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Topological and Ergodic Aspects of Partially Hyperbolic Diffeomorphisms and Nonhyperbolic Step Skew Products

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Abstract—We review some ergodic and topological aspects of robustly transitive partially hyperbolic diffeomorphisms with one-dimensional center direction. We also discuss step skew-product maps whose fiber maps are defined on the circle which model such dynamics. These dynamics are genuinely nonhyperbolic and exhibit simultaneously ergodic measures with positive, negative, and zero exponents as well as intermingled horseshoes having different types of hyperbolicity. We discuss some recent advances concerning the topology of the space of invariant measures and properties of the spectrum of Lyapunov exponents.

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1. INTRODUCTION

The paper [30] opens with the following general question:

- *To what extent is the behavior of a generic dynamical system hyperbolic*?

The authors of [30] observe that a substantial number of problems in dynamical systems theory are just reformulations of this question. In the beginning of the theory in the late 1960s, the paper by Abraham and Smale [2] showed that hyperbolic systems are not dense in the space of dynamical systems. Indeed, there are open sets in the space of diffeomorphisms consisting of nonhyperbolic ones. These findings showed the necessity of weaker notions of hyperbolicity and lead to the ones of nonuniform hyperbolicity due to Pesin [42] and partial hyperbolicity [32], among others.

Consider a differentiable dynamical system $F\colon M\to M$ defined on a closed and compact manifold M. Recall that a closed and F-invariant *transitive* (existence of a point in Γ whose orbit is dense in Γ) set $\Gamma\subset M$ is *hyperbolic* if there exists a dF-invariant splitting $E^{\mathrm{s}}\oplus E^{\mathrm{u}}=T_\Gamma M$ of the tangent bundle and constants $C>0$ and $\lambda>1$ such that for every $x\in\Gamma$ and every $n\ge 0$ we have

$$\|dF^n_x(v)\|\le C\lambda^n\|v\|\quad\forall\, v\in E^{\mathrm{s}}_x\qquad\text{and}\qquad\|dF^{-n}_x(w)\|\le C\lambda^n\|w\|\quad\forall\, w\in E^{\mathrm{u}}_x.$$

Any variation of hyperbolicity is based on the notion of Lyapunov exponents. Recall that a point $x\in M$ is called *Lyapunov regular* if there exist a positive integer $s(x)$, numbers $\chi_1(x)<\ldots<\chi_{s(x)}(x)$, and a dF-invariant splitting $T_xM=\bigoplus_{i=1}^{s(x)}E^i_x$ of the tangent space at x such that for all $i=1,\ldots,s(x)$ and $v\in E^i_x\setminus\{0\}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log\|dF^n_x(v)\|=\chi_i(x),\tag{1.1}$$

and these numbers $\chi_1(x)<\ldots<\chi_{s(x)}(x)$ are called the *Lyapunov exponents* of x.

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By the Oseledets multiplicative ergodic theorem (see [40]), given an F-invariant ergodic probability measure μ, the set of Lyapunov regular points has full measure and $s(\cdot) = s(\mu)$ and $\chi_i(\cdot) = \chi_i(\mu)$, $i = 1, \dots, s(\mu)$, are constant μ-almost everywhere; the latter numbers are called the *Lyapunov exponents* of μ. If there is ℓ such that $\chi_\ell(\mu) = 0$, then the measure μ is called *nonhyperbolic*; otherwise it is called *hyperbolic*. We call the cardinality of negative Lyapunov exponents of μ its (*stable*) *index*. When talking about nonhyperbolic measures, we always assume ergodicity and hence exclude nontrivial convex combinations of ergodic measures. The easiest examples of ergodic measures are the ones supported on a periodic orbit (we will call such measures simply *periodic*). The *index* of a hyperbolic periodic orbit is the index of the (unique) invariant measure supported on its orbit. Below we will call a measure *nontrivial* if its support is uncountable; hence such a measure cannot be periodic. The easiest example of a nonhyperbolic measure is a periodic one supported on the orbit of a nonhyperbolic periodic point. A hyperbolic periodic orbit which has both positive and negative exponents will also be called a *saddle*.

The discussion above leads to the following question:

- *To what extent does ergodic theory detect nonhyperbolic dynamics*?

This is just a reformulation of the opening question above (though note that the term "nonhyperbolic dynamics" is vague and used differently in different contexts). The answer to this question is negative, as there are examples of nonhyperbolic systems (in the sense that the nonwandering set is not hyperbolic) for which all ergodic measures are hyperbolic with Lyapunov exponents uniformly bounded away from zero [4, 16]. Note that these examples are fragile in the sense that they can be destroyed by perturbations. On the other hand, by the Kupka–Smale genericity theorem (see, for instance, [41, Ch. 3]), generically[1] periodic points are all hyperbolic. Hence, considering nonhyperbolic periodic measures, one can get systems with nonhyperbolic *ergodic* measures only densely in the complement of the hyperbolic ones. Thus, to go beyond dense subsets, one needs to investigate nonhyperbolic measures which are not periodic. This was first done in [30], where the method of periodic approximations was introduced to construct nontrivial ergodic measures as weak$*$ limits of periodic ones.

In dimension strictly greater than 2 one needs to take into account that *a priori* different types of hyperbolicity may coexist together with nonhyperbolicity. Indeed, one may have hyperbolic periodic orbits of different *stable indices* (dimensions of the stable bundle) which are robustly contained in the same *transitive set* (existence of a dense orbit); this leads to intermingled types of hyperbolicity in the same transitive set. For instance, this is exactly what happens in the dynamics analyzed in [30].

Having this in mind, we rephrase the above question:

- *To what extent does ergodic theory distinguish the different types of hyperbolicity in nonhyperbolic dynamics*?

In what follows we will restrict ourselves to systems which are transitive in the whole ambience (this prevents the existence of attractors and repellers). In this setting, the set of ergodic measures $\mathcal{M}_{\mathrm{erg}}$ will split into several disjoint components, which will be analyzed separately (see (2.1)).

2. PARTIALLY HYPERBOLIC DYNAMICS

At the present state of the art, to advance in answering the above questions, we need to assume more structure on the dynamics. We have to require the existence of a globally defined splitting of the tangent space into continuously varying invariant subbundles (a *dominated splitting*) which

[1] A *generic* property is a property satisfied on a residual subset, that is, a set which contains a countable intersection of open and dense subsets.

as a consequence incorporates the subspaces of the Oseledets splitting. In more specific terms, we will require that the dynamical system is *partially hyperbolic* having three such bundles $TM = E^{ss} \oplus E^{c} \oplus E^{uu}$, where E^{ss} is uniformly contracting and E^{uu} is uniformly expanding. Hence, zero Lyapunov exponents are automatically associated with the *central bundle* E^{c}. The exponent associated with E^{c} will be simply called the *central exponent*. We denote by $\mathbf{PH}^1(M)$ the set of C^1 diffeomorphisms defined on a compact closed manifold M having a partially hyperbolic splitting as above with three nontrivial directions, the central one having dimension 1.[2]

In what follows we will consider $F\colon M \to M$ a C^1 diffeomorphism of a Riemannian manifold which is partially hyperbolic and transitive. We will assume that the dynamics is *robustly* transitive, *robustly* nonhyperbolic, and there is some F-invariant compact closed curve $\gamma = F(\gamma)$. The latter property also turns out to be robust, by normal hyperbolicity. We denote this open set by $\mathbf{RTPH}^1(M)$ (here we consider the uniform topology in the space of C^1 diffeomorphisms). Note that this implies that the curve γ is tangent to E^{c}, and we refer to it as a *compact central leaf*. Note that the robust nonhyperbolicity hypothesis excludes pathological cases such as that of diffeomorphisms on $\mathbb{T}^3$ being direct products of an Anosov diffeomorphism on $\mathbb{T}^2$ and an irrational rotation. Note that for $F \in \mathbf{RTPH}^1(M)$ the set of ergodic measures $\mathcal{M}_{\rm erg}$ splits into three disjoint components

$$\mathcal{M}_{\rm erg} = \mathcal{M}_{{\rm erg},<0} \cup \mathcal{M}_{{\rm erg},0} \cup \mathcal{M}_{{\rm erg},>0}, \tag{2.1}$$

where the measures in $\mathcal{M}_{{\rm erg},0}$ are nonhyperbolic and those in $\mathcal{M}_{{\rm erg},<0}$ and $\mathcal{M}_{{\rm erg},>0}$ are hyperbolic. Further, the measures in $\mathcal{M}_{{\rm erg},<0}$ have $\dim E^{ss} + 1$ negative and $\dim E^{uu}$ positive exponents, and the measures in $\mathcal{M}_{{\rm erg},>0}$ have $\dim E^{uu} + 1$ positive and $\dim E^{ss}$ negative exponents.

2.1. Hyperbolic and nonhyperbolic measures. First observe that for a C^1 open and dense subset in $\mathbf{RTPH}^1(M)$, there exist hyperbolic periodic points which are contracting in the central direction and hyperbolic periodic points which are expanding in the central direction. Besides that there are horseshoes which are contracting and horseshoes which are expanding in the central direction, respectively. Hence there are also hyperbolic ergodic measures with positive entropy in the sets $\mathcal{M}_{{\rm erg},<0}$ and $\mathcal{M}_{{\rm erg},>0}$. Indeed, in this setting the existence of a hyperbolic ergodic measure with positive entropy implies the existence of horseshoes and hence of hyperbolic periodic orbits with the corresponding type by Katok's horseshoe construction (see [34, 26]).

The existence of nonhyperbolic measures is a bit more subtle. Indeed, densely in $\mathbf{RTPH}^1(M)$ there are diffeomorphisms with nonhyperbolic periodic orbits and hence with trivial nonhyperbolic measure (see [3]).[3] Since generic diffeomorphisms have hyperbolic periodic orbits, one can obtain *at most* a dense subset in $\mathbf{RTPH}^1(M)$ with trivial nonhyperbolic measures. Hence, to get larger sets of diffeomorphisms with nonhyperbolic measures, it is necessary to investigate the occurrence of nontrivial nonhyperbolic measures. In [30] the authors introduce the method of periodic approximations that produces a nontrivial nonhyperbolic (ergodic) measure which is a weak$*$ limit of hyperbolic periodic measures, and they apply it to some specific step skew-product examples. This method builds on the existence of controlled transitions between saddles of different indices. Using it, the authors of [24, 7, 17] obtained a C^1-generic set of C^1 diffeomorphisms with nonhyperbolic nontrivial (ergodic) measures (see also the variant in [12]). With this method, some specific open sets of robustly nonhyperbolic diffeomorphisms defined on $\mathbb{T}^3$ with such measures are provided in [35]. A different approach, using the so-called *flip-flop configuration* which relies on the concept

[2] In what follows, for simplicity, we will assume that the splitting is defined on the whole ambient space; a similar approach can be adopted when the splitting is only locally defined.

[3] First note that in a C^1 open and dense subset in $\mathbf{RTPH}^1(M)$, the diffeomorphisms have periodic points. On the other hand, recall that the set $\mathcal{F}^1(M)$ is defined as the C^1 interior of the set of diffeomorphisms which have only hyperbolic periodic orbits. By [3], the set $\mathcal{F}^1(M)$ is contained in the set of Axiom A diffeomorphisms. The claim now follows by observing that the diffeomorphisms in $\mathbf{RTPH}^1(M)$ are not Axiom A.

of a blender (also explained below), was followed in [5] to prove that there is an open and dense set of diffeomorphisms in $\mathbf{RTPH}^1(M)$ that have nonhyperbolic measures with positive entropy. Indeed, it is shown that there is a compact invariant set with positive topological entropy consisting of points whose central Lyapunov exponent is zero, and hence we can apply the variational principle in [52] to get such measures. We remark that the method of periodic approximations can only lead to measures with zero entropy (see [36]).

Summarizing, we say that in an open and dense subset of $\mathbf{RTPH}^1(M)$ each of the components in (2.1) is nonempty and contains measures with positive entropy. Therefore, a natural question is what type of behavior (negative, zero, or positive exponent) predominates. In our context, it is natural to quantify this in terms of entropy. Here we have two ways of doing so. First, given a central Lyapunov exponent α in the possible spectrum of all exponents, determine the maximal entropy of ergodic measures with that exponent:

$$\sup\{h_\mu(F)\colon\ \mu\in\mathcal{M}_{\mathrm{erg}},\ \chi_{\mathrm{c}}(\mu)=\alpha\}. \tag{2.2}$$

Or, given an exponent, determine the topological entropy of the set of Lyapunov regular points with that exponent:

$$h_{\mathrm{top}}(F,\mathcal{L}(\alpha)),\qquad \text{where}\qquad \mathcal{L}(\alpha)\overset{\mathrm{def}}{=}\{x\colon\ \chi_{\mathrm{c}}(x)=\alpha\}. \tag{2.3}$$

The former is related to restricted variational principles and implicitly determines the latter when one performs a multifractal analysis (see Theorem 3.5 below). As there is an intimate relation between ergodic measures and the corresponding generic points, in a reasonable context we expect that the two quantities introduced above coincide (see Subsection 3.3 for a full discussion).

2.2. Invariant foliations. Let us finally briefly describe geometrical features of the diffeomorphisms in $\mathbf{RTPH}^1(M)$ which are also essential to study the ergodic properties discussed above and the level sets (2.2) and (2.3). In particular, they motivate the model we will study in Section 3. The existence of the partially hyperbolic splitting $TM=E^{\mathrm{ss}}\oplus E^{\mathrm{c}}\oplus E^{\mathrm{uu}}$ implies that there are invariant foliations $\mathcal{F}^{\mathrm{ss}}$ and $\mathcal{F}^{\mathrm{uu}}$ tangent to E^{ss} and E^{uu} and called the *strong stable* and *strong unstable foliations*, respectively (see [32]). By [9, 46], because by assumption E^{c} is one-dimensional and there is a compact central leaf, there is a C^1 open and dense subset $\mathbf{ORTPH}^1(M)$ of $\mathbf{RTPH}^1(M)$ consisting of diffeomorphisms for which both foliations are *minimal* (i.e., every leaf of the foliation is dense in the whole space). A special case occurs when there is a center foliation (tangent to E^{c}) whose leaves are *all* compact. Such systems are topologically of skew-product type. An important example which still inspires many open questions is the example in [47].[4] We will further discuss this topic in Subsection 4.1.

The above geometric features are in the realm of the large family of step skew products we introduce and discuss in Section 3. On the other hand, the properties of this family seem to capture the essential dynamical properties of diffeomorphisms in $\mathbf{RTPH}^1(M)$ that allow us to study the level sets above and several ergodic properties as well as to analyze the topology of the space of invariant measures. The latter will be further discussed in the next subsection.

2.3. Topology of the space of measures: Framework. Let us now have a look at the topology of the space of Borel probability measures $\mathcal{M}$ invariant under a continuous map of a compact metric space. Equipped with the weak$*$ topology, it is a compact metrizable topological space [52, § 6.1]. We denote by $\mathcal{M}_{\mathrm{erg}}\subset\mathcal{M}$ the subset of ergodic measures. Recall that $\mathcal{M}$ is a nonempty

[4] For example, in [28] the authors ask if a general ergodic volume-preserving diffeomorphism sufficiently C^1 close to a partially hyperbolic linear automorphism $L\colon\mathbb{T}^3\to\mathbb{T}^3$ given by $L(x,y,z)=(A(x,y),z)$, with A a linear Anosov diffeomorphism, has dense ergodic measures. This would include the systems in [47].

Choquet simplex (see [52, § 6.2]). In particular, it is convex and compact. The extreme points of $\mathcal{M}$ are the ergodic measures.

In general, when one studies the topology of $\mathcal{M}$, there are many properties that can be of interest such as density and entropy density of ergodic measures (in $\mathcal{M}$) as well as connectedness of the set of ergodic measures. The density of ergodic measures has very strong immediate consequences. Indeed, then $\mathcal{M}$ is either a singleton (the map is uniquely ergodic) or a nontrivial Choquet simplex in which extreme points are dense, and one then calls $\mathcal{M}$ a *Poulsen simplex*. Poulsen [44] was the first who constructed an example of a space with such properties; by [38] any two metrizable nontrivial simplices with dense extreme points are equivalent up to affine homeomorphisms, and hence one can regard $\mathcal{M}$ as the Poulsen simplex. Note that, for example, $\mathcal{M}_{\rm erg}$ is then arcwise connected (see [38, Sect. 3, property 4]). To conclude, recall that one says that ergodic measures are *entropy dense* if for any $\mu \in \mathcal{M}$ and any $\varepsilon > 0$, any neighborhood of μ contains an ergodic measure ν such that $h_\nu(F) > h_\mu(F) - \varepsilon$.[5]

Density of ergodic (even periodic) measures was first shown in [48, 49] under the assumption that the map satisfies the periodic specification property (in [50] connectedness was concluded for shift spaces, which is however an immediate consequence of density by [38], as explained above). Recall that for smooth dynamical systems, periodic specification holds for any basic set of an Axiom A diffeomorphism (see [13]). In a more general context, in [1] it was shown that for $\Lambda \subset M$ being an isolated nontrivial transitive set of a C^1-generic diffeomorphism, periodic measures are dense (and also have many further properties, see [27]). Below we will give more details on two more recent results [11, 31].

All known results on properties such as (entropy) density and connectedness involve approximations of hyperbolic ergodic measures by either periodic measures or Markov ergodic measures supported on horseshoes. We will see that this can also be achieved in some nonhyperbolic context, in particular when the set of ergodic measures contains measures of different indices as well as nonhyperbolic measures.

Let us observe that connectedness and (entropy) density of ergodic measures are not always guaranteed. Note that [27] provides a number of counterexamples in shift spaces, though in the following we would like to focus on partially hyperbolic systems. Therefore, we point out the porcupine-like examples of compact invariant sets of partially hyperbolic transitive C^1 diffeomorphisms studied in [25, 37, 20, 21] which have a spectrum of central Lyapunov exponents with at least two disjoint components, and at least two connected components of ergodic measures. In particular, ergodic measures are not dense.

2.4. Topology of the space of measures: Intersection and homoclinic classes. To understand the topology of the space $\mathcal{M}$, it proves useful to consider the so-called intersection and homoclinic classes of hyperbolic periodic points. To state the results more precisely, let us briefly define them. We say that two hyperbolic periodic points of the same index are *homoclinically related* if the invariant sets of their orbits meet cyclically (note that in our partially hyperbolic setting with one-dimensional center transversality is not involved; see Section 5 for details). Note that this is an equivalence relation on the set of hyperbolic periodic points, and by *intersection classes* we mean the equivalence classes for the homoclinic relation. Given a hyperbolic periodic point P, we denote by $\mathrm{Int}(P)$ its intersection class.

The *intersection class* of a hyperbolic periodic point was first considered in [39], where it was called an *h-class* and used to obtain the so-called spectral decomposition of Axiom A

[5]Here our definition follows, for example, [43] and is in principle slightly more general than the one in [31]. Note however that by [19] any C^1 diffeomorphism F which is partially hyperbolic with one-dimension central bundle is h-expansive and hence by [14] the entropy map $\mu \mapsto h_\mu(F)$ is upper semi-continuous. Hence, in our setting the two definitions coincide.

diffeomorphisms. With this terminology, a *homoclinic class* (called *h-closure* in [39]) is the closure of the intersection class it contains. Note that a homoclinic class is always a transitive invariant set. Moreover, a homoclinic class of a hyperbolic periodic point may contain periodic points which are not homoclinically related to it and, consequently, may contain several distinct intersection classes. Indeed, a homoclinic class which is not hyperbolic may support ergodic measures of different indices and/or nonhyperbolic ergodic measures [30, 24, 5]. Moreover, there are examples of homoclinic classes which are not hyperbolic, whose ergodic measures are all hyperbolic, but which simultaneously support ergodic measures with negative and positive central Lyapunov exponent (see [25, 37]).

We call a (not necessarily ergodic) measure $\mu \in \mathcal{M}$ *hyperbolic with negative central Lyapunov exponent* if μ-almost every point has a negative central Lyapunov exponent and denote by $\mathcal{M}_{<0}$ the set of all such measures. Similarly, we define a measure to be *hyperbolic with positive central Lyapunov exponent* and introduce $\mathcal{M}_{>0}$. By convergence *in the weak$*$ topology and in entropy* we mean that the sequence of measures converges in the weak$*$ topology and their entropies also converge to the entropy of the limit measure.

By [8, Theorem E], the C^1 open and dense subset $\mathbf{ORTPH}^1(M)$ of $\mathbf{RTPH}^1(M)$ above can be chosen to consist of diffeomorphisms such that any two saddles with the same index are homoclinically related, that is, share the same intersection class. Thus, for $F \in \mathbf{ORTPH}^1(M)$ there are precisely two intersection classes, which we will denote by $\mathrm{Int}_{<0}$ and $\mathrm{Int}_{>0}$, where

$$\mathrm{Int}_{<0} \stackrel{\mathrm{def}}{=} \big\{P \in M\colon\ P \text{ hyperbolic periodic point},\ \chi_{\mathrm{c}}(P) < 0\big\}$$

and $\mathrm{Int}_{>0}$ is defined analogously. Moreover, each of those two sets is dense in M. Given a hyperbolic (not necessarily ergodic or periodic) measure $\mu \in \mathcal{M}$, we define its *intersection class*, denoted by $\mathrm{Int}(\mu)$, as the intersection class of hyperbolic periodic orbits such that μ is accumulated by periodic measures from that class. Hence, for $F \in \mathbf{ORTPH}^1(M)$ either $\mathrm{Int}(\mu) = \mathrm{Int}_{<0}$ or $\mathrm{Int}(\mu) = \mathrm{Int}_{>0}$. By [11], the intersection class of μ is indeed well-defined.

We can finally formulate two results which address the topics mentioned above. We briefly restate them in our more specific setting of $\mathbf{RTPH}^1(M)$. Note that a key argument to show that an *ergodic* hyperbolic measure μ is accumulated by hyperbolic periodic measures is Katok's horseshoe construction. Note that this method holds for either $C^{1+\alpha}$ diffeomorphisms (see [34]) or C^1 diffeomorphisms which have a dominated splitting (hence, in particular, for $\mathbf{RTPH}^1(M)$; see [26]) and note that this construction enables approximation in the weak$*$ topology and entropy.

First, the question of density of ergodic measures in $\mathcal{M}$ is partially answered in [11] by showing that for $F \in \mathbf{RTPH}^1(M)$ every $\mu \in \mathcal{M}_{<0}$ (not necessarily ergodic) is approached in the weak$*$ topology by ergodic measures if, and only if, almost all ergodic measures in the ergodic decomposition of μ (with respect to the corresponding distribution supported on the set of ergodic measures) share one intersection class (necessarily, μ must have the same index as the hyperbolic periodic orbits of that class). As a consequence, for $F \in \mathbf{ORTPH}^1(M)$, we have that ergodic measures are dense in $\mathcal{M}_{<0}$ and $\mathcal{M}_{>0}$. The structure of $\mathcal{M}_0$, however, seems to be much more complicated.

Second, by [31], given $F \in \mathbf{ORTPH}^1(M)$ and a saddle P of F, the set of ergodic measures supported on $\overline{\mathrm{Int}(P)}$ with the same index as P is arcwise connected. Observe that in [31] the authors originally assume $C^{1+\alpha}$ to apply Katok's result, which can be replaced by C^1 plus partial hyperbolicity as explained above. Another crucial assumption is that $\overline{\mathrm{Int}(P)}$ is isolated (observe that this guarantees that approximating measures stay in the same space of measures as μ). Note that in our setting this is automatically guaranteed since the entire ambient space M is a homoclinic class. One further assumption is that any two saddles with the same index are homoclinically related, which also holds in our case.

3. STEP SKEW-PRODUCT MODEL

We now turn to the step skew-product setting. For simplicity, we restrict ourselves to a setting with two symbols only. Let $\sigma\colon \Sigma \to \Sigma$ be the usual shift map on the space $\Sigma = \{0,1\}^{\mathbb{Z}}$ of two-sided sequences, equipped with the standard metric. Consider C^1 diffeomorphisms $f_0, f_1\colon \mathbb{S}^1 \to \mathbb{S}^1$ and the associated step skew product

$$F\colon \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1, \qquad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)), \qquad \text{where} \quad \xi = (\xi_i)_{i\in\mathbb{Z}}. \tag{3.1}$$

We denote by $\mathbf{SP}^1(\Sigma \times \mathbb{S}^1)$ the family of maps F as in (3.1) and call f_0 and f_1 also their *fiber maps*.

As motivated in Section 2 (see also the discussions in [29, 33]), a step skew product can be seen as a model of a particular example of a partially hyperbolic diffeomorphism with compact center leaves (homeomorphic to circles).

Before describing precisely our setting, let us introduce some notation. Given a sequence $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \Sigma$, we write $\xi = \xi^-.\xi^+$, where $\xi^- = (\dots\xi_{-1}) \in \Sigma^- \overset{\text{def}}{=} \{0,1\}^{-\mathbb{N}}$ and $\xi^+ = (\xi_0\xi_1\dots) \in \Sigma^+ \overset{\text{def}}{=} \{0,1\}^{\mathbb{N}_0}$ are the corresponding one-sided infinite sequences. We also consider the cylinders

$$[\eta_k \dots \eta_{k+r}] \overset{\text{def}}{=} \bigl\{\xi = (\xi_i)_{i\in\mathbb{Z}}\colon\ \xi_i = \eta_i,\ i = k,\dots,k+r\bigr\},$$

where $r \ge 0$, and define the cylinders $[\xi^-.\xi_0\dots\xi_r]$ and $[\xi_{-r}\dots\xi_{-1}.\xi^+]$ in the natural way.

Given *finite* sequences $(\xi_0\dots\xi_n)$ and $(\xi_{-m}\dots\xi_{-1})$, we let

$$f_{[\xi_0\dots\xi_n]} \overset{\text{def}}{=} f_{\xi_n} \circ \dots \circ f_{\xi_1} \circ f_{\xi_0} \qquad \text{and} \qquad f_{[\xi_{-m}\dots\xi_{-1}.]} \overset{\text{def}}{=} \bigl(f_{\xi_{-1}} \circ \dots \circ f_{\xi_{-m}}\bigr)^{-1} = \bigl(f_{[\xi_{-m}\dots\xi_{-1}]}\bigr)^{-1}.$$

We can naturally define the *central Lyapunov exponent* of a point $X = (\xi, x)$ by

$$\chi_{\mathrm{c}}(X) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log\bigl|f'_{[\xi_0\dots\xi_{n-1}]}(x)\bigr| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\bigl|f'_{\xi_k}\bigl(f_{[\xi_0\dots\xi_{k-1}]}(x)\bigr)\bigr|,$$

whenever the limit exists, which is nothing but a Birkhoff average of a continuous function (with respect to F). If the step skew product F is regarded as a model of a certain partially hyperbolic diffeomorphism, this exponent corresponds to some Lyapunov exponent in (1.1).

We will require that the fiber maps f_0 and f_1 of the map F *satisfy Axioms* CEC+ *and* Acc+, that is, there is a closed interval $J^+ \subset \mathbb{S}^1$, called a *forward blending interval*, with the following properties:

Axiom CEC+(J^+) (controlled expanding forward covering relative to J^+)**.** There exist positive constants $K_1, \dots, K_5$ such that for every interval $H \subset \mathbb{S}^1$ intersecting J^+ and satisfying the inequality $|H| < K_1$ the following conditions hold:

- (controlled covering) there exists a finite sequence $(\eta_0\dots\eta_{\ell-1})$ for some positive integer $\ell \le K_2\bigl|\log|H|\bigr| + K_3$ such that
$$f_{[\eta_0\dots\,\eta_{\ell-1}]}(H) \supset B(J^+, K_4),$$
where $B(J^+,\delta)$ is the δ-neighborhood of the set J^+;
- (controlled expansion) for every $x \in H$ we have
$$\log\bigl|\bigl(f_{[\eta_0\dots\,\eta_{\ell-1}]}\bigr)'(x)\bigr| \ge \ell K_5.$$

We let

$$\mathcal{O}^+(x) \overset{\text{def}}{=} \bigcup_{n\ge 0} \ \bigcup_{(\theta_0\dots\theta_{n-1})} f_{[\theta_0\dots\theta_{n-1}]}(x).$$

Axiom Acc+(J^+) (forward accessibility relative to J^+)**.**

$$\mathcal{O}^+(\operatorname{int} J^+) = \bigcup_{x \in \operatorname{int} J^+} \mathcal{O}^+(x) = \mathbb{S}^1.$$

Analogously, *F satisfies Axioms* CEC− *and* Acc− if there is a closed interval $J^- \subset \mathbb{S}^1$, called a *backward blending interval*, such that the inverse maps f_0^{-1} and f_1^{-1} satisfy Axioms CEC+ and Acc+ (with J^-).

The overall assumption for our consideration is transitivity.

Axiom T (transitivity)**.** There is a point $x \in \mathbb{S}^1$ such that the sets $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are both dense in $\mathbb{S}^1$.

In what follows we denote by $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$ the subset of $\mathbf{SP}^1(\Sigma \times \mathbb{S}^1)$ of skew products satisfying Axioms CEC+(J^+), Acc+(J^+), CEC−(J^-), Acc−(J^-), and T for some intervals J^+ and J^-. Under these assumptions it is possible to choose a common blending interval (see [22, Sect. 2.2] for a discussion about relations between the choice of (common) blending intervals and transitivity).

Lemma 3.1 (common blending interval [22, Lemma 2.3])**.** *Let $F \in \mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$. Then for every $x \in \mathbb{S}^1$ and every δ sufficiently small the interval $J = \overline{B(x, \delta)}$ satisfies Axioms* CEC+(J), Acc+(J), CEC−(J), *and* Acc−(J).

Returning to our setting, for maps in $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$ one can define homoclinic relations and intersection classes as in Subsection 2.4 (for details see Section 5 below). By Proposition 5.1, there are precisely two intersection classes:

$$\mathrm{Int}_{<0} \stackrel{\mathrm{def}}{=} \bigl\{P \in \Sigma \times \mathbb{S}^1 \colon\ P \text{ hyperbolic periodic point, } \chi_{\mathrm{c}}(P) < 0\bigr\}$$

and the analogously defined set $\mathrm{Int}_{>0}$. Moreover, we have

$$\overline{\mathrm{Int}_{<0}} = \overline{\mathrm{Int}_{>0}} = \Sigma \times \mathbb{S}^1 = \text{Homoclinic class}(Q), \tag{3.2}$$

where Q is any hyperbolic periodic point in $\Sigma \times \mathbb{S}^1$. Below we will conclude that each intersection class corresponds to one connected component of ergodic measures. Given a hyperbolic ergodic (not necessarily periodic) measure $\mu \in \mathcal{M}$, we define its *intersection class*, denoted by $\mathrm{Int}(\mu)$, as in Subsection 2.4. By Proposition 5.1, either $\mathrm{Int}(\mu) = \mathrm{Int}_{<0}$ or $\mathrm{Int}(\mu) = \mathrm{Int}_{>0}$.

3.1. Hyperbolic measures. The discussion in Subsection 2.4 can also be adapted to the set $\mathbf{SP}^1(\Sigma \times \mathbb{S}^1)$ (see [22, Sect. 3]). The following result is a version of [11, Theorem 2] and [31, Theorems 1.1, 1.4] for $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$.

Theorem 3.2. *Let $F \in \mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$. Every measure $\mu \in \mathcal{M}_{<0}$ is accumulated by measures $\nu_n \in \mathcal{M}_{\mathrm{erg},<0}$ in the weak$*$ topology and* $\mathrm{Int}(\mu) = \mathrm{Int}_{<0}$. *Moreover, every measure $\mu \in \mathcal{M}_{\mathrm{erg},<0}$ is accumulated by measures $\nu_n \in \mathcal{M}_{\mathrm{erg},<0}$ in the weak$*$ topology and in entropy. The analogous result is true for $\mathcal{M}_{>0}$ and $\mathcal{M}_{\mathrm{erg},>0}$, respectively.*

In particular, each of the sets $\mathcal{M}_{\mathrm{erg},<0}$ and $\mathcal{M}_{\mathrm{erg},>0}$ is arcwise connected.

The first of the above-stated results was shown in the case of a C^1 diffeomorphism with a dominated splitting $E \oplus F$ in [11], and the above statement is a translation to our setting (we recall again that the isolation condition is satisfied since we study the dynamics on the entire ambient space $\Sigma \times \mathbb{S}^1$). We point out that the essential ingredients in [11] are the C^1 dominated Pesin theory as well as the Kingman subadditive and the maximal ergodic theorems, which all have their natural translation to our setting. We refrain from repeating this part of the proof. To see that $\mathcal{M}_{\mathrm{erg},<0}$ is arcwise connected, we follow the steps of proof in [31]. In this paper the $C^{1+\alpha}$-regularity of the diffeomorphisms is used only to apply Katok's result for approximating an ergodic hyperbolic measure by hyperbolic periodic measures. As mentioned before, this also holds true for a C^1 diffeomorphism with a dominated splitting $E \oplus F$ (see [26]) and has its natural translation to our setting. The two main properties assumed in [31] are as follows:

(i) every pair of hyperbolic periodic points of the same stable index is homoclinically related (true by Proposition 5.1), and

(ii) the homoclinic class is isolated (true because of (3.2)).

However, we prefer to sketch the argument. Assume that $\mu^0, \mu^1 \in \mathcal{M}_{\mathrm{erg},<0}$. Then μ^i is accumulated by a sequence of hyperbolic periodic measures $\nu_n^i \in \mathcal{M}_{\mathrm{erg},<0}$ supported on the orbits of hyperbolic periodic points P_n^i, $i = 0, 1$. Since P_1^0 and P_1^1 are homoclinically related, there exists a continuous path $\mu_0\colon [1/3, 2/3] \to \mathcal{M}_{\mathrm{erg},<0}$ joining the measures ν_1^0 and ν_1^1. For any pair of measures ν_n^0, ν_{n+1}^0 and any neighborhood U of their convex combination $\{s\nu_n^0 + (1-s)\nu_{n+1}^0,\ s \in [0,1]\}$, one can choose a basic set Γ_n^0 such that all measures supported on it are contained in U. Hence, in particular, there exists a continuous path $\mu_n^0\colon [1/3^{n+1}, 1/3^n] \to \mathcal{M}_{\mathrm{erg},<0} \cap U$ joining the measure ν_n^0 with ν_{n+1}^0. The same applies to the measures ν_n^1, defining paths $\mu_n^1\colon [1 - 1/3^n, 1 - 1/3^{n+1}] \to \mathcal{M}_{\mathrm{erg},<0}$ which stay in neighborhoods converging to μ^1. Defining $\mu_\infty|_{(0,1)}\colon (0,1) \to \mathcal{M}_{\mathrm{erg},<0}$ by concatenating the domains of those paths, we complete the definition of the path μ_∞ by letting $\mu_\infty(0) = \lim_{n\to\infty} \mu_n^0(1/3^n)$ and $\mu_\infty(1) = \lim_{n\to\infty} \mu_n^1(1 - 1/3^n)$.

3.2. Nonhyperbolic ergodic measures. For nonhyperbolic ergodic measures, the above ergodic approximation methods do not apply in general. However, in our setting, for a step skew product in $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$, the special orbit structure enables us to somehow extend these methods. The following was shown in [22].

Theorem 3.3. *Let $F \in \mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$. Every measure in $\mathcal{M}_{\mathrm{erg},0}$ is accumulated by measures in $\mathcal{M}_{\mathrm{erg},<0}$ in the weak$*$ topology and in entropy and by measures in $\mathcal{M}_{\mathrm{erg},>0}$ in the weak$*$ topology and in entropy.*

In particular, any such measure is arcwise connected with any ergodic measure in $\mathcal{M}_{\mathrm{erg},<0}$ and in $\mathcal{M}_{\mathrm{erg},>0}$.

The rough idea of the proof of the above result is to first follow the essential ingredients in Katok's horseshoe construction (see [34; 26; 22, Sect. 3]), that is, given an *ergodic* measure, find sufficiently many orbits (whose number growths roughly exponentially with a factor given by the entropy $h_\mu(F)$) whose (noninvariant) orbital measures roughly approach μ in the weak$*$ topology. The second ingredient is, based on the special orbit structure of our maps, to choose so-called *skeletons* which connect the previously obtained orbit pieces to almost recurrent orbits, which can be shadowed by periodic ones. The main issue is to carefully control distortion (recall that we only assume that the maps are C^1 and that we shadow orbits which roughly have a central Lyapunov exponent equal to zero). This collection of hyperbolic (with exponent close to zero) periodic orbits allows to construct horseshoes; the only obstacle in this final step is that they have periods which can vary in between some numbers, which is unavoidable because the "orbit-gluing" step is only achieved by the topological constraints (in particular, guaranteed by Axioms Acc$\pm$). To bypass this problem, we construct so-called *multi-variable-time horseshoes.* This way we construct by hand horseshoes which support only measures which are weak$*$ close to μ and whose entropy is close to μ.

Theorems 3.2 and 3.3 immediately imply the following result.

Corollary 3.4. *Let $F \in \mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$. Then $\mathcal{M}_{\mathrm{erg}}$ is arcwise connected.*

3.3. Entropies of the spectrum of Lyapunov exponents. We will now answer the question raised in Section 2 as to what type of behavior (negative, zero, or positive central exponent) predominates? We will do this in terms of entropy.

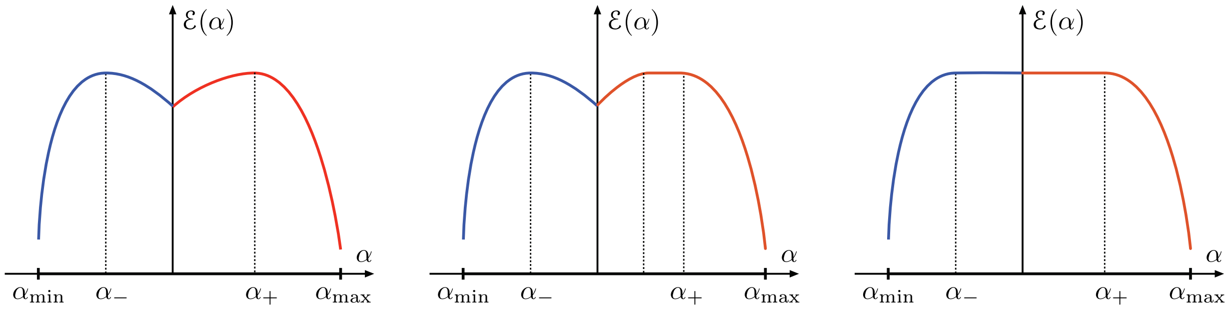

Fig. 1. Possible shapes of entropies $\mathcal{E}(\alpha)$. The left graph occurs under the assumption of proximality.

Taking first an orbitwise point of view when studying the sets $\mathcal{L}(\alpha)$ as in (2.3), we obtain the following *multifractal decomposition*:

$$\Sigma \times \mathbb{S}^1 = \bigcup_{\alpha\in\mathbb{R}} \mathcal{L}(\alpha) \cup \mathcal{L}_{\mathrm{irr}},$$

where $\mathcal{L}_{\mathrm{irr}}$ is the set of points where the central Lyapunov exponent is not well-defined (the limit does not exist). Note that each level set $\mathcal{L}(\alpha)$ is nonempty in some range of α which decomposes into three natural nonempty parts

$$\{\alpha\colon\ \mathcal{L}(\alpha)\neq\varnothing\} = [\alpha_{\min},0)\cup\{0\}\cup(0,\alpha_{\max}].$$

It is easy to verify that the maximum and minimum are indeed attained. As an immediate consequence of the fact that any pair of hyperbolic periodic orbits with the same stable index is homoclinically related (and hence is contained in a common horseshoe), we find that for every $\alpha\in(\alpha_{\min},0)\cup(0,\alpha_{\max})$ there exists an ergodic measure with positive entropy and central Lyapunov exponent equal to α. The corresponding result for $\alpha=0$ is a consequence of [5]. We will determine the "size" of those level sets in terms of *topological entropy.* Since these sets are invariant but in general noncompact, we will rely on the general concept of topological entropy h_{top} introduced by Bowen [15].

The following result from [23] now makes a connection to the above via restricted variational principles for the quantity (2.2), and Fig. 1 illustrates the theorem.

Denote by $h_\mu(F)$ the *entropy* of a measure μ. Recall that the system of fiber maps $\{f_0,f_1\}$ is *proximal* if for every $x,y\in\mathbb{S}^1$ there exists at least one sequence $\xi\in\Sigma$ such that $|f_\xi^n(x)-f_\xi^n(y)|\to 0$ as $|n|\to\infty$. Note that the system is proximal if, for example, f_0 is a Morse–Smale map whose nonwandering set contains only one attractor and one repeller and f_1 is an irrational rotation (see also Proposition 4.2 below).

Theorem 3.5. *For every $\alpha\in[\alpha_{\min},\alpha_{\max}]$ we have $\mathcal{L}(\alpha)\neq\varnothing$. Moreover, for every $\alpha\in(\alpha_{\min},0)\cup(0,\alpha_{\max})$ we have*

$$h_{\mathrm{top}}(\mathcal{L}(\alpha)) = \sup\bigl\{h_\mu(F)\colon\ \mu\in\mathcal{M}_{\mathrm{erg}},\ \chi_{\mathrm{c}}(\mu)=\alpha\bigr\}, \tag{3.3}$$

the function $\alpha\mapsto h_{\mathrm{top}}(\mathcal{L}(\alpha))$ is continuous on $[\alpha_{\min},\alpha_{\max}]$, and $h_{\mathrm{top}}(\mathcal{L}(0))>0$.

There exist (finitely many) ergodic measures μ_+ and μ_- of maximal entropy $h_{\mu_\pm}(F)=\log 2$ and with $\chi_{\mathrm{c}}(\mu_-)<0<\chi_{\mathrm{c}}(\mu_+)$. Moreover, under the proximality assumption, there exist unique ergodic F-invariant probability measures $\mu_-\in\mathcal{M}_{\mathrm{erg},<0}$ and $\mu_+\in\mathcal{M}_{\mathrm{erg},>0}$ of maximal entropy $h_{\mu_\pm}(F)=\log 2$ and we have

$$h_{\mathrm{top}}(\mathcal{L}(\alpha_-)) = h_{\mathrm{top}}(\mathcal{L}(\alpha_+)) = \log 2, \qquad h_{\mathrm{top}}(\mathcal{L}(\alpha)) < \log 2 \quad \textit{for all}\ \ \alpha\neq\alpha_-,\alpha_+,$$

where $\alpha_-=\chi_{\mathrm{c}}(\mu_-)$ and $\alpha_+=\chi_{\mathrm{c}}(\mu_+)$.

Assuming proximality, one can also show (see [23]) that no measure which is a nontrivial convex combination of the two ergodic measures of maximal entropy is (simultaneously) a weak$*$ and in-entropy limit of ergodic measures.

We note that in [45] in a somewhat similar setting the authors obtain the finiteness of entropy-maximizing measures and study their properties. Note that this is related to the so-called invariance principle, which is also studied in [51], where a similar phenomenon as in Theorem 3.5 (entropy achieving its maximum away from zero exponent) was observed (for certain ergodic measures of C^2 partially hyperbolic diffeomorphisms).

Our main approach to showing Theorem 3.5 is to treat positive, negative, and zero spectra separately. First, it is an immediate consequence of [15] that the restricted variational entropy (3.3) provides a lower bound for $h_{\rm top}(\mathcal{L}(\alpha))$. Using the fact that by Proposition 5.1 for any pair of uniformly hyperbolic sets with negative (positive) fiber exponents we can find a larger hyperbolic set containing both of them, one can conclude that these values can be expressed via the Legendre–Fenchel transform of a certain restricted pressure function (negative and positive values should be treated separately). Finally, for any α with a level set of given entropy h one can choose so-called *skeletons* established in [22, Sect. 4] to construct hyperbolic sets with entropy close to h and with almost homogeneous exponents close to α. Hence, $h_{\rm top}(\mathcal{L}(\alpha))$ is limited from above by entropies of ergodic measures with exponents close to α. Concavity of the Legendre–Fenchel transform implies its continuity, which concludes the main argument.

4. EXAMPLES

In this section, we first return to the differentiable setting and provide some details about the objects that characterize the diffeomorphisms in $\mathbf{ORTPH}^1(M)$. Further, we also will see how they motivate the axioms for the skew product in Section 3.

4.1. Blender-horseshoes and minimal foliations. The following discussion does not aim for generality; it applies primarily to diffeomorphisms in $\mathbf{PH}^1(M^3)$, where the manifold is three-dimensional, and is adapted to the setting where there are globally defined strong unstable and stable foliations.

An *unstable blender-horseshoe* (see [6, Sect. 3]) is a hyperbolic and partially hyperbolic set Λ of a diffeomorphism F conjugate to a complete shift of two symbols, such that there is a splitting $E^{\rm ss}\oplus E^{\rm c}\oplus E^{\rm uu}$ over Λ (where $E^{\rm ss}$ is its stable bundle and $E^{\rm c}\oplus E^{\rm uu}$ its unstable one) and Λ is isolated in an open neighborhood (a "cube") $\mathbf{C}$, that is, $\Lambda=\bigcap_{n\in\mathbb{Z}}F^n(\mathbf{C})$. Moreover, the splitting is defined in the whole neighborhood $\mathbf{C}$ and there is $\lambda>1$ such that $\|dF_x(v)\|\ge\lambda\|v\|$ for every $x\in\mathbf{C}$ and every vector $v\in E^{\rm c}\oplus E^{\rm uu}$. A *stable blender-horseshoe* is an unstable blender-horseshoe for F^{-1}.

For the next discussion see Fig. 2. First consider the local stable set of Λ defined by $W^{\rm s}_{\rm loc}(\Lambda)\stackrel{\rm def}{=}\bigcap_{n\le 0}F^n(\mathbf{C})$. Naively, the cube $\mathbf{C}$ has six faces: two opposite stable–center faces "tangent" to $E^{\rm ss}\oplus E^{\rm c}$, two opposite center–unstable faces "tangent" to $E^{\rm c}\oplus E^{\rm uu}$, and two opposite stable–unstable faces "tangent" to $E^{\rm ss}\oplus E^{\rm uu}$. A *strong unstable curve* is a closed curve contained in some strong unstable leaf of $\mathcal{F}^{\rm uu}$ whose boundary is contained in the stable–center faces of the cube (thus the curve joins these two faces). Similarly, a *strong stable curve* is a closed curve contained in some strong stable leaf of $\mathcal{F}^{\rm ss}$ whose boundary is contained in the center–unstable faces of the cube. An *unstable strip* (or shortly *strip*) S is a "rectangle" that is foliated by strong unstable curves. The *width* of a strip S, denoted by $w(S)$, is the supremum of the numbers w such that there is some curve η tangent to $E^{\rm c}$ and contained in S with length w. Note that there is a number $\kappa>0$ such that every strip contained in $\mathbf{C}$ has width at most κ.

By hypothesis, the hyperbolic set Λ has two *fixed points* P and Q, and we consider their local stable manifolds $W^{\rm s}_{\rm loc}(P)$ and $W^{\rm s}_{\rm loc}(Q)$ (the connected components of $W^{\rm s}_{\rm loc}(\Lambda)$ containing P and Q, respectively). There are now two isotopy classes of strong unstable curves disjoint from $W^{\rm s}_{\rm loc}(P)$,

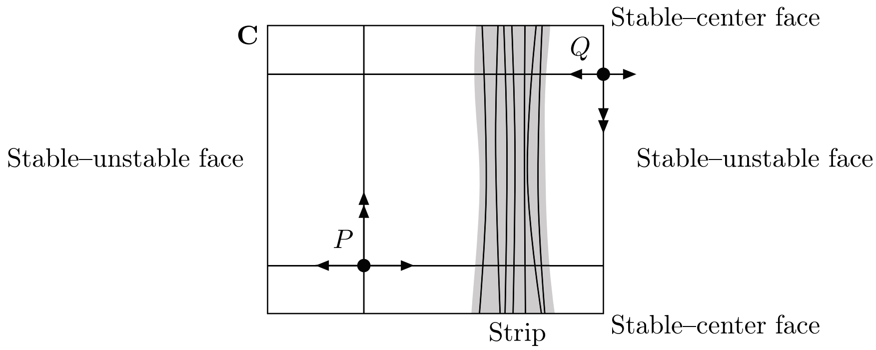

Fig. 2. Blender-horseshoe: strong unstable curves and strips (projection along the strong stable direction).

say to the right and to the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, and there are two similar classes for $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. Then the local strong unstable leaves of points of $\Lambda \setminus \{P, Q\}$ are strong unstable curves which are to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and to the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. A strip S foliated by strong unstable curves to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ is called a *strip to the right of* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$.

The key property of a blender-horseshoe is the following: for every strip S to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ there are two possibilities (that may occur simultaneously):

(1) the set $F(S)$ contains a strip S' (called the *successor* of S) to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ such that $w(S') > \lambda w(S)$ (where $\lambda > 1$ is as above), or

(2) $F(S)$ contains a strip S' that intersects $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ at a point lying at some uniform distance ρ from the boundary of S'.

Note that case (1) can occur at most $\ell(S)$ consecutive times where $\ell = \ell(S)$ is the least number with $\lambda^{\ell} w(S) > \kappa$ (with κ as above). Here by "consecutive" we mean that case (1) also holds for the successor of S and so on. In case (2) let $k = k(\rho)$ be the least number with $\lambda^k \rho > \kappa$; considering now k additional iterates, we have that $F^k(S')$ contains a strip that crosses the two stable–unstable faces of the cube $\mathbf{C}$. Summarizing, given any unstable strip S to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, we have that $F^m(S)$ contains a strip that crosses the two stable–unstable faces of the cube $\mathbf{C}$ for some $m \le \ell(S) + k$.

The blender-horseshoe has the following geometric property that we state using the approach in [5]. The family of strong unstable curves $\mathcal{D}$ defined as the ones to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and to the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ satisfies the following invariance and covering properties: every $D \in \mathcal{D}$ contains a subset D_0 such that $F(D_0) \in \mathcal{D}$. This implies that the local stable set of Λ, $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$, intersects every curve of the family $\mathcal{D}$ (see [6, Remark 3.10] and [5, Lemma 3.13]). We call $\mathcal{D}$ the *distinctive family of curves of the blender*. Finally, let us also observe that blender-horseshoes have well-defined continuations: if Λ is a blender-horseshoe for F, then for every G close enough to F the continuation Λ_G of Λ is also a blender-horseshoe (see [6, Lemma 3.9]).

As mentioned above, there is a C^1 open and dense subset $\mathbf{ORTPH}^1(M^3)$ of $\mathbf{RTPH}^1(M^3)$ whose strong stable and strong unstable foliations are both minimal. The main step of proving this is the following (see [9]). There is an unstable blender-horseshoe Λ^+ with associated cube $\mathbf{C}^+$, a stable blender-horseshoe Λ^- with associated cube $\mathbf{C}^-$, and a constant $\varrho > 0$ such that

- every curve α contained in some leaf of $\mathcal{F}^{\mathrm{uu}}$ with length $\ell(\alpha) \ge \varrho$ contains strong unstable curves $\alpha^+ \subset \mathbf{C}^+$ and $\alpha^- \subset \mathbf{C}^-$; moreover, α^+ is in the distinctive family of curves of Λ^+;
- every curve β contained in some leaf of $\mathcal{F}^{\mathrm{ss}}$ with length $\ell(\beta) \ge \varrho$ contains strong stable curves $\beta^+ \subset \mathbf{C}^+$ and $\beta^- \subset \mathbf{C}^-$; moreover, β^- is in the distinctive family of curves of Λ^-.

The uniform expansion of the bundle E^{uu} implies that for every curve α contained in some leaf of $\mathcal{F}^{\mathrm{uu}}$ there is $n = n(\alpha)$ such that $\ell(F^n(\alpha)) > \varrho$. Therefore, $\alpha_0 = F^n(\alpha)$ contains curves $\alpha_0^{\pm}$ as above. Similarly, the uniform contraction of the bundle E^{ss} implies that for every curve β contained in some leaf of $\mathcal{F}^{\mathrm{ss}}$ there is $m=m(\beta)$ such that $\beta_0=F^{-m}(\beta)$ contains curves $\beta_0^{\pm}$ as above. In particular, this implies that there is a number n_0 such that for every $n\geq n_0$ and every strong unstable curve α contained in either $\mathbf{C}^+$ or $\mathbf{C}^-$ the curve $F^n(\alpha)$ contains a strong unstable curve in $\mathbf{C}^+$ and a strong unstable curve in $\mathbf{C}^-$. A similar statement holds for strong stable curves β in $\mathbf{C}^+$ or $\mathbf{C}^-$ and F^{-n}. This means that there are *transitions along the strong unstable and strong stable foliations in finite time n_0 between the cubes of the blenders.*

4.2. Blender-horseshoes in step skew products. We now reformulate the ingredients from above for step skew products (see [33, Sect. 5] and [22, Sect. 8.3] for dictionaries stabilizing the relations between step skew products and partially hyperbolic diffeomorphisms). We will describe the blender-horseshoes in terms of the underlying one-dimensional dynamics. We consider only the strong unstable foliation; the translation for the strong stable foliation is straightforward and follows by considering negative iterations.

Note that the "local strong unstable leaf" $\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(\xi,x)$ of a point (ξ,x) is the set $[\xi^-.]\times\{x\}$, where $\xi=\xi^-.\xi^+$, and the iteration of this leaf is completely governed by the fiber dynamics:

$$F^k(\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(\xi,x))\subset\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(F^k(\xi,x))=[\xi^-\xi_0\dots\xi_{k-1}.]\times\bigl\{f_{[\xi_0\dots\xi_{k-1}]}(x)\bigr\}.$$

Therefore, the equivalent of a *curve contained in some strong unstable leaf* is a set of the form $[\xi^-.\xi_0\dots\xi_{k-1}]\times\{x\}$. Note that

$$F^k\bigl([\xi^-.\xi_0\dots\xi_{k-1}]\times\{x\}\bigr)=[\xi^-\xi_0\dots\xi_{k-1}.]\times\bigl\{f_{[\xi_0\dots\xi_{k-1}]}(x)\bigr\}$$

is a local strong unstable leaf. Thus, to study the dynamics of a local strong unstable leaf, it is enough to consider the forward orbit of the central coordinate for the iterated function system generated by the fiber maps f_0 and f_1. This also means that to obtain blenders in step skew products, it is enough to consider the dynamics in the fiber coordinate. We now define a blender-horseshoe for a step skew-product map F as in (3.1) with fiber maps f_0 and f_1 (see Fig. 3) using the terminology commonly adopted for blenders (see [10, Sect. 6.2]).

Definition 4.1 (unstable blender-horseshoe for a step skew product)**.** The skew-product map F in (3.1) has an *unstable blender-horseshoe* if there are $\beta>1$, an interval $[p,q]\subset\mathbb{S}^1$, points $a,b\in[p,q]$, $a<b$, and finite sequences $(\xi_0\dots\xi_r)$ and $(\eta_0\dots\eta_r)$, $\xi_i,\eta_j\in\{0,1\}$, such that the maps $f_{[\xi_0\dots\xi_r]}$ and $f_{[\eta_0\dots\eta_r]}$ have the following properties:

- (uniform expansion) $(f_{[\xi_0\dots\xi_r]})'(x)\geq\beta$ for all $x\in[p,b]$ and $(f_{[\eta_0\dots\eta_r]})'(x)\geq\beta$ for all $x\in[a,q]$;
- (fixed points) $f_{[\xi_0\dots\xi_r]}(p)=p$ and $f_{[\eta_0\dots\eta_r]}(q)=q$;
- (covering and invariance) $f_{[\xi_0\dots\xi_r]}([p,b])=f_{[\eta_0\dots\eta_r]}([a,q])=[p,q]$.

We say that $[p,q]$ is the *domain of definition* of the blender and that $[a,b]$ is the *superposition interval* of the blender.

The step skew-product map F has a *stable blender-horseshoe* provided F^{-1} has an unstable blender-horseshoe.

To consider the corresponding set for the cube $\mathbf{C}^+$ in the skew-product setting, we consider the union $\widehat{\mathbf{C}}^+$ of the sets $[.\xi_0\dots\xi_r]\times[p-\varepsilon,b]$ (for some small $\varepsilon>0$) and $[.\eta_0\dots\eta_r]\times[a,q]$ and define Λ^+ as the maximal invariant set of F^{r+1} in $\widehat{\mathbf{C}}^+$. In this case, the fixed points of the blender are $P=((\xi_0\dots\xi_r)^{\mathbb{Z}},p)$ and $Q=((\eta_0\dots\eta_r)^{\mathbb{Z}},q)$, and the strong unstable "curves" to the right (of the local stable set) of P are of the form $[.\xi_0\dots\xi_r]\times\{x\}$ if $x\in[p,b]$ or $[.\eta_0\dots\eta_r]\times\{x\}$ if $x\in[a,q]$. With this definition it is immediate that the image under F^{r+1} of any strong unstable curve to the right of P contains a strong unstable curve to the right of P. The stable blender-horseshoe has an associated "cube" $\widehat{\mathbf{C}}^-$ given by the union of the sets $[\xi_{-k}\dots\xi_{-1}.]\times[p-\varepsilon,b]$ (for some small $\varepsilon>0$) and $[\eta_{-k}\dots\eta_{-1}.]\times[a,q]$.

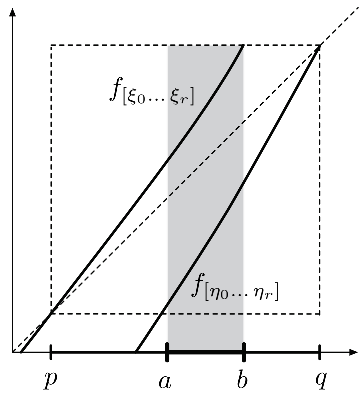

Fig. 3. Unstable blender-horseshoe.

The forward transition from $\widehat{\mathbf{C}}^+$ to $\widehat{\mathbf{C}}^-$ means that for each $x \in [p, b]$ there is a finite sequence of the form $(\xi_0 \ldots \xi_r \ldots \xi_{r+m})$, $m \geq 0$, such that $f_{[\xi_0 \ldots \xi_r \ldots \xi_{r+m}]}(x) \in (p, q)$ and for each $x \in [a, q]$ there is a finite sequence of the form $(\eta_0 \ldots \eta_r \ldots \eta_{r+n})$, $n \geq 0$ such that $f_{[\eta_0 \ldots \eta_r \ldots \eta_{r+n}]}(x) \in (p, q)$. The forward transition from $\widehat{\mathbf{C}}^-$ to $\widehat{\mathbf{C}}^+$ is defined similarly. The two backward transitions are the corresponding reformulation for backward iterates.

Finally, for the conditions in Section 3 to hold, the two blenders must capture all the dynamics of the map F (e.g., Axioms Acc$\pm$). For this we require that every point $x \in \mathbb{S}^1$ has some forward and backward iterates under the iteration of the fiber maps in the intervals (p, q) and (p', q') associated to the blenders. For a complete discussion of these constructions we refer to [22, Sect. 8.1].

4.3. Contraction–expansion–rotation in step skew products. The next result does not aim for generality and is just a reformulation of the constructions in [30, Theorem 2], where the assumption of forward minimality is replaced by a density-like hypothesis. It also restates [22, Proposition 8.8] in a slightly different way.

Proposition 4.2. *Consider a step skew-product map F as in* (3.1) *with fiber maps $f_0, f_1\colon \mathbb{S}^1 \to \mathbb{S}^1$. Suppose that*

- *there are $\delta > 0$ and finite sequences $(\xi_0 \ldots \xi_r)$ and $(\eta_0 \ldots \eta_s)$ such that $f_{[\xi_0 \ldots \xi_r]}$ has an attracting fixed point p and is uniformly contracting in $[p - \delta, p + \delta]$ and $f_{[\eta_0 \ldots \eta_s]}$ has a repelling fixed point q and is uniformly expanding in $[q - \delta, q + \delta]$;*
- *every point $x \in \mathbb{S}^1$ has some forward and some backward iterates in $(p - \delta, p + \delta)$ and some forward and some backward iterates in $(q - \delta, q + \delta)$.*

Then there are intervals $J^+, J^- \subset \mathbb{S}^1$ such that the fiber maps of F satisfy Axioms CEC+(J^+) *and* Acc$\pm(J^+)$ *and Axioms* CEC$-(J^-)$ *and* Acc$\pm(J^-)$.

To get the hypothesis of the orbits visiting the neighborhoods of p and q, it is enough to have a finite sequence $(\zeta_0 \ldots \zeta_t)$ such that $f_{[\zeta_0 \ldots \zeta_t]}$ is an irrational rotation or such that every orbit of the system is "sufficiently dense" in $\mathbb{S}^1$. In particular, if some map $f_{[\zeta_0 \ldots \zeta_t]}$ is an irrational rotation, then small perturbations of the skew product satisfy the hypotheses of Proposition 4.2.

5. HOMOCLINIC AND INTERSECTION CLASSES

We briefly discuss the homoclinic relations in the setting of skew products (for details see, for instance, [18, Sect. 2.1]). For skew-product maps F as in (3.1) that are only differentiable in the fiber direction, we call a periodic point $P = ((\xi_0 \ldots \xi_{\pi-1})^{\mathbb{Z}}, p)$ *hyperbolic* if

$$\big(f_{[\xi_0 \ldots \xi_{\pi-1}]}\big)'(p) \neq \pm 1$$

and call it *contracting* if this derivative has modulus less than one and *expanding* otherwise. As in the hyperbolic case, these points have well-defined and uniquely defined *continuations* for maps G close to F, that is, for $G(\xi,x)=(\sigma(\xi),g_{\xi_0}(x))$ where each g_i is close to f_i.

Given a hyperbolic fixed point p of the map $f_{[\xi_0\dots\,\xi_{\pi-1}]}$, consider its local invariant manifolds $W^{\mathrm{s/u}}_{\mathrm{loc}}(p,f_{[\xi_0\dots\xi_{\pi-1}]})$. If p is contracting, then $W^{\mathrm{u}}_{\mathrm{loc}}(p,f_{[\xi_0\dots\xi_m]})=\{p\}$ and $W^{\mathrm{s}}_{\mathrm{loc}}(p,f_{[\xi_0\dots\xi_m]})$ is an open interval containing p. When p is expanding, the situation is similar.

In what follows let $P=((\xi_0\dots\xi_{\pi-1})^{\mathbb{Z}},p)$ be a hyperbolic periodic point of F. Note that the stable and unstable sets of the orbit $\mathcal{O}(P)$ of P are defined, respectively, by

$$\begin{aligned}W^{\mathrm{s}}(\mathcal{O}(P),F)=\big\{(\eta,x)\colon\ \eta=\big(\dots\eta_{-1}.\eta_0\dots\eta_k(\xi_0\dots\xi_{\pi-1})^{\mathbb{N}}\big),\ k\ge 0,&\\ \text{and}\ \ f_{[\eta_0\dots\,\eta_k]}(x)\in W^{\mathrm{s}}_{\mathrm{loc}}\big(p,f_{[\xi_0\dots\xi_{\pi-1}]}\big)\big\}&,\\ W^{\mathrm{u}}(\mathcal{O}(P),F)=\big\{(\eta,x)\colon\ \eta=\big((\xi_0\dots\xi_{\pi-1})^{\mathbb{N}}\,\eta_{-k}\dots\eta_{-1}.\eta_0\dots\big),\ k\ge 0,&\\ \text{and}\ \ f^{-1}_{[\eta_{-1}\dots\,\eta_{-k}]}(x)\in W^{\mathrm{u}}_{\mathrm{loc}}\big(p,f_{[\xi_0\dots\xi_{\pi-1}]}\big)\big\}&.\end{aligned}$$

We now adapt the definitions of a *homoclinic class* and *homoclinic relations* of differentiable dynamics to the skew-product setting.

First, two hyperbolic periodic points P and Q of the same index are *homoclinically related* if the invariant manifolds of their orbits intersect cyclically, $W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(Q),F)\neq\varnothing$ and $W^{\mathrm{u}}(\mathcal{O}(Q),F)\cap W^{\mathrm{s}}(\mathcal{O}(P),F)\neq\varnothing$. The *intersection class* of P is the set of all hyperbolic periodic points homoclinically related to P. A point $X\in W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(P),F)$ is called a *homoclinic point* of P. As the transverse ones in the differentiable case, these points have well defined continuations. The *homoclinic class* of P is the closure of the homoclinic points of the orbit of P. Note that this definition does not involve transversality. As in the case of differentiable dynamics, the homoclinic class of P is a transitive set that coincides with the closure of its intersection class.

Proposition 5.1. *Suppose that the skew-product map F in* (3.1) *satisfies the conditions in Section* 3. *Then*

- *every pair of hyperbolic periodic points of the same index is homoclinically related*;
- *every homoclinic class is the whole set $\Sigma\times\mathbb{S}^1$.*

We sketch the proof of this proposition. Consider two hyperbolic periodic points $P=((\xi_0\dots\xi_{\pi_P-1})^{\mathbb{Z}},p)$ and $Q=((\eta_0\dots\eta_{\pi_Q-1})^{\mathbb{Z}},q)$ such that there is a point $c\in W^{\mathrm{u}}_{\mathrm{loc}}(p,f_{[\xi_0\dots\xi_{\pi_P-1}]})$ and a finite sequence $(\beta_0\dots\beta_r)$ with

$$f_{[\beta_0\dots\,\beta_r]}(c)\in W^{\mathrm{s}}_{\mathrm{loc}}\big(q,f_{[\eta_0\dots\,\eta_{\pi_Q-1}]}\big).$$

Then

$$C=\big((\xi_0\dots\xi_{\pi_P-1})^{\mathbb{N}}\,.\,\beta_0\dots\beta_r(\eta_0\dots\eta_{\pi_Q-1})^{\mathbb{N}},c\big)\in W^{\mathrm{s}}(\mathcal{O}(Q),F)\cap W^{\mathrm{u}}(\mathcal{O}(P),F).$$

This fact implies that, under the conditions in Section 3, any homoclinic class (of a hyperbolic periodic point) is the whole $\Sigma\times\mathbb{S}^1$ and that any pair of hyperbolic periodic points of the same index is homoclinically related. To see why this is so, assume that P and Q are both expanding and show that $W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(Q),F)\neq\varnothing$. Consider now a blending interval J containing q in its interior, as in Lemma 3.1. By Axiom Acc$-(J)$ there is a small $\delta>0$ and a finite sequence $(\tau_0\dots\tau_k)$ such that

$$(p-\delta,p+\delta)\subset W^{\mathrm{u}}_{\mathrm{loc}}\big(p,f_{[\xi_0\dots\,\xi_{\pi_P-1}]}\big)\qquad\text{and}\qquad f_{[\tau_0\dots\,\tau_k]}(p-\delta,p+\delta)\subset J.$$

Now Axiom CEC+(J) provides a finite sequence $(\eta_0 \dots \eta_\ell)$ such that

$$J \subset f_{[\tau_0 \dots \tau_k \eta_0 \dots \eta_\ell]}(p-\delta, p+\delta).$$

Hence there is $c \in (p-\delta, p+\delta)$ such that

$$q = f_{[\tau_0 \dots \tau_k \eta_0 \dots \eta_\ell]}(c).$$

Taking $C = \big((\xi_0 \dots \xi_{\pi_P - 1})^{\mathbb{N}} . \tau_0 \dots \tau_k \eta_0 \dots \eta_\ell (\eta_0 \dots \eta_{\pi_Q - 1})^{\mathbb{N}}, c\big)$, we get

$$C \in W^{\mathrm{s}}(\mathcal{O}(Q), F) \cap W^{\mathrm{u}}(\mathcal{O}(P), F).$$

The intersection $W^{\mathrm{s}}(\mathcal{O}(P), F) \cap W^{\mathrm{u}}(\mathcal{O}(Q), F) \neq \varnothing$ is obtained by reversing the roles of P and Q. The fact that the homoclinic class is the whole set $\Sigma \times \mathbb{S}^1$ follows from similar arguments.

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