^n x) = \Psi(N_k) + o(\Psi(N_k))$$

and the rest of the proof is the same.

We can repeat the same arguments in Section 3 and show that for potential g, the set $E_\Psi(\beta)$ is empty if $\Psi(n) = 2^{n^\gamma}$ $(\gamma \geq 1/2)$.

In fact, by definition, for $x \in E_\Psi(\beta)$, if x has binary expansion

$$x = [0^{n_1 - 1} 1 0^{n_2 - 1} 1 \cdots 0^{n_\ell - 1} 1 0^{n_{\ell+1} - 1} 1 \cdots],$$

then

$$\frac{S_{n_1 + n_2 + \cdots + n_\ell} g(x)}{\Psi(n_1 + n_2 + \cdots + n_\ell)} \to \beta, \quad \frac{S_{n_1 + n_2 + \cdots + n_\ell + 1} g(x)}{\Psi(n_1 + n_2 + \cdots + n_\ell + 1)} \to \beta.$$

Thus

$$\frac{S_{n_1 + n_2 + \cdots + n_\ell} g(x)}{S_{n_1 + n_2 + \cdots + n_\ell + 1} g(x)} \to 1.$$

Observing $\varphi \leq g \leq 2\varphi$, we have

$$\frac{2^{n_{\ell+1}}}{S_{n_1 + n_2 + \cdots + n_\ell} g(x)} \to 0,$$

which then implies

$$\frac{S_{n_1 + n_2 + \cdots + n_\ell} g(x)}{S_{n_1 + n_2 + \cdots + n_{\ell+1}} g(x)} \to 1.$$

By the definition of $x \in E_\Psi(\beta)$, we have

$$\frac{\Psi(n_1 + n_2 + \cdots + n_{\ell+1})}{\Psi(n_1 + n_2 + \cdots + n_\ell)} \to 1.$$

This further implies the same inequality with (3.11) and the rest of proof is the same by noting $\varphi \leq g \leq 2\varphi$. □

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