Dimension estimates for invariant measures of contracting-on-average iterated function systems

Michał Rams *

Institute of Mathematics, Polish Academy of Sciences

ul. Śniadeckich 8, 00-950 Warszawa, Poland

e-mail rams@impan.gov.pl

Abstract

We estimate from above and below the dimension of invariant measure for contracting-on-average iterated function systems in $\mathbb{R}^d$.

1 Introduction

The goal of this paper is to give dimension estimates for invariant measures of contracting-on-average self-conformal iterated function systems with point-depending probabilities.

The motivation for this work comes from the recent results of Szarek and Myjak [S], [MS] and also Nicol, Sidorov and Broomhead [NSB] and Fan, Simon and Toth [FST] who gave bounds for the Hausdorff dimension of the invariant measure for some contracting-on-average iterated function systems.

Lately, Johansson and Öberg computed the precise value of the Hausdorff dimension for bounded (not necessarily contracting) iterated function systems on the line satisfying the open set condition, [JO]. Similar questions come from other branches of mathematics, compare [Wa].

There are two kinds of results on the dimension of an invariant measure for (contracting) iterating function systems. The upper bound is easy to obtain,

*supported by Polish KBN Grant No 2P0 3A 034 25

1991 *Mathematics Subject Classification*: Primary 28A78,34D45 depending on assumptions it takes form of Moran formula (for similitudes) or Bowen (pressure) formula (for smooth conformal maps). For systems with probabilities we get Eckmann-Ruelle formula, i.e. ratio of entropy to the Lyapunov exponent. To get the lower bound for an iterated function system one typically needs a version of the open set condition (like in [LN], [R]).

We are working here with systems that aren't everywhere contracting. We need to assume some contractiveness-on-average condition to guarantee existence and uniqueness of the invariant measure, as was proved in [BDEG].

We prove the Eckmann-Ruelle formula for the upper bound on the Hausdorff dimension of the invariant measure. It is the same as the bound in [JO] and [FST], but we work on different class of systems, in particular our method works for measures with unbounded supports. As for the lower bound, we can only obtain it under some (bit stronger) version of OSC.

The rest of the paper is divided as follows. In the second section we state some known results on the iterated function systems and formulate our theorem. In the third section we prove some properties of contracting-on-average iterated function systems that we will need in the future. As the results of this section might be of independent interest, they are formulated in more general situation, i.e. for iterated function system acting on a Polish space and satisfying Dini condition. In the fourth section we finish our preparation by formulating some geometric lemmas about typical behaviour of orbits of IFS. Finally, in the fifth section we present the proof of Theorem 2.1.

2 Introduction to iterated function systems

For informations on dimension theory and properties of contracting iterated function systems we refer the reader to standard textbooks like [F]. The existence and most important properties of invariant measures for contracting-on-average iterated function systems are to be found in [BDEG] and [E].

Consider an iterated function system consisting of k $C^{1+\alpha}$ maps $h_1, \dots, h_k : \mathbb{R}^d \to \mathbb{R}^d$ and a probability vector field $\vec{p}(x) = (p_1(x), \dots, p_k(x))$ defined on $\mathbb{R}^d$. For a map h we denote $h'(x) = ||Dh(x)||$. We assume h_i' are uniformly bounded away from 0 and ∞. We also assume that p_i are C^α and uniformly bounded away from 0. Hence, $\log p_i$ and $\log h_i'$ are C^α.

The system acts as follows: on every iteration we randomly choose $i \in \{1, \dots, k\}$ according to the vector $\vec{p}(x)$ (depending on the point $x \in \mathbb{R}^d$ we are in) and then apply the map h_i. If all the maps h_i are contracting, there exists a unique probability measure solving equation

$$\nu = F(\nu) := \sum_i h_{i*}(p_i \cdot \nu), \tag{2.1}$$

it is called *invariant measure* for the iterated function system. Moreover, the invariant measure satisfies

$$\overline{\dim}_H(\nu) \le \frac{\int \sum_i p_i(x) \log p_i(x) d\nu(x)}{\int \sum_i p_i(x) \log |h_i'(x)| d\nu(x)} \tag{2.2}$$

where the upper Hausdorff dimension of the measure is defined as the infimum of the Hausdorff dimension over sets of full measure. Let us remind here that the lower Hausdorff dimension $\underline{\dim}_H$ is defined as the infimum of the Hausdorff dimensions of sets of positive measure and if both are equal, the common value is called Hausdorff dimension of the measure and denoted $\dim_H$.

Denote $\Sigma = \{1, \dots, k\}^{\mathbb{N}}$. Given a sequence $\omega \in \Sigma$, we will denote by ω_n the n-th element and by ω^n the first n elements of ω. We will also denote $\omega^{-n} = \omega_n \omega_{n-1} \dots \omega_1$. The *n-th level cylinder* C_{ω^n} is the set of all sequences from Σ that begin with ω^n. Let σ, σ_i be the left shift map and the k right shift maps on Σ, i.e.

$$\sigma\omega = \omega_2\omega_3 \dots$$

$$\sigma_i\omega = i\omega_1\omega_2 \dots$$

Define $h_{\omega^n} = h_{\omega_n} \circ \dots \circ h_{\omega_1}$. Let

$$p_{\omega^n}(x) = \prod_{i=1}^{n} p_{\omega_i}(h_{\omega^{i-1}}(x))$$

and let p_x be the unique probability measure satisfying

$$p_x(C_{\omega^n}) = p_{\omega^n}(x)$$

The limit

$$\pi(\omega) = \lim_{n\to\infty} h_{\omega^{-n}}(x_0) \tag{2.3}$$

exists for all $\omega \in \Sigma$ and does not depend on $x_0 \in \mathbb{R}^d$. The invariant measure ν can be written as

$$\nu = \pi_*(\mu_-) \tag{2.4}$$

where μ_- is some σ-invariant probability measure on Σ.

In this paper we assume only that the IFS is *contracting-on-average*, ie

$$\sum_1^k p_j(x) \log \frac{d(h_j(x), h_j(y))}{d(x,y)} < -b < 0 \tag{2.5}$$

uniformly in x and y. The existence and uniqueness of the invariant measure ν for such systems was established in [BDEG]. In addition, it was proved that the measure is attractive, i.e. that $F^n(\tilde{\nu}) \to \nu$ for all probability measures $\tilde{\nu}$.

To get a reasonable lower bound on the Hausdorff dimension of the invariant measure we will need stronger geometric assumptions. Namely, we will assume h_i are conformal maps (we will call it *the conformal case*). The conformality assumption is void for $d = 1$ but very strong in higher dimension: it implies analyticity in $\mathbb{R}^2$ and only Möbius transformations are conformal in $\mathbb{R}^d$, $d \geq 3$ (Liouville Thm).

In conformal case, if IFS is contracting and if in addition there exists an open set U such that $h_i(U) \subset U$ and $h_i(U) \cap h_j(U) = \emptyset$ for $i \neq j$ (it is called *open set condition*, OSC), we have equality in (2.2) ([F]).

Our main result is that the results on the Hausdorff dimension of invariant measures for contracting-on-average iterated function systems are almost the same as for contracting ones. The one difference is that instead of OSC we have to use a bit stronger geometrical conditions.

If the IFS satisfies OSC for some open set U and if (for some R_1)

$$\operatorname{dist}(h_i(U), h_j(U)) \geq R_1 > 0, \tag{2.6}$$

the system satisfies *strong open set condition* (SOSC). If (in addition to OSC) U satisfies (for some R_2 and R_3)

$$\forall_{r<R_2} \forall_{x\in U} \operatorname{vol}(B_r(x) \cap U) \geq R_3 r^d, \tag{2.7}$$

the system satisfies *regular open set condition* (ROSC).

Denote the right hand side of (2.2) by s.

Theorem 2.1. *Let $(\mathbb{R}^d, h_i, p_i)$ be an iterated function system. Assume h_i are C^α and with bounded $\log h_i'$, p_i are C^α and bounded away from 0. Assume that (2.5) is satisfied. Denote the invariant measure by ν. Then*

$$\overline{\dim}_H(\nu) \leq s$$

If in addition all h_i are conformal and either SOSC or ROSC are satisfied,

$$\dim_H(\nu) = s$$

3 Coding map and ergodicity

Our main goal in this section is to prove the following proposition:

Proposition 3.1. *Let X be a Polish space and fix $x_0 \in X$. Let (X, h_i, p_i) be an iterated function system satisfying (2.5), with p_i satisfying Dini condition and bounded away from 0 and with h_i Lipschitz. Let ν be the invariant measure of this system. Then there exists a σ-invariant probabilistic measure μ_- on Σ such that (2.3) is well defined μ_--almost everywhere and (2.4) holds.*

Similar result was proved by Werner [We], but he used a stronger assumption instead of (2.5).

We start with some notations. Let

$$\mu_+(d\omega) = \int_X p_x(d\omega)d\nu(x)$$

It is a probability σ-invariant measure on Σ. We denote $\tilde{\Sigma} = \{1, \ldots, k\}^{\mathbb{Z}}$ and let $\tilde{\sigma}$ be the left shift on $\tilde{\Sigma}$, i.e. $(\tilde{\sigma}\omega)_i = \omega_{i+1}$. The space $(\tilde{\Sigma}, \tilde{\sigma})$ has a natural equivalence with $(\Sigma \times \Sigma, S)$, where

$$S(\tau, \omega) = (\sigma_{\omega_1}\tau, \sigma\omega)$$

On $\tilde{\Sigma}$ there exists a unique $\tilde{\sigma}$-invariant measure μ such that μ_+ is the marginal measure of μ with respect to the positive coordinates:

$$\mu_+(\cdot) = \int \mu(d\tau, \cdot)$$

The measure μ is called *natural extension* of μ_+.

We define the measure μ_- as the marginal measure of μ with respect to the nonpositive coordinates, i.e.

$$\mu_-(\cdot) = \int \mu(\cdot, d\omega)$$

The explicit formula for μ_- can be written as

$$\mu_-(C_{\omega^n}) = \mu_+(C_{\omega^{-n}}) \tag{3.1}$$

Let π be given by (2.3) (wherever it is defined). In order to prove (2.4) we need only to prove that π is well defined for μ_--almost all $\omega \in \Sigma$. Indeed, assume it is so. The measure $\pi_*(\mu_-)$ is then a probability measure on X. Moreover,

$$h_i(\pi(\omega)) = \pi(\sigma_i\omega)$$

hence $\pi_*(\mu_-)$ is invariant under the action of the IFS. By uniqueness of the invariant measure, it must be equal to ν and (2.4) holds.

We will need some estimations on the ergodic sums of integrable functions over typical trajectories of the IFS. The ergodic theorem proved in [E] gives us some estimations, but we need something stronger. The following lemma is a large deviation result for typical trajectories:

Lemma 3.2. *Choose arbitrary bounded $A \subset X$. Then there exists $L(A)$ such that for all $x \in X$, $n > 0$ and $K > 0$*

$$p_x(\omega \in \Sigma; \operatorname{diam}(h_{\omega^n}(A)) \leq K\exp(-nb)) > 1 - \frac{L(A)(1 + \log^+(d(x, x_0)))n}{(\log K)^2}$$

Proof. For $j \in \{1, \dots, k\}$ and $y \in X$ let

$$\tilde{H}(y, j) = \sup_{z \in X} \log \frac{d(h_j(y), h_j(z))}{d(y, z)}$$

and

$$H(y, j) = \tilde{H}(y, j) - b - \sum_{i=1}^{k} p_i(y) \tilde{H}(y, i)$$

By (2.5), $H(y, j) > \tilde{H}(y, j)$ for all y and j. In addition, as $\tilde{H}$ is uniformly bounded, so is H.

Define a sequence of random variables of fixed expected value b:

$$H_n(\omega) = H(h_{\omega^{n-1}}(x), \omega_n)$$

where $\omega \in (\Sigma, p_x)$. By boundedness of H_n, we have a bound on its variation as well:

$$\operatorname{var} H_n < v$$

The random variables H_n and $\sum_{j=1}^{n-1} H_j$ are uncorrelated (but not necessarily independent). Hence

$$\operatorname{var}(\sum_{j=1}^{n} H_j) < vn$$

and by Tchebysheff inequality

$$p_x(\omega; \sum_{j=1}^{n} H_j(\omega) > -nb + \log K) < \frac{v}{(\log K)^2} \tag{3.2}$$

Now, we can estimate

$$\operatorname{diam}(h_{\omega^n}(A)) \le 2 \sup_{y \in A} d(h_{\omega^n}(x), h_{\omega^n}(y)) \le 2 \exp(\sum_{j=1}^{n} H_j(\omega))(d(x, x_0) + d(x_0, A) + \operatorname{diam} A)$$

(here we use the inequality $\tilde{H} < H$ and (2.5)). The assertion follows from (3.2). ☐

Lemma 3.3.

$$\int_X \log^+(d(x, x_0)) d\nu(x) \le M(x_0) < \infty$$

Proof. Let $\tilde{\nu}$ be any probability measure on X with bounded support. Let

$$M_0 = \max_i d(x_0, h_i(x_0))$$

We have for all $x \in X$

$$d(x_0, h_i(x)) \leq M_0 + d(h_i(x_0), h_i(x))$$

By (2.5) and Lipschitz property of h_i we conclude that there exists $M_1 > 0$ such that for $d(x_0, x) > M_1$

$$\sum p_i(x) \log^+(d(x_0, h_i(x))) \leq (1 - b/2) \log(d(x_0, x)) \tag{3.3}$$

Let

$$M_2 = \sup_{d(x,x_0)<M_1} \max_i d(h_i(x), x_0)$$

We divide $\tilde{\nu}$ into two parts: $\tilde{\nu}_1$, supported on a ball $B_{M_1}(x_0)$ and $\tilde{\nu}_2$, supported on the outside. Clearly,

$$F(\tilde{\nu}) = F(\tilde{\nu_1}) + F(\tilde{\nu}_2)$$

We have

$$\int_X \log^+(d(x, x_0)) F(\tilde{\nu}_1)(dx) \leq \log(M_2)$$

and (by (3.3))

$$\int_X \log^+(d(x, x_0)) F(\tilde{\nu}_2)(dx) \leq (1 - b/2) \int_X \log^+(d(x, x_0)) d\tilde{\nu}(x)$$

Hence,

$$\int_X \log^+(d(x, x_0)) F(\tilde{\nu})(dx) \leq \log(M_2) + (1 - b/2) \int_X \log^+(d(x, x_0)) d\tilde{\nu}(x)$$

The sequence $1/n \sum_{i=1}^n F^i(\tilde{\nu})$ converges weakly to ν while the upper limit of $\int_X \log^+(d(x, x_0)) F^n(\tilde{\nu})(dx)$ is not greater than $2 \log(M_2)/b$. $\square$

Let h be the supremum of $h_i'(x)$ over all i and all x. Let

$$A = \{x_0, h_1(x_0), \ldots, h_k(x_0)\}$$

Choose two positive constants $b'' < b' < b$ and let $\delta > 1$ be such that

$$b' - (\delta - 1) \log h > b'' \delta \tag{3.4}$$

Let $n_j = \lfloor \delta^j \rfloor$ and

$$Z_j = \{\omega \in \Sigma; \mathrm{diam}(h_{\omega^{-n_j}}(A)) < \exp(-n_j b')\}$$

Similarly, let

$$\tilde{Z}_j = \{\omega \in \Sigma; \mathrm{diam}(h_{\omega^{n_j}}(A)) < \exp(-n_j b')\}$$

Both Z_j and $\tilde{Z}_j$ are disjoint unions of n_j-th level cylinders, hence by (3.1)

$$\mu_-(Z_j) = \mu_+(\tilde{Z}_j)$$

By Lemma 3.2,

$$p_x(\tilde{Z}_j) \geq 1 - \frac{L(A)(1 + \log^+(d(x, x_0)))}{(b - b')^2 n_j}$$

Using Lemma 3.3 we get

$$\mu_+(\tilde{Z}_j) = \int_X p_x(\tilde{Z}_j) d\nu(x) \geq 1 - \frac{L(A)(1 + M(x_0))}{(b - b')^2 n_j}$$

Hence, the series $\sum(1 - \mu_-(Z_j)) = \sum(1 - \mu_+(\tilde{Z}_j))$ is convergent. Denote $Z = \liminf Z_j$. By Borel-Cantelli Lemma

$$\mu_-(Z) = 1$$

As

$$\mathrm{diam}(h_j \circ h_{\omega^n}(A)) \leq h \,\mathrm{diam}(h_{\omega^n}(A)),$$

(3.4) implies that for $\omega \in \bigcap_{j \geq m} Z_j$ and $n > n_m$

$$\mathrm{diam}(h_{\omega^{-n}}(A)) < \exp(-nb'')$$

In particular, for all $\omega \in Z$ the sequence $\mathrm{diam}(h_{\omega^{-n}}(A))$ is summable.

We can estimate

$$d(h_{\omega^{-(n+1)}}(x_0), h_{\omega^{-n}}(x_0)) = d(h_{\omega^{-n}}(h_{\omega_{n+1}}(x_0)), h_{\omega^{-n}}(x_0)) \leq \mathrm{diam}(h_{\omega^{-n}}(A))$$

hence $h_{\omega^{-n}}(x_0)$ form a Cauchy sequence. We proved thus that the limit in (2.3) exists for all $\omega \in Z$, i.e. for a set of full measure μ_-. As stated above, (2.4) follows and we are done with Proposition 3.1. $\square$

Consider the space $X \times \Sigma$ with measure

$$d\bar{\nu}(x, \omega) = p_x(d\omega) d\nu(x)$$

This measure is invariant under the map

$$g(x, \omega) = (h_{\omega_1}(x), \sigma\omega)$$

By Elton [E], Lemma 1, this map is ergodic. An immediate corollary of Elton's Lemma and Proposition 3.1 is the following

Proposition 3.4. *Under assumptions of Proposition 3.1, the map* $(\tilde{\Sigma}, \tilde{\sigma}, \mu)$ *is ergodic.*

Proof. Denote the ergodic average for any $f \in L^1(\tilde{\Sigma}, \mu)$ by $\bar{f}$. To prove the assertion we need to prove that $\bar{f}$ exists and is equal to $\mu(f)$ μ-almost everywhere.

Note first that $(X \times \Sigma, g, \bar{\nu})$ is a factor of $(\Sigma \times \Sigma, S, \mu)$ under the projection (π, Id). Hence, the assertion is true for functions of the form $f(\tau, \omega) = f(\omega)$, i.e. depending only on the future. As the ergodic average doesn't depend on the finite number of initial summands, same is true for functions of the form $f(\tau, \omega) = f(\tau^n, \omega)$ for any n. Those functions are dense in $L^1(\sigma \times \Sigma, \mu)$, hence the assertion follows for all functions by standard density argument. □

We remind that as $(\tilde{\Sigma}, \tilde{\sigma})$ is bijective, the map $(\tilde{\Sigma}, \tilde{\sigma}^{-1}, \mu)$ is also ergodic.

4 Ergodic properties of typical orbits

Let us define the Lyapunov exponent and (minus) entropy of the system:

$$\lambda = \int \sum_{j=1}^{k} p_j(x) \log |h_j'(x)| d\nu(x),$$

$$\eta = \int \sum_{j=1}^{k} p_j(x) \log p_j(x) d\nu(x)$$

By (2.5) and ergodicity of $(\mathbb{R}^d \times \Sigma, g)$, $\lambda < 0$.

For any $z_1, \ldots, z_k \in L^1(\mathbb{R}^d, \nu)$ we denote

$$z = \int_{\mathbb{R}^d} \sum_{j=1}^{k} p_j(x) z_j(x) d\nu(x)$$

and

$$z_{\omega^n}(x) = \sum_{i=1}^{n} z_{\omega_i}(h_{\omega^{i-1}}(x)).$$

The following two lemmas are direct consequences of the Birkhoff Ergodic Theorem; the first by application to the map $(\tilde{\Sigma}, \tilde{\sigma}, \mu)$ and the function $f(\tau, \omega) = z_{\omega_1}(\pi(\tau))$, the second is obtained for the same f but for map $(\tilde{\Sigma}, \tilde{\sigma}^{-1}, \mu)$. Note that

$$\mu(f) = \bar{\nu}(f \circ (\pi^{-1}, \mathrm{Id})) = \int_{\mathbb{R}^d} \sum_{j=1}^{k} z_j(x) p_j(x) d\nu(x) = z$$

Lemma 4.1. *For any $\varepsilon > 0$, μ_+-almost every $\omega \in \Sigma$ and ν-almost all $x \in \mathbb{R}^d$ one can find a constant $K > 0$ such that for all n*

$$K^{-1}\exp(n(1+\varepsilon)z) \leq \exp(z_{\omega^n}(x)) \leq K\exp(n(1-\varepsilon)z).$$

Lemma 4.2. *For any $\varepsilon > 0$ and μ_--almost every $\omega \in \Sigma$ there exists $K > 0$ such that for all n*

$$K^{-1}\exp(n(1+\varepsilon)z) \leq \exp(z_{\omega^{-n}} \circ \pi(\sigma^n\omega)) \leq K\exp(n(1-\varepsilon)z).$$

The following is a standard bounded distortion lemma.

Lemma 4.3. *For every $\varepsilon > 0$ and $K > 0$ there exists $L(K)$ such that if for some $x \in \mathbb{R}^d$ and $\omega \in \Sigma$ and for all $n > 0$*

$$K^{-1}\exp(n(1+\varepsilon)\lambda) \leq h'_{\omega^n}(x) \leq K\exp(n(1-\varepsilon)\lambda) \tag{4.1}$$

then

$$\sum_{m=0}^{n} \operatorname{diam}(h_{\omega^m}(B_{L(K)}(x))) \leq 1$$

Proof. Choose some L. We have

$$\operatorname{diam} h_{\omega^n}(B_L(x)) \leq L \cdot \sup_{y \in B_L(x)} h'_{\omega^n}(y) \tag{4.2}$$

As $\log|h'_i|$ is C^α, for all $y \in B_L(x)$

$$|h'_{\omega^n}(y) - h'_{\omega^n}(x)| \leq c\sum_{i=0}^{n-1}(d(h_{\omega^i}(x), h_{\omega^i}(y)))^\alpha h'_{\omega^n}(x) \tag{4.3}$$

Denote

$$d_0 = 2L$$

and

$$d_m = 2LK\exp(m(1-\varepsilon)\lambda)(1 + c\sum_{j=0}^{m-1} d_j^\alpha)$$

By (4.2) and (4.3) we have

$$d_m \geq \operatorname{diam}(h_{\omega^m}(B_L(x)))$$

Consider the series

$$M(L) = \sum_{i=0}^{\infty}(L(1+c)K\exp(i(1-\varepsilon)\lambda))^\alpha$$

For L small enough (depending on K) $M(L)$ is smaller than 1. For such L for all n

$$\sum_{m=0}^{n} d_m^\alpha \le M(L)$$

It implies that

$$\sum_{m=0}^{n} \mathrm{diam}(h_{\omega^m}(B_{L(K)}(x))) \le 2LK(1 + cM(L)) \sum_{m=0}^{n} \exp(m(1-\varepsilon)\lambda) \le$$

$$\le \frac{2LK(1 + cM(L))}{e^{-\lambda} - 1}$$

and the right hand side is arbitrarily small (independent of n) for L sufficiently small. ☐

Corollary 4.4. *Under assumptions of Lemma 4.3, for any C^α functions z_i the difference $z_{\omega^n}(x) - z_{\omega^n}(y)$ is uniformly bounded for all $y \in B_{L(K)}(x)$.*

The following bounded distortion lemma implies that both (2.6) and(2.7) are preserved under iterations (up to rescaling).

Lemma 4.5. *Fix $x \in \mathbb{R}^d$, $R > 0$ and ω^n. Assume that for some $K > 0$ and $\varepsilon > 0$ and for all $m \le n$*

$$K^{-1} \exp(m(1+\varepsilon)\lambda) \le h'_{(\sigma^{n-m}\omega)^m}(h_{\omega^m}(x)) \le K \exp(m(1-\varepsilon)\lambda)$$

Then there exists $l(K,R)$ such that for all $r \le l(K) \exp(n(1+2\varepsilon)\lambda)$ we have

$$\sum_{m=0}^{n} \mathrm{diam}(h^{-1}_{(\sigma^{n-m}\omega)^m}(B_r(h_{\omega^n}(x)))) \le R$$

Proof. Choose some l and let

$$r = l \exp(n(1+2\varepsilon)\lambda)$$

Like in the proof of Lemma 4.3, we can estimate

$$\mathrm{diam}(h^{-1}_{(\sigma^{n-m}\omega)^m}(B_r(h_{\omega^n}(x)))) \le d_m$$

where

$$d_0 = 2r$$

and

$$d_m = 2rK \exp(-m(1+\varepsilon)\lambda)(1 + c\sum_{j=0}^{m-1} d_j^\alpha)$$

Consider the series

$$M(l) = \sum_{j=0}^{\infty} (2lK(1+c)\exp(j\varepsilon\lambda))^{\alpha}$$

For l small enough (depending on K),

$$M(l) < 1$$

and

$$\sum_{m=0}^{n-1} d_m^{\alpha} < M(l)$$

Hence,

$$\sum_{m=0}^{n} \operatorname{diam}(h^{-1}_{(\sigma^{n-m}\omega)^m}(B_r(h_{\omega^n}(x)))) \le 2lK(1+M(l))e^{n(1+2\varepsilon)\lambda} \sum_{m=0}^{n} e^{-m(1+\varepsilon)\lambda} \le$$

$$\le \frac{2e^{-\lambda} lK(1+c)}{e^{-(1+\varepsilon)\lambda} - 1}$$

and for l small enough the right hand side is smaller than any constant, independently of n. □

Corollary 4.6. *Under assumptions of Lemma 4.5, for any C^{α} functions z_i the difference $z_{\omega^n}(x) - z_{\omega^n}(y)$ is uniformly bounded for all $y \in h^{-1}_{\omega^n}(B_r(x))$.*

Before we begin the proof of Theorem 2.1, let us show an example of iterated function system satisfying the regular open set condition. The maps are not similitudes (it is impossible to have OSC for similitudes that are not all contracting) but their derivatives are constant in U. Outside U it may well happen that all the maps are expanding at some point, hence (2.5) is satisfied only on U. The system is thus not necessarily contracting-on-average. However, Theorem 2.1 still holds.

Example 4.7. Let $B_n = (10^n, 3 \cdot 10^n)$ for $n \ge 0$. Let

$$h_1(x) = \frac{1}{20}x + 15 \cdot 10^{n-2}$$

for $x \in B_n$, $n > 0$,

$$h_1(x) = \frac{1}{20}x + 1$$

for $x \le 3$,

$$h_2(x) = 5x + 5 \cdot 10^n$$

for $x \in B_n$, $n > 0$,

$$h_2(x) = 5x + 5$$

for $x \leq 3$. We extend maps h_1, h_2 to be homeomorphisms of $\mathbb{R}$. The system satisfies the regular open set condition for $U = \bigcup B_n$ (regularity is satisfied because all the components of U have diameters bounded away from 0). The system is contracting-on-average (on U only) for

$$p_1(x) = p_1 = 1 - p_2 > \frac{\log(50/7)}{\log(500/17)}$$

For negative x the contracting map h_1 is stronger than the dilatating map h_2 (in addition it is applied more often for typical ω) and the fixed point for h_2 is smaller than the fixed point for h_1. Similarly, for positive x both h_1 and h_2 are equally strong (former moves B_i into B_{i-1}, latter into B_{i+1}) but the former happens more often for typical ω. Hence, wherever we start, the typical trajectory lands inside B_0 after finite number of iterations. The invariant measure (if it exists) is thus supported on U, hence independent from the way we define h_1 and h_2 outside U.

Note that we may define h_1 and h_2 in such a way that the system satisfies (2.5) on whole $\mathbb{R}$. Hence, the invariant measure indeed exists and is unique. Its Hausdorff dimension equals

$$\dim_H(\nu) = \frac{p_1 \log p_1 + p_2 \log p_2}{-2p_1 \log 2 - (p_1 - p_2) \log 5}. \tag{4.4}$$

More detailed study shows that the necessary and sufficient condition for existence of invariant measure is $p_1 > 1/2$, even though the iterated function system doesn't satisfy (2.5) in this case. The invariant measure is always unique and its Hausdorff dimension is still given by (4.4).

5 Proof of Theorem 2.1

5.1 Upper bound

Fix small $\varepsilon > 0$. Choose some big $K > 0$ and $n > 0$.

Let $\Sigma(n, K, \varepsilon)(x)$ be the set of sequences $\omega \in \Sigma$ for which

$$K^{-1} \exp(n(1+\varepsilon)\eta) \leq p_{\omega^n}(x) \leq K \exp(n(1-\varepsilon)\eta)$$

and

$$K^{-1} \exp(n(1+\varepsilon)\lambda) \leq |h'_{\omega^n}(x)| \leq K \exp(n(1-\varepsilon)\lambda). \tag{5.1}$$

By Lemma 4.1, $\mu(\Sigma(n,K,\varepsilon)(x))$ is (for sufficiently big K, independently of n) greater than $(1-\varepsilon)$ for all x except some set $A(n,K,\varepsilon)$ of ν-measure smaller than ε. In addition, $\nu([-K,K]^d) > 1-\varepsilon$ for K big enough.

Let us assume K is sufficiently big that all the above is satisfied. Note that (as $\sum_{\omega^n} p_{\omega^n}(x) = 1$) ω^n can take at most $K\exp(-n(1+\varepsilon)\eta)$ values inside each $\Sigma(n,K,\varepsilon)(x)$. Given ω^n, let $B(\omega^n,K,\varepsilon)$ denote the set of x for which $\omega^n = \tau^n$ for some $\tau \in \Sigma(n,K,\varepsilon)(x)$. Denote

$$\nu_n = \sum_{\omega^n} h_{\omega^n *}(p_{\omega^n} \cdot \chi_{B(\omega^n,K,\varepsilon)\cap[-K,K]^d} \cdot \nu)$$

As

$$\nu = \sum h_{\omega^n *}(p_{\omega^n}\nu),$$

we have $\nu_n \le \nu$.

At the same time, for any $x \in [-K,K]^d \setminus A(n,K,\varepsilon)$ we have

$$p_x(\omega; x \in B(\omega^n,K,\varepsilon)) \ge 1-\varepsilon$$

Hence,

$$|\nu_n| \ge \nu([-K,K]^d \setminus A(n,K,\varepsilon)) \cdot \inf_{x\in[-K,K]^d\setminus A(n,K,\varepsilon)} p_x(\omega; x\in B(\omega^n,K,\varepsilon)) \ge 1-3\varepsilon.$$

Let $(E_i^{\varepsilon,n})$ be a family of sets of diameter $L(K)$, covering $[-K,K]^d \setminus A(n,K,\varepsilon)$. This family may be chosen to have at most $(2dK/L(K))^d$ elements. Let us choose a point $x_i^{\varepsilon,n}$ inside each $E_i^{\varepsilon,n} \setminus A(n,K,\varepsilon)$ whenever this set is nonempty. Let $(F_j^{\varepsilon,n})$ be a family of sets of the form $h_{\omega^n}(E_i^{\varepsilon,n})$ for $\omega \in \Sigma(n,K,\varepsilon)(x_i^{\varepsilon,n})$. This family has at most $2^d d^d K^{d+1} L(K)^{-d} \exp(-n(1+\varepsilon)\eta)$ elements. By Lemma 4.3, Corollary 4.4 and (5.1) each of sets $F_j^{\varepsilon,n}$ has diameter not greater than $K_0 L(K) K \exp(n(1-\varepsilon)\lambda)$.

As the support of ν_n is contained in $\bigcup_{\omega^n} h_{\omega^n}([-K,K]^d \cap B(\omega^n,K,\varepsilon))$, it is also contained in $\bigcup F_j^{\varepsilon,n}$. The sum

$$\sum(\operatorname{diam} F_j^{\varepsilon,n})^{s(1+3\varepsilon)} \le 2^d d^d K_0^{s(1+3\varepsilon)} K^{d+1+s(1+3\varepsilon)} L(K)^{s(1+3\varepsilon)-d} \exp(n\eta\varepsilon(1-3\varepsilon))$$

is arbitrarily small for big n. Let us choose n for which this sum is smaller than ε and denote $Z_\varepsilon = \bigcup F_j^{\varepsilon,n}$.

We can now repeat the procedure for a smaller $\varepsilon_m = 2^{-m}\varepsilon$. The set $Z = \limsup Z_{\varepsilon_m}$ has full ν-measure and we have constructed a family of its covers $(G^M) = \bigcup_{m>M} \bigcup F_j^{\varepsilon_m}$ such that

$$\sum_{G_l^M \in G^M} (\operatorname{diam} G_l^M)^{s(1+3\varepsilon)} \le 1.$$

Hence,

$$\dim_H(Z) \leq s(1+3\varepsilon)$$

and ε can be arbitrarily small. $\square$

5.2 Lower bound

We claim that if we can find a family of sets $\Sigma_K \in \Sigma$ such that $\mu(\Sigma_K) \nearrow 1$ and the measures $\nu_K = \pi_*(\mu_{|\Sigma_K})$ have lower Hausdorff dimension not smaller than s_0 then the lower Hausdorff dimension of ν will be not smaller than s_0.

Suppose it is not true, i.e. there exists a set $X \subset \mathbb{R}^d$ such that $\nu(X) > 0$ and $\dim_H(X) < s_0$. For some K, $\nu(X) + \mu(\Sigma_K) > 1$. Hence, the set $Y = \pi^{-1}(X) \cap \Sigma_K$ has positive measure μ. It follows that $\pi(Y)$ has positive measure ν_K and its Hausdorff dimension is smaller than s_0 – a contradiction.

Thus, we need only to prove the existence of such families of sets Σ_K (or equivalently, of measures ν_K) of lower Hausdorff dimension arbitrarily close to s.

Fix $\varepsilon > 0, K > 0$. Let Σ_K be the set of $\omega \in \Sigma$ for which for all n

$$K^{-1} \exp(-n(1-\varepsilon)\lambda) \leq |(h_{\omega^{-n}}^{-1})' \circ \pi(\omega)| \leq K \exp(-n(1+\varepsilon)\lambda) \tag{5.2}$$

and

$$K^{-1} \exp(n(1+\varepsilon)\eta) \leq p_{\omega^{-n}} \circ \pi(\sigma^n \omega) \leq K \exp(n(1-\varepsilon)\eta). \tag{5.3}$$

For K big enough $\mu_-(\Sigma_K)$ is arbitrarily close to 1 by Lemma 4.2.

Let $x = \pi(\omega)$ for some $\omega \in \Sigma_K$ and

$$r_n = K^{-1} l(K, R) \exp(n(1+2\varepsilon)\lambda),$$

where $R = R_1$ in SOSC case or $R = R_2$ in ROSC case. The ball $B_{r_n}(x)$ may in general intersect many of the sets $h_{\tau^n}(U)$. Let T be the set of all τ^n for which $\nu_K(B_{r_n}(x) \cap h_{\tau^n}(U)) > 0$.

By Lemma 4.5, Corollary 4.6 and (5.3) we have

$$p_{\tau^n}(h_{\tau^n}^{-1}(y)) \leq K_0 K \exp(n(1-\varepsilon)\eta)$$

for ν_K-almost all $y \in B_{r_n}(x) \cap h_{\tau^n}(U)$. Hence,

$$\nu_K(h_{\tau^n}(U) \cap B_{r_n}(x)) \leq \nu(h_{\tau^n}(U) \cap B_{r_n}(x)) \leq K \exp(n(1-\varepsilon)\eta)$$

for all $\tau^n \in T$.

We claim that T has uniformly bounded (independently of n) number of elements. Assume first that the strong open set condition is satisfied. Choose any point $y \in U$ and any $\tau^n \neq \omega^n$. Let $m = \min\{j \geq 1; \omega_j \neq \tau_j\} - 1$. By SOSC,

$$d(h_{(\sigma^m\omega)^{-(n-m)}}(h^{-1}_{\omega^{-n}}(x)), h_{(\sigma^m\tau)^{-(n-m)}}(y)) > R_1$$

Hence, by Lemma 4.5

$$h_{\tau^{-n}}(y) \notin B_{r_n}(x)$$

and thus $\tau^{-n} \notin T$. Hence, T has just one element ω^{-n}.

Consider now the regular open set condition situation. Choose $\tau^n \in T$ and ν_K-typical $y \in h_{\tau^n}(U) \cap B_{r_n}(x)$. By Lemma 4.5 and Corollary 4.6,

$$B_{K_0 r}(h^{-1}_{\tau^n}(y)) \subset h^{-1}_{\tau^n}(B_{r_n}(x)) \subset B_r(h^{-1}_{\tau^n}(y))$$

for some $r < R_2$. By ROSC

$$\mathrm{vol}(U \cap h^{-1}_{\tau^n}(B_{r_n}(x))) \geq R_3 K_0^d \,\mathrm{vol}(h^{-1}_{\tau^n}(B_{r_n}(x)))$$

and (by Corollary 4.6 again)

$$\mathrm{vol}(h_{\tau^n}(U) \cap B_{r_n}(x)) \geq K_1 \,\mathrm{vol}(B_{r_n}(x))$$

As $h_{\tau^n}(U)$ are pairwise disjoint, T can have at most K_1^{-1} elements.

Hence, if either SOSC or ROSC are satisfied,

$$\nu_K(B_{r_n}(x)) \leq K_2 \exp(n(1-\varepsilon)\eta)$$

with K_2 depending on K but not on x or n. Hence,

$$\limsup_{r\to 0} \frac{\nu_K(B_r(x))}{(2r)^{s(1-4\varepsilon)}} = 0 \tag{5.4}$$

We only assumed above (5.2) and (5.3), so (5.4) is satisfied ν_K-almost everywhere, which (by Frostman Lemma) implies that the lower Hausdorff dimension of ν_K is not smaller than $s(1-4\varepsilon)$. $\square$

References

[BDEG] M. F. Barnsley, S. G. Demko, J. H. Elton, J. S. Geronimo, Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities, *Ann. Inst. H. Poincare Prob. Stat.* 24 (1988), 367-394.

[E] J. H. Elton, An ergodic theorem for iterated maps, *Erg. Th. and Dyn. Sys.* 7 (1987), 481-488.

[F] K. Falconer, *Techniques in Fractal Geometry*, John Wiley and Sons, Chichester 1997.

[FST] A. H. Fan, K. Simon, H. R. Toth, Contracting on average iterated function systems with expanding fixpoint, preprint.

[JO] A. Johansson, A. Öberg, Exact dimension of Cantor type measures generated by iterated function systems, preprint.

[LN] Ka-Sing Lau, Sze-Man Ngai, Multifractal Measures and a Weak Separation Condition, *Adv. in Math.* 141 (1999), 45-96.

[MS] J. Myjak, T. Szarek, On Hausdorff dimension of invariant measures arising from non-contractive iterated function systems, *Ann. Mat. Pura Appl. (4)* 181 (2002), 223-237.

[NSB] M. Nicol, N. Sidorov, D. Broomhead, On the Fine Structure of Stationary Measures in Systems which Contract-on-Average, *J. Theoretical Prob.* 15 (2002), 715-730.

[R] M. Rams, Packing dimension estimation for exceptional parameters, *Israel J. of Math.* 130 (2002), 125-144.

[S] T. Szarek, The dimension of self-similar measures, *Bull. Polish Acad. Sci. Math.* 48 (2000), 293-302.

[Wa] P. Walczak, Hausdorff dimension of Markov invariant sets, *J. Math. Soc. Japan* 48 (1996), 125-133.

[We] I. Werner, Coding map for a contractive Markov system, preprint 2005.