MASS TRANSFERENCE PRINCIPLE: FROM BALLS TO ARBITRARY SHAPES

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Abstract. The mass transference principle, proved by Beresnevich and Velani in 2006, is a strong result that gives lower bounds for the Hausdorff dimension of limsup sets of balls. We present a version for limsup sets of open sets of arbitrary shape.

1. Introduction

For (A_i) a sequence of subsets of $\mathbb{R}^d$, define the limsup set

$$\limsup A_i = \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} A_i.$$

The geometry of limsup sets is of great importance in dimension theory, as large classes of fractal sets, including attractors of iterated function systems and random covering sets, are limsup sets. See [AT] and [FJJS] for discussion and references. Our main focus is on the following, fundamental result on dimensions of limsup sets, from a 2006 article of Beresnevitch and Velani [BV]:

Theorem 1.1 (Mass transference principle)**.** *Let (B_i) be a family of balls in $[0,1]^d$ such that $\lambda(\limsup B_i) = 1$. Let $a > 1$ and for each i, let E_i be a ball with the same center as B_i but of diameter $(\operatorname{diam} B_i)^a$. Then*

$$\dim_{\mathscr{H}} \limsup E_i \geq \frac{d}{a}.$$

Here λ denotes the Lebesgue measure in $\mathbb{R}^d$. This theorem has found great many applications in calculating the Hausdorff dimension of limsup sets, in metric number theory and fractal geometry. It has also been generalized in several directions. For recent development see [AB], where versions of this result with different, more general assumptions on B_i and $\limsup B_i$ are established.

Of particular interest for us is the generalisation of Wang, Wu, and Xu [WWX]. In their work, under the assumption $\lambda(\limsup B_i) = 1$, the authors let the sets E_i to be not balls of diameter $(\operatorname{diam} B_i)^a$ but ellipsoids with semiaxes $(\operatorname{diam} B_i)^{a_j}$, $1 \leq a_1 \leq \ldots \leq a_d$. They obtain the lower bound

$$\min_{1 \leq j \leq d} \left\{ \frac{d + j a_j - \sum_{i=1}^{j} a_i}{a_j} \right\} \tag{1.1}$$

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for the Hausdorff dimension of $\limsup E_i$. In [WWX, Section 6] they also briefly address the related problem of relaxing the condition on the shapes of the sets B_i.

In this note we will generalize this result to arbitrary shapes E_i: we will only assume that $E_i \subset B_i$ and that they are open and nonempty. We will provide a Hausdorff dimension bound for these sets, and also calculate their packing dimension directly. (The packing dimension claim also follows from the fact that $\limsup E_i$ is a dense G_δ-set, see [SV].) The argument involves a generalization of what in the dimension theory of iterated function systems is known as Falconer's singular value function, see [F2] and Section 2.

The paper is organized as follows. In Section 2 we introduce the generalized singular value function and discuss its properties, in particular its relation to the Hausdorff content. Except for Lemma 2.2, this section is not necessary for the proof of our main result Theorem 3.1, but it explains why Corollary 3.4 follows. Our results are formulated in Section 3, where we also present an example to show why the full Lebesgue measure assumption is necessary (in a sense, see [AB]). The results are proved in Sections 4-6.

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2. Singular value function

In 1988 Kenneth Falconer [F2] introduced a function, the *singular value function*, which for an ellipsoid $E \subset \mathbb{R}^d$ with semiaxes $\alpha_1 \geq \ldots \geq \alpha_d$ and parameter $s \in [0, d]$ assigns the value

$$\varphi^s(E) = \alpha_1 \alpha_2 \ldots \alpha_m \alpha_{m+1}^{s-m},$$

where $m = \lfloor s \rfloor$ is the largest integer not larger than s. This notion was of crucial importance in calculating in [F2] the dimension of certain self-affine sets.

Observe that the formula (1.1) is equivalent to $\varphi^{s_0}(E_i) = \operatorname{diam}(B_i)^d$. This is not a coincidence, the singular value function played important role in [WWX]. Up to a multiplicative constant, the singular value function agrees with the better known quantity of *Hausdorff content*

$$\mathscr{H}^s_\infty(A) = \inf\{\sum_{i=1}^{\infty} (\operatorname{diam} D_i)^s \mid A \subset \cup_{i=1}^{\infty} D_i\},$$

where the sets D_i are, say, closed balls. In order to find a mass transference principle for general shapes, we look for a version of the singular value function that applies to all Borel sets and is also equivalent to the Hausdorff content. We come to the following formula for a Borel set $E \subset \mathbb{R}^d$

$$\varphi^s(E) = \sup_{\mu} \inf_{x \in E} \inf_{r>0} \frac{r^s}{\mu(B_r(x))}, \tag{2.1}$$

where the supremum is taken over Borel probability measures supported on E and $B_r(x)$ denotes the ball of radius r and center x. It is not hard to see that when E is an ellipsoid this formula returns Falconer's singular value function (up to a multiplicative constant), so that our definition is indeed a generalization.

We note here that the function φ^s is not the only way of approximating the Hausdorff content. While studying large intersection properties of some random covering sets, Persson [P] defined an approximating function g_s, which was later related to the Hausdorff content under a positive density assumption, see [FJJS, Lemmas 3.2, 3.9]. Persson's definition is applicable for sets of positive Lebesgues measure.

The following proposition relates the singular value function and the Hausdorff content for all bounded Borel sets E. We have formulated it for a bounded Borel set in a Euclidean space, but note that the proof for Suslin sets (analytic sets) [C] in compact metric spaces is the same. The second inequality is actually Frostman's Lemma [M, Theorem 8.8], only with better constant: by a simple bootstrapping argument we show that if Frostman Lemma holds, it holds with constant 6^s.

Proposition 2.1. *Let E be a bounded Borel set. Then*

$$\varphi^s(E) \le \mathscr{H}^s_\infty(E) \le 6^s \varphi^s(E).$$

Proof. Let $\varepsilon > 0$. Find a probability measure μ such that for every $r > 0$ and every $x \in E$ we have

$$\mu(B_r(x)) < \frac{r^s}{\varphi^s(E) - \varepsilon}.$$

As every ball intersecting E is contained in a ball centred in E of twice the radius, without assuming $x \in E$ we still have

$$\mu(B_r(x)) < \frac{2^s r^s}{\varphi^s(E) - \varepsilon} = \frac{(\operatorname{diam} B_r(x))^s}{\varphi^s(E) - \varepsilon}.$$

Thus, for any collection of balls D_i covering E we have

$$\sum_i (\operatorname{diam} D_i)^s > (\varphi^s(E) - \varepsilon) \sum_i \mu(D_i) \ge \varphi^s(E) - \varepsilon.$$

This proves the first inequality.

We will need some preparation to prove the second inequality. First, for $\eta > 0$, let μ_η be a Borel probability measure supported on E such that

$$\inf_{x \in E} \inf_{r>0} \frac{r^s}{\mu_\eta(B_r(x))} > \varphi^s(E)(1-\eta), \tag{2.2}$$

and denote

$$Z = \inf_{x \in E} \inf_{r>0} \frac{r^s}{\mu_\eta(B_r(x))} \le \varphi^s(E).$$

For $\varepsilon > 0$ let $A = A_{\eta,\varepsilon}$ be the set of those points $x \in E$ for which the following is true: for all $y \in E$ and $r > 0$, if $x \in B_r(y))$ then $\mu_\eta(B_r(y)) < (1-\varepsilon) r^s / \varphi^s(E)$.

By Frostman's Lemma ([C, Section II], [M, Theorem 8.8]), there exists a constant c_d only depending on d such that whenever $\mathscr{H}^s(K) > 0$ for a bounded Borel set K, there is a Borel probability measure m supported on K such that

$$m(B_r(x)) \le \frac{1}{c_d \mathscr{H}^s_\infty(K)} r^s$$

for all $r > 0, x \in \mathbb{R}^d$, where c_d only depends on d.

We wish to prove the following fact; that $\mathscr{H}^s_\infty(A)$ is relatively small.

Fact:

$$\mathscr{H}^s_\infty(A) \leq \frac{\varphi^s(E)}{c_d(1-\varepsilon+\frac{\varepsilon}{\eta})}.$$

Proof of Fact. Assume without loss of generality that $\mathscr{H}^s_\infty(A) > 0$. Then also $\mathscr{H}^s(A) > 0$ and by [C, Section II], [M, Theorem 8.8], as above, there exists a measure ν supported on A such that

$$\nu(B_r(x)) \leq \frac{1}{c_d \mathscr{H}^s_\infty(A)} r^s$$

for all $r > 0, x \in \mathbb{R}^d$. For $\delta \in (0,1)$ let $\nu_\delta = (1-\delta)\mu_\eta + \delta\nu$ and denote

$$Z_\delta = \inf_{x\in E} \inf_{r>0} \frac{r^s}{\nu_\delta(B_r(x))}.$$

Choose some $x \in E, r > 0$. If $B_r(x) \cap A = \emptyset$, then from (2.2)

$$\begin{aligned}\nu_\delta(B_r(x)) &= (1-\delta)\mu_\eta(B_r(x)) \\ &\leq \frac{1-\delta}{1-\eta} \cdot \frac{r^s}{\varphi^s(E)} =: G_1(\delta).\end{aligned}$$

Otherwise, by definitions of A and ν

$$\begin{aligned}\nu_\delta(B_r(x)) &= (1-\delta)\mu_\eta(B_r(x)) + \delta\nu(B_r(x)) \\ &\leq r^s \left(\frac{(1-\delta)(1-\varepsilon)}{\varphi^s(E)} + \frac{\delta}{c_d \mathscr{H}^s_\infty(A)} \right) =: G_2(\delta).\end{aligned}$$

For $\delta > \eta$ we have $G_1(\delta) < r^s/\varphi^s(E)$. If, contrary to the claim, we have

$$\mathscr{H}^s_\infty(A) > \varphi^s(E) \cdot (c_d(1-\varepsilon+\frac{\varepsilon}{\eta}))^{-1} \tag{2.3}$$

then $G_2(\eta) < r^s/\varphi^s(E)$. It follows that for some $\delta > \eta$

$$Z_\delta \geq \max(r^s/G_1(\delta), r^s/G_2(\delta)) > \varphi^s(E),$$

which is a contradiction with the definition of φ^s. Thus, (2.3) cannot hold. □

We can now compare $\varphi^s(E)$ to $\mathscr{H}^s_\infty(E)$. Let (D_i) be the family of balls $B_r(x)$ with $x \in E$ for which $\mu_\eta(B_r(x)) \geq (1-\varepsilon)r^s/Z$. This family can be infinite (even uncountable), but it contains a ball of maximal radius (possibly more than one). We will inductively construct a subfamily $(E_j) \subset (D_i)$ in the following way. We take the largest ball from (D_i), which is the first ball in (E_j). We then inductively add to (E_j) the largest ball from (D_i) not contained in $\bigcup_j 3E_j$ (where $3E_j$ means the ball with the same center as E_j but three times larger radius). This way we construct a family of disjoint balls $B_r(x)$ satisfying $\mu_\eta(B_r(x)) \geq (1-\varepsilon)r^s/Z$ and such that $\bigcup B_{3r}(x) \supset E \setminus A$.

From now on, consider the sequences $\varepsilon_n \to 0$ and $\eta_n = \varepsilon_n^2$ fixed. Then, by Fact above,

$$\mathscr{H}^s_\infty(A_{\eta_n,\varepsilon_n}) \leq \frac{\varphi^s(E)}{c_d(1-\varepsilon_n+\frac{\varepsilon_n}{\eta_n})} \leq \frac{\varphi^s(E)}{c_d(1-\varepsilon_n+\frac{1}{\varepsilon_n})} =: \ell_n.$$

By subadditivity of $\mathscr{H}^s_\infty$, for all n

$$\mathscr{H}^s_\infty(E) \le \mathscr{H}^s_\infty(E \setminus A_{\eta_n,\varepsilon_n}) + \mathscr{H}^s_\infty(A_{\eta_n,\varepsilon_n}) \le \sum_{B_r(x)\in E_j} (6r)^s + \ell_n$$
$$\le 6^s Z/(1-\varepsilon_n) \sum \mu_\eta(B_r(x)) + \ell_n \le 6^s Z/(1-\varepsilon_n) + \ell_n.$$

As $\varepsilon_n, \ell_n \to 0$ and $Z \le \varphi^s(E)$, this finishes the proof of the proposition. □

We finish the section with a lemma showing that for open sets E the supremum in (2.1) is attained over absolutely continuous measures.

Lemma 2.2. *There exists $\kappa_1 > 0$ such that for every open bounded set $E \subset \mathbb{R}^d$ there exists an absolutely continuous probability measure η of bounded density, such that the support of η is a finite union of disjoint d-dimensional cubes contained in E and*

$$\varphi^s(E) \le \kappa_1 \cdot \inf_{x\in E} \inf_{r>0} \frac{r^s}{\eta(E \cap B_r(x))}.$$

Proof. Fix $\varepsilon > 0$. Let μ_1 be a probability measure supported on E such that

$$\varphi^s(E) \le (1+\varepsilon) \cdot \inf_{x\in E} \inf_{r>0} \frac{r^s}{\mu_1(E \cap B_r(x))}.$$

For $\delta > 0$ let E_δ denote the points in E lying at distance greater than δ from ∂E. We choose δ so small that $\mu_1(E_\delta) \ge 1 - \varepsilon$ and define

$$\mu_2 = \frac{1}{\mu_1(E_\delta)} \mu_1|_{E_\delta}.$$

Note

$$\inf_{x\in E} \inf_{r>0} \frac{r^s}{\mu_2(E \cap B_r(x))} \ge (1-\varepsilon) \inf_{x\in E} \inf_{r>0} \frac{r^s}{\mu_1(E \cap B_r(x))}.$$

Let f_δ be the normalized characteristic function of $B_\delta(0)$ and define

$$d\mu_3(x) = \int f_\delta(x-y) d\mu_2(y).$$

This is an absolutely continuous probability measure with density bounded by $(\lambda(B_\delta(0)))^{-1}$. For $x \in E$ and $r \ge \delta$ we have

$$\mu_3(B_r(x)) \le \mu_2(B_{r+\delta}(x)) \le \mu_2(B_{2r}(x)). \tag{2.4}$$

For $x \in E$ and $r < \delta$ we have

$$\mu_3(B_r(x)) \le \frac{r^d}{\delta^d} \mu_2(B_{r+\delta}(x)) \le \frac{r^d}{\delta^d} \mu_2(B_{2\delta}(x)). \tag{2.5}$$

Hence, for every $x \in E$ and $r > 0$ one can find $r' > 0$ such that

$$\frac{r^s}{\mu_3(B_r(x))} \geq 2^{-s}\frac{(r')^s}{\mu_2(B_{r'}(x))}. \tag{2.6}$$

We obtain the following statement: for every $x \in E$ and $r > 0$ one can find $r' > 0$ such that

$$\frac{r^s}{\mu_3(B_r(x))} \geq 2^{-s}\frac{(r')^s}{\mu_2(B_{r'}(x))}. \tag{2.7}$$

Finally, we choose some finite union $F \subset E$ of disjoint cubes such that $\mu_3(F) \geq 1-\varepsilon$ and define

$$\eta = \frac{1}{\mu_3(F)}\mu_3|_F.$$

We have

$$\inf_{x\in E}\inf_{r>0}\frac{r^s}{\eta(E\cap B_r(x))} \geq (1-\varepsilon)\inf_{x\in E}\inf_{r>0}\frac{r^s}{\mu_3(E\cap B_r(x))}$$

which ends the proof with κ_1 arbitrarily close to 2^s. □

3. Statement of results

The following is the main theorem of this article.

Theorem 3.1. *Let (B_i) be a sequence of balls in $[0,1]^d \subset \mathbb{R}^d$ such that $\lambda(\limsup_{i\to\infty} B_i) = 1$. Let (E_i) be a sequence of open sets, such that $E_i \subset B_i$. Define*

$$s = \sup\{t \mid \lambda(\limsup\{B_i \mid \varphi^t(E_i) \geq \lambda(B_i)\}) = 1\}.$$

Then

$$dim_{\mathscr{H}} \limsup E_i \geq s$$

and

$$dim_{\mathscr{P}} \limsup E_i = d.$$

The claim $\dim_{\mathscr{P}} \limsup E_i = d$ also follows by observing that $\limsup E_i$ is a dense G_δ-set [SV, Fact 12], but we give a direct proof. We will actually prove the following result; it is clear that Theorem 3.1 is an immediate corollary.

Theorem 3.2. *Let (B_i) be a sequence of balls in $[0,1]^d \subset \mathbb{R}^d$ such that $\lambda(\limsup_{i\to\infty} B_i) = 1$. Let (E_i) be a sequence of open sets, such that $E_i \subset B_i$. Assume that for some $s \geq 0$ each pair (B_i, E_i) satisfies*

$$\varphi^s(E_i) \geq \lambda(B_i).$$

Then

$$dim_{\mathscr{H}} \limsup E_i \geq s$$

and

$$dim_{\mathscr{P}} \limsup E_i = d.$$

Remark 3.3. In particular, the sets (E_i) being balls as in [BV] or ellipsoids as in [WWX] satisfy the assumptions of Theorem 3.1, so that Theorem 3.1 recovers these dimension results. Furthermore, as is the case in [BV, WWX], the lower bound we provide can be sharp, see e.g. [WWX, Corollary 5.1].

By Proposition 2.1, we have the following corollary:

Corollary 3.4. *Let (B_i) be a family of balls in $[0,1]^d$, such that $\lambda(\limsup B_i) = 1$. For some $s \in (0,d)$ for every i let $E_i \subset B_i$ be an open set satisfying $\mathscr{H}^s_\infty(E_i) \geq \lambda(B_i)$. Then $\dim_{\mathscr{H}} \limsup E_i \geq s$ and $\dim_{\mathscr{P}} \limsup E_i = d$.*

Example 3.5. The following example shows that the assumption that $\limsup B_i$ have full Lebesgue measure cannot be relaxed to positive Lebesgue measure.

Denote by Σ_* the countable set $\bigcup_{n=0}^{\infty}\{0,1\}^n$. For a word $\omega \in \Sigma_*$ let $|\omega|$ denote its length. We will construct a countable family of closed intervals $B_\omega \in [0,1], \omega \in \Sigma_*$ such that $\lambda(\limsup B_\omega) > 0$ but for every $a > 1$ $\limsup E_\omega = \emptyset$, where E_ω is an interval with the same center as B_ω but with diameter $|B_\omega|^a$. In particular, there is no nontrivial lower bound for the dimension of $\limsup E_\omega$

Let $a_n = 1/2(n+1)^2$. Let $B_\emptyset = [0,1]$. Inductively, for $\omega \in \Sigma_*$, define $J_\omega = a_{|\omega|}B^o_\omega$. Then let$B_{\omega 0}$ and $B_{\omega 1}$ be the left and right components of $B_\omega \setminus J_\omega$.

For every $n \geq 0$ we have $\lambda(\bigcup_{\omega\in\{0,1\}^n} B_\omega) = \prod_{i=0}^{n-1}(1-a_i) > \kappa := \prod_{i=0}^{\infty}(1-a_i) > 0$. In particular, $|B_\omega| \geq \kappa \cdot 2^{-|\omega|}$ and $\lambda(\limsup B_\omega) = \kappa > 0$ as desired.

Choose $a > 1$ and define E_ω. There is $N = N(a)$ such that for all $n > N$ we have $(\kappa 2^{-n})^a < a_n \kappa 2^{-n}$. Thus, for $|\omega| > N$ we have $E_\omega \subset J_\omega$, hence E_ω eventually become disjoint with all $E_\nu, |\nu| > |\omega|$. This implies $\limsup E_\omega = \emptyset$.

The strategy of the proof of Theorem 3.1 is as follows: We will construct a large Cantor subset F of $\limsup E_i$, define a mass distribution μ on the construction tree of F and estimate the local dimension of μ. This will give a lower bound to the dimension.

4. Construction of the Cantor subset

We note that we can freely assume that the size of balls B_i forms a nonincreasing sequence converging to 0. Indeed, the statement of the theorem does not depend on the order of B_i's, and moreover if the size of the balls B_i has a non-zero lower bound and if $\varphi^s(E_{n_i}) > \lambda(B_{n_i})$ for some $s > 0$ and some subsequence E_{n_i} then by the definition of φ we will have a nonzero lower bound for $\lambda(E_{n_i})$, and hence for $\lambda(\limsup E_{n_i})$ as well.

For a ball B, denote by MB a ball of the same center and M times the radius. The following lemma has been proven as [BV, Lemma 5], but, as it is a crucial ingredient in the construction of the Cantor set F, for completeness we present a proof.

Lemma 4.1. *Assume $\lambda(\limsup B_i) = 1$. Then there exists $\kappa_2 > 0$ such that for every cube $C \subset [0,1]^d$ there exists a finite family of balls $B_{n_i} \subset C$ such that the balls $3B_{n_i}$ are pairwise disjoint and that*

$$\sum \lambda(B_{n_i}) \geq \kappa_2 \lambda(C).$$

Proof. Let r denote the side of C. As the diameter of balls B_i converges to 0, for any positive ε we know that

$$\bigcup_{i;B_i\subset C} 3B_i \supset \bigcup_{i;B_i\subset C} B_i \supset (1-\varepsilon)C,$$

where $(1-\varepsilon)C$ denotes a cube of the same center as C but of side $(1-\varepsilon)r$.

Applying the $5r$-covering theorem [M, Theorem 2.1], we find a (finite or countable) subfamily of balls $B_{i_k} \subset C$ such that

$$\bigcup 15B_{i_k} \supset (1-\varepsilon)C,$$

and that the balls $3B_{i_k}$ are disjoint. Hence,

$$\sum \lambda(B_{i_k}) \geq r^d(1-\varepsilon)^d 15^{-d}$$

and we can choose a finite subfamily such that

$$\sum \lambda(B_{i_k}) \geq r^d(1-2\varepsilon)^d 15^{-d}.$$

As $\lambda(C) = r^d$, we are done. □

We will now begin the construction of the Cantor set F. First, for every set E_i denote by η_i the absolutely continuous measure provided by Lemma 2.2 and by ℓ_i the supremum of its density. We will denote by $\tilde{E}_i$ the support of η_i, which by Lemma 2.2 is a finite union of disjoint cubes. It suffices to give the lower bound for $\dim_{\mathscr{H}} \limsup \tilde{E}_i$ and $\dim_{\mathscr{P}} \limsup \tilde{E}_i$.

We will now inductively construct a family of sets $F_0 \supset F_1 \supset \dots$ such that each $F_j; j \geq 1$ is a finite union of some $\tilde{E}_i$'s. Clearly,

$$F := \bigcap F_j \subset \limsup \tilde{E}_i \subset \limsup E_i.$$

In the next section we will proceed to distribute a measure μ on F.

Start with the cube $F_0 = [0,1]^d$. Applying Lemma 4.1 to the cube F_0 we can find a finite family of disjoint balls $\mathscr{F}_1 \subset \{B_i\}$ such that

$$\sum_{B_i\in\mathscr{F}_1} \lambda(B_i) > \kappa_2,$$

where κ_2 is from Lemma 4.1. Let $F_1 = \bigcup_{B_i\in\mathscr{F}_1} \tilde{E}_i$.

Fix some sequence $\varepsilon_j \searrow 0$ and recall that each $\tilde{E}_i$ for $B_i \in \mathscr{F}_1$ is a union of cubes. Denote by r_1 the diameter of the smallest of these cubes. Further, let

$$\tilde{r}_1 = \min(r_1, (\kappa_2 \cdot \min\{\frac{1}{\ell_i\lambda(B_i)}; B_i \in \mathscr{F}_1\})^{1/\varepsilon_1}).$$

Now divide all the components of all $\tilde{E}_i$ with $B_i \in \mathscr{F}_1$ into cubes $D_1^{(1)}, \dots, D_{N_1}^{(1)}$ of diameter between $\tilde{r}_1/2$ and $\tilde{r}_1$ (notice that different components might need to be divided into cubes of different size). These cubes will be where the construction continues.

We carry on inductively. Assume that the notions $\mathscr{F}_{j-1}$, $\tilde{r}_{j-1}$,

$$F_{j-1} = \bigcup_{B_i\in\mathscr{F}_{j-1}} \tilde{E}_i$$

and the cubes $D_1^{(j-1)}, \dots, D_{N_{j-1}}^{(j-1)} \subset F_{j-1}$ of diameter between $\tilde{r}_{j-1}/2$ and $\tilde{r}_{j-1}$ as above have been defined.

Now apply Lemma 4.1 to each $D_k^{(j-1)}$, $k = 1, \dots, N_{j-1}$. Obtain in this way a family $\mathscr{F}_j$ of balls B_i such that for $D_k^{(j-1)}$, $k = 1, \dots, N_{j-1}$,

$$\sum_{B_i \in \mathscr{F}_j; B_i \subset D_k^{(j-1)}} \lambda(B_i) \geq \kappa_2 \lambda(D_k^{(j-1)}).$$

Let $F_j = \bigcup_{B_i \in \mathscr{F}_j} \tilde{E}_i$.

Finally, define r_j as the smallest diameter of cube components of F_j. Set

$$\tilde{r}_j = \min(r_j, \tilde{r}_{j-1} \cdot (\kappa_2 \cdot \min\{\frac{1}{\ell_i \lambda(B_i)}; B_i \in \mathscr{F}_j\})^{1/\varepsilon_j}),$$

and subdivide F_j into cubes of diameter between $\tilde{r}_j/2$ and $tilder_j$ as above to continue.

5. Construction of the mass distribution

We will now construct a mass distribution on F, and proceed in the next section by investigating its local dimension. Begin by setting the notations

$$\mathscr{F}_j(E) = \{B_i \in \mathscr{F}_j \mid B_i \subset E\} \text{ and } F_j(E) = \bigcup_{B_i \in \mathscr{F}_j(E)} \tilde{E}_i$$

for $E \subset F_0$.

We start with μ_0 defined as the Lebesgue measure λ restricted to F_0. As an intermediate step in the definition of μ_1, in the first level of construction F_1, define

$$\nu_1|_{B_i} = \frac{\mu_0|_{B_i}}{\sum_{B_k \in \mathscr{F}_1} \mu_0(B_k)}$$

for all i such that $B_i \in \mathscr{F}_1$, and no mass elsewhere. Then define, for $B_i \in \mathscr{F}_1$ and $\tilde{E}_i \subset B_i$, the measure μ_1 supported on F_1 by setting

$$\mu_1|_{\tilde{E}_i} = \nu_1(B_i) \cdot \eta_i.$$

Continue in this way; assume that μ_{n-1} has been defined on the sets $\tilde{E}_i$ with $B_i \in \mathscr{F}_{n-1}$. Let $B_k \in \mathscr{F}_n$, $B_k \subset D_j^{(n-1)}$, where $D_j^{(n-1)}$ is a cube of side length approximately $\tilde{r}_{n-1}$ from the cube decomposition of $\tilde{E}_i$. Then define

$$\nu_n|_{B_k} = \frac{\mu_{n-1}(D_j^{(n-1)})\lambda|_{B_k}}{\sum_{B_\ell \in \mathscr{F}_n(E_i); B_\ell \subset D_j^{(n-1)}} \lambda(B_\ell)},$$

and for each $\tilde{E}_k \subset B_k \in \mathscr{F}_n$

$$\mu_n|_{\tilde{E}_k} = \nu_n(B_k) \cdot \eta_k,$$

obtaining a measure supported on F_n.

Notice that (μ_n) is a sequence of probability measures supported on the compact set $[0,1]^d$, so that it has a weakly convergent subsequence. Denote the limit of this subsequence by μ, and notice that it is by construction supported on F. In fact, $\mu_n(B_i) = \mu_{n+k}(B_i)$ for all $k \geq 0$, for all $B_i \in \mathscr{F}_n$, and similarly for $\tilde{E}_i \subset B_i \in \mathscr{F}_n$.

6. Estimating the local dimension

We now bound the local dimension of μ. Pick a point $x \in F$ and $r > 0$. We want to give an estimate to the μ-measure of the ball $B_r(x)$. Let n be such that $\tilde{r}_n < r \le \tilde{r}_{n-1}$. Since $x \in F$, we can write $x \in B_{i_n} \subset B_{i_{n-1}} \subset \ldots \subset B_{i_1}$, with $B_{i_k} \in \mathscr{F}_k$ for all k.

There are two cases to consider: $\operatorname{diam} B_{i_n} \le r < \tilde{r}_{n-1}$ and $\tilde{r}_n \le r < \operatorname{diam} B_{i_n}$.

Case 1: $\operatorname{diam} B_{i_n} \le r < \tilde{r}_{n-1}$**.** Recall that in the construction we divide the set $\tilde{E}_{i_{n-1}}$ into the $(n-1)$-st generation cubes $D_j^{(n-1)}, j = 1, \ldots, N_{n-1}$ of diameter approximately $\tilde{r}_{n-1}$, and for each of them

$$\mu(D_j^{(n-1)}) = \mu_{n-1}(D_j^{(n-1)}).$$

Further,

$$\mu_{n-1}(D_j^{(n-1)}) = \nu_{n-1}(D_j^{(n-1)})\eta_{i_{n-1}}(D_j^{(n-1)}) \le \nu_{n-1}(D_j^{(n-1)})\ell_{i_{n-1}}\lambda(D_j^{(n-1)}).$$

To continue, set the notation

$$D_j^{(n-1)} \subset \tilde{E}_{i_{n-1}} \subset B_{i_{n-1}} \subset D_k^{(n-2)} \subset \tilde{E}_{i_{n-2}} \subset B_{i_{n-2}}.$$

Then by the definition of ν_{n-1},

$$\nu_{n-1}(D_j^{(n-1)}) = \frac{\mu_{n-1}(D_k^{(n-2)})}{\sum_{B_\ell \in \mathscr{F}_{n-1}; B_\ell \subset D_k^{(n-1)}} \lambda(B_\ell)} \cdot \lambda(B_{i_{n-1}}),$$

where $\mathscr{F}_{n-1}$ was chosen using Lemma 4.1 so that

$$\sum_{B_\ell \in \mathscr{F}_{n-1}; B_\ell \subset D_k^{(n-2)}} \lambda(B_\ell) \ge \kappa_2 \lambda(D_k^{(n-2)}).$$

Combining the above, we obtain

$$\mu_{n-1}(D_j^{(n-1)}) \le \frac{\mu_{n-1}(D_k^{(n-2)})}{\lambda(D_k^{(n-2)})} \cdot \frac{\ell_{i_{n-1}}\lambda(B_{i_{n-1}})}{\kappa_2} \cdot \lambda(D_j^{(n-1)}).$$

Using this inductively, we end up with

$$\mu(D_j^{(n-1)}) = C_{n-1}(D_j^{(n-1)}) \cdot \lambda(D_j^{(n-1)}), \tag{6.1}$$

where

$$C_{n-1}(D_j^{(n-1)}) \le \frac{\lambda(B_{i_1}) \cdots \lambda(B_{i_{n-1}}) \cdot \ell_{i_1} \cdots \ell_{i_{n-1}}}{\kappa_2^{n-1}}. \tag{6.2}$$

Let now $D^{(n-1)}$ be the $(n-1)$-st generation cube containing x. We will write $C_{n-1}(x)$ for $C_{n-1}(D^{(n-1)})$.

Recall that $\tilde{r}_{n-1}$ was chosen in such a way that

$$\begin{aligned} |\log \tilde{r}_{n-1}| &\le 1/\varepsilon_{n-1} \log\Big(\frac{\ell_{i_{n-1}}\lambda(B_{i_{n-1}})}{\kappa_2}\Big) + \log \tilde{r}_{n-2} \\ &= 1/\varepsilon_1 \log\Big(\frac{\ell_{i_1}\lambda(B_{i_1})}{\kappa_2}\Big) + \cdots + 1/\varepsilon_{n-1} \log\Big(\frac{\ell_{i_{n-1}}\lambda(B_{i_{n-1}})}{\kappa_2}\Big). \end{aligned}$$

In particular, by (6.2) and the choice of $1/\varepsilon_n \to \infty$, we have

$$\lim_{n\to\infty} \max_y \frac{|\log C_n(y)|}{|\log \tilde{r}_n|} = 0. \tag{6.3}$$

Using, essentially, (6.1), in the cube $D^{(n-1)}$ we find a collection of balls $B_i \in \mathscr{F}_n$ such that each of them satisfies

$$\mu(B_i) = \nu_n(B_i) \le \frac{1}{\kappa_2} \cdot C_{n-1}(x)\lambda(B_i) \tag{6.4}$$

and $3B_i$ are disjoint. Observe that, since $r < \tilde{r}_{n-1}$, the ball $B_r(x)$ can intersect at most 5^d of the $(n-1)$-st generation cubes. Furthermore, if indeed there is some $D_j^{(n-1)}$ such that $y \in D_j^{(n-1)} \cap F \cap B_r(x)$ then $B_r(x) \cap D_j^{(n-1)} \subset B_{2r}(y) \cap D_j^{(n-1)}$. Hence, $\mu(B_r(x)) \le 10^d \mu(B_r(x) \cap D^{(n-1)})$, and we continue with the latter. We have

$$\mu(B_r(x) \cap D^{(n-1)}) \le \sum_{B_i \in \mathscr{F}_n; B_i \cap B_r(x) \neq \emptyset} \mu(B_i).$$

However, by the construction, balls $3B_{i_n}$ and $3B_i$ are disjoint for any $i \neq i_n$, and in particular, since $x \in B_{i_n}$, we have $x \notin 3B_i$. Hence, if $B_r(x)$ intersects B_i then $\operatorname{diam} B_i \le r$, and we have from (6.4) and the disjointness of B_i

$$\mu(B_r(x) \cap D^{(n-1)}) \le \lambda(B_{2r}(x)) \cdot \frac{C_{n-1}(x)}{\kappa_2}.$$

Summing up the estimates from above, we get

$$\mu(B_r(x)) \le 20^d \cdot \frac{1}{\kappa_2} \max_y C_{n-1}(y) r^d$$

and hence by (6.3), for $\operatorname{diam} B_{i_n} \le r < \tilde{r}_{n-1}$ we have

$$\frac{\log \mu(B_r(x))}{\log r} \ge d - q_n \tag{6.5}$$

with $q_n \to 0$.

Case 2: $\tilde{r}_n \le r < \operatorname{diam} B_{i_n}$**.** In this case $B_r(x)$ is not going to intersect any $B_i \in \mathscr{F}_n, i \neq i_n$. Hence, $\mu(B_r(x)) = \mu(B_r(x) \cap \tilde{E}_{i_n})$.

Consider the distribution of measure μ on $\tilde{E}_{i_n}$. We have

$$\mu_n|_{\tilde{E}_{i_n}} \le C_{n-1}(x)\kappa_2^{-1}\lambda(B_{i_n}) \cdot \eta_{i_n}. \tag{6.6}$$

Hence, for each of the n-th level cubes $D_j^{(n)}$ we have

$$\mu(D_j^{(n)}) = \mu_n(D_j^{(n)}) \le C_{n-1}(x)\kappa_2^{-1}\lambda(B_{i_n}) \cdot \eta_{i_n}(D_j^{(n)}).$$

We note that these are n-th generation cubes, of size approximately $\tilde{r}_n$, not the $(n-1)$-st generation cubes we considered in the previous case. However, we do not yet know how exactly μ is distributed on each $D_j^{(n)}$ – this will be decided on the following stages of the construction. Nevertheless, we can write

$$\mu(B_r(x)\cap\tilde{E}_{i_n})\le\sum_{D_j^{(n)};D_j^{(n)}\cap B_r(x)\neq\emptyset}\mu(D_j^{(n)})$$

and we also know that if $D_j^{(n)}\cap B_r(x)\neq\emptyset$ then $D_j^{(n)}\subset B_{r+\tilde{r}_n}(x)$. Combining this with (6.6) we get

$$\mu(B_r(x))\le C_{n-1}(x)\kappa_2^{-1}\lambda(B_{i_n})\eta_{i_n}(B_{r+\tilde{r}_n}(x)\cap\tilde{E}_{i_n}).$$

Note that $r+\tilde{r}_n\le 2r$. By the definition of η_{i_n} and the assumption $\lambda(B_i)\ge\varphi^s(E_i)$ we have

$$\eta_{i_n}(B_{2r}(x))\le\frac{(2r)^s\kappa_1}{\lambda(B_{i_n})},$$

and, using (6.3)

$$\frac{\log\mu(B_r(x))}{\log r}\ge\frac{s\log 2+\log C_{n-1}(x)+\log\kappa_1-\log\kappa_2}{\log r}+s\ge s+q_n \tag{6.7}$$

with $q_n\to 0$.

We finish the proof of Theorem 3.2 applying the mass distribution principle [F, Proposition 2.3] to (6.5) and (6.7).

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