TOPOLOGICAL AND ERGODIC ASPECTS OF PARTIALLY HYPERBOLIC DIFFEOMORPHISMS AND NONHYPERBOLIC STEP SKEW-PRODUCTS

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Abstract. We review some ergodic and topological aspects of robustly transitive partially hyperbolic diffeomorphisms with one-dimensional center direction. We also discuss step skew-product maps whose fiber maps are defined on the circle which model such dynamics. These dynamics are genuinely nonhyperbolic and exhibit simultaneously ergodic measures with positive, negative, and zero exponents as well as intermingled horseshoes having different types of hyperbolicity. We discuss some recent advances concerning the topology of the space of invariant measures and properties of the spectrum of Lyapunov exponents.

Contents

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1. Introduction

The paper [27] opens with the following general question:

To what extent is the behaviour of a generic dynamical system hyperbolic?

and observes that a substantial number of problems in Dynamical Systems Theory are just reformulations of this question. In the beginning of the theory in the late 60's the paper by Abraham-Smale [2] showed that nonhyperbolic systems are not dense in the space of dynamical systems. Indeed there are open sets in the space of diffeomorphisms consisting of nonhyperbolic ones. These findings showed the necessity of weaker notions of hyperbolicity and lead to the ones of nonuniform hyperbolicity due to Pesin [40] and partial hyperbolicity [29], among others.

Consider a differentiable dynamical system $F\colon M\to M$ defined on a closed and compact manifold M. Recall that a closed and F-invariant *transitive* (existence of a point in Γ whose orbit is dense in Γ) set Γ is *hyperbolic* if there exists a dF-invariant splitting $E^{\rm s}\oplus E^{\rm u}=T_\Gamma M$ of the tangent bundle and constants $C>0$ and $\lambda>1$ such that for every $x\in\Gamma$ for every $n\ge 0$ we have

$$\|dF^n_x(v)\|\le C\lambda^n\|v\| \text{ for all } v\in E^{\rm s}_x \text{ and } \|dF^{-n}_x(w)\|\le C\lambda^n\|w\| \text{ for all } w\in E^{\rm u}_x.$$

Any variation of hyperbolicity is based on the notion of Lyapunov exponents. Recall that a point $x\in M$ is called *Lyapunov regular* if there exist a positive integer $s(x)$, numbers $\chi_1(x)<\ldots<\chi_{s(x)}(x)$, and a dF-invariant splitting $T_xM=\oplus_{i=1}^{s(x)}E^i_x$ of the tangent space at x such that for all $i=1,\ldots,s(x)$ and $v\in E^i_x\setminus\{0\}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log\|dF^n_x(v)\|=\chi_i(x) \tag{1.1}$$

and these numbers $\chi_1(x)<\ldots<\chi_{s(x)}(x)$ are called the *Lyapunov exponents* of x.

By the Oseledets multiplicative ergodic theorem (see [38]), given an F-invariant ergodic probability measure μ, the set of Lyapunov regular points has full measure and $s(\cdot)=s(\mu)$ and $\chi_i(\cdot)=\chi_i(\mu)$, $i=1,\ldots,s(\mu)$, are constant μ-almost everywhere and the latter numbers are called the *Lyapunov exponents* of μ. If there is ℓ such that $\chi_\ell(\mu)=0$ then the measure μ is called μ *nonhyperbolic*, otherwise it is called *hyperbolic* and then we will refer to $s(\mu)$ as its (*stable*) *index*. When talking about nonhyperbolic measures, we always assume ergodicity and hence exclude nontrivial convex combination of ergodic measures. The easiest examples of ergodic measures are the ones supported on a periodic orbit (we will call such measures simply *periodic*). The *index* of a hyperbolic periodic orbit is the index of the (unique) invariant measure supported on its orbit. Below we will call a measure *nontrivial* if its support is uncountable, hence such a measure cannot be periodic. The easiest examples of a nonhyperbolic measure is a periodic one supported on the orbit of a nonhyperbolic periodic point. A hyperbolic periodic orbit which has both positive and negative exponents we will also call a *saddle*.

The discussion above leads to the following question:

To what extent does Ergodic Theory detect nonhyperbolic dynamics?

that is just a reformulation of the opening question above (though note that the term "nonhyperbolic dynamics" is vague and differently used in different contexts). The answer to this question is negative as there are examples of nonhyperbolic systems (in the sense that the nonwandering set is not hyperbolic) for which all ergodic measures are hyperbolic with Lyapunov exponents uniformly bounded away from zero [3, 14]. Note that these examples are fragile in the sense that they can be destroyed by perturbations. On the other hand, by the Kupka-Smale genericity theorem, see for instance [39, Chapter 3], generically[1] periodic points are all hyperbolic. Hence, considering nonhyperbolic periodic measures, one can get systems with nonhyperbolic *ergodic* measures only densely in the complement of the hyperbolic ones. Thus, to go beyond dense subsets, one needs to investigate nonhyperbolic measures which are not periodic. This was first done in [27] where it is introduced the method of periodic approximations to construct nontrivial ergodic measures as weak$*$ limits of periodic ones.

In dimension strictly bigger than two one needs to take into account that *a priori* different types of hyperbolicity may coexist together with nonhyperbolicity. Indeed one may have hyperbolic periodic orbits of different *stable index* (dimension of the stable bundle) which robustly are part of the same *transitive set* (existence of a dense orbit), this leads to intermingled types of hyperbolicity in the same transitive set. For instance, this is exactly what happens in the dynamics analysed in [27].

Having this in mind we rephrase the above question:

To what extent does Ergodic Theory distinguishes the different types of hyperbolicity in nonhyperbolic dynamics?

In what follows we will restrict to systems which are transitive in the whole ambience (this prevents the existence of attractors and repellers). In this setting, the set of ergodic measures $\mathcal{M}_{\mathrm{erg}}$ will split into several disjoint components, which will be analyzed separately (see (2.1)).

2. Partially hyperbolic dynamics

At the present state of the art, to advance in answering the above questions, we need to assume more structure on the dynamics. We have to require the existence of a globally defined splitting of the tangent space into continuously varying invariant subbundles (a *dominated splitting*) which as a consequence incorporates the subspaces of the Oseledets splitting. In more specific terms, we will require that the dynamical system is partially hyperbolic having three such bundles $TM = E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$, where E^{ss} is uniformly contracting and E^{uu} is uniformly expanding. Hence, zero Lyapunov exponents are automatically associated with the *central bundle* E^{c}. The exponent associated with E^{c} we will simply call *central exponent*. We denote by $\mathbf{PH}^1(M)$ the set of C^1 diffeomorphisms defined on a compact closed manifold M having a partially hyperbolic splitting as above with three nontrivial directions, the central one having dimension one.[2]

In what follows we will consider $F\colon M \to M$ a C^1 diffeomorphism of a Riemannian manifold which is partially hyperbolic and transitive. We will assume that the dynamics is *robustly* transitive, *robustly* nonhyperbolic, and there is some F-invariant compact closed curve $\gamma = F(\gamma)$. The latter property turns out to be also robust, by normal hyperbolicity. We denote this open set by $\mathbf{RTPH}^1(M)$ (here we consider the uniform topology in the space of C^1-diffeomorphisms). Note that this implies that this curve γ is tangent to E^{c} and we refer to it as a *compact*

[1] A *generic* property is a property satisfied on a residual subset, that is, a set which contains a countable intersection of open and dense subsets.

[2] In what follows, to simplify, we will assume that the splitting is defined on the whole ambient space, similar approach can be done when the splitting is only locally defined.

central leaf. Note that the robust nonhyperbolicity hypothesis excludes pathological cases as the one of diffeomorphisms on $\mathbb{T}^3$ being a direct product of a Anosov diffeomorphism on $\mathbb{T}^2$ and a irrational rotation. Note that for $F \in \mathbf{RTPH}^1(M)$ the set of ergodic measures $\mathcal{M}_{\rm erg}$ splits into three disjoint components

$$\mathcal{M}_{\rm erg} = \mathcal{M}_{{\rm erg},<0} \cup \mathcal{M}_{{\rm erg},0} \cup \mathcal{M}_{{\rm erg},>0}, \tag{2.1}$$

where the measures in $\mathcal{M}_{{\rm erg},0}$ are nonhyperbolic and the ones $\mathcal{M}_{{\rm erg},<0}$ and $\mathcal{M}_{{\rm erg},>0}$ are hyperbolic. Further, the measures in $\mathcal{M}_{{\rm erg},<0}$ have $\dim E^{\rm ss}+1$ negative and $\dim E^{\rm uu}$ positive exponents and the measures in $\mathcal{M}_{{\rm erg},>0}$ have $\dim E^{\rm uu}+1$ positive and $\dim E^{\rm ss}$ negative exponents.

2.1. **Hyperbolic and nonhyperbolic measures.** First observe that C^1-open and -densely in the set $\mathbf{RTPH}^1(M)$, there exist hyperbolic periodic points which are contracting in the central direction and hyperbolic periodic points which are expanding in the central direction. Besides that there are horseshoes which are contracting and horseshoes which are expanding in the central direction, respectively. Hence there are also hyperbolic ergodic measures with positive entropy in the sets $\mathcal{M}_{{\rm erg},<0}$ and $\mathcal{M}_{{\rm erg},>0}$. Indeed, in this setting the existence of a hyperbolic ergodic measure with positive entropy implies the existence of horseshoes and hence of hyperbolic periodic orbits with corresponding type by Katok's horseshoe construction (see [31, 23]).

The existence of nonhyperbolic measures is a bit more subtle. Indeed, densely in $\mathbf{RTPH}^1(M)$ there are diffeomorphisms with nonhyperbolic periodic orbits and hence with trivial nonhyperbolic measure (see [36])[3]. Since generic diffeomorphisms have hyperbolic periodic orbits, one can obtain *at most* a dense subset in $\mathbf{RTPH}^1(M)$ with trivial nonhyperbolic measures. Hence, to get larger sets of diffeomorphisms with nonhyperbolic measures, it is necessary to investigate the occurrence of nontrivial nonhyperbolic measures. In [27] the authors introduce the method of periodic approximations that produces a nontrivial nonhyperbolic (ergodic) measure which is a weak$*$ limit of hyperbolic periodic measures and apply it to some specific step skew-product examples. This method builds on the existence of controlled transitions between saddles of different indices. Using it, [21, 9, 15] obtained a C^1-generic set of C^1-diffeomorphisms with nonhyperbolic nontrivial (ergodic) measures (see also the variant [11]). Using this method, [33] provides some specific open sets of robustly nonhyperbolic diffeomorphisms defined on $\mathbb{T}^3$ with such measures. Following a different approach, using the so-called *flip-flop configuration* which relies on the concept of a blender also explained below, [4] prove that open and densely in $\mathbf{RTPH}^1(M)$ there are diffeomorphisms with nonhyperbolic measures with positive entropy. Indeed, it is shown that there is a compact invariant set with positive topological entropy consisting of points whose central Lyapunov is zero and hence we can apply the variational principle in [49] to get

[3]This is a consequence of the so-called ergodic closing lemma in [36]. A recurrent point of a diffeomorphism is called *well-closable* if its orbit can be closed by a small C^1-perturbation in such a way that the resulting periodic point shadows the original orbit along the periodic point's entire orbit. The ergodic closing lemma claims that almost every point of any invariant measure is well-closable and implies that when the nonwandering set is nonhyperbolic then nonhyperbolic period points can be created by perturbation. Note that the ergodic closing lemma also implies that C^1-generically in $\mathbf{RTPH}^1(M)$ the hyperbolic periodic points having negative (positive, respectively) central exponent are dense in M.

such measures. We remark that the method of periodic approximations can only lead to measures with zero entropy, see [32].

Summarizing, open and densely in $\mathbf{RTPH}^1(M)$ each of the components in (2.1) is nonempty and contains measures with positive entropy. Therefore, a natural question is what type of behavior (negative, zero, or positive exponent) predominates? In our context, it is natural to quantify this in terms of entropy. Here we have two ways of doing so. First, given an central Lyapunov exponent α in the possible spectrum of all exponents, determine the maximal entropy of ergodic measures with that exponent:

$$\sup\{\mu \in \mathcal{M}_{\rm erg} \colon h_\mu(F) = \alpha\}. \tag{2.2}$$

Or, given an exponent, determine the topological entropy of the set of Lyapunov regular points with that exponent:

$$h_{\rm top}(F, \mathcal{L}(\alpha)), \quad \text{where} \quad \mathcal{L}(\alpha) \stackrel{\rm def}{=} \{x \colon \chi_{\rm c}(x) = \alpha\}. \tag{2.3}$$

The former is related to restricted variational principles and implicitly determines the latter when performing a multifractal analysis (see Theorem 3.5). As there is an intimate relation between ergodic measures and the corresponding generic points, in a reasonable context we expect that both above introduced quantities coincide, see Section 3.3 for a full discussion.

2.2. **Invariant foliations.** Let us finally describe quickly geometrical features of the diffeomorphisms in $\mathbf{RTPH}^1(M)$ which are also essential to study the above discussed ergodic properties and the level sets (2.2) and (2.3). In particular, they motivate the model we will study in Section 3. The existence of the partially hyperbolic splitting $TM = E^{\rm ss} \oplus E^{\rm c} \oplus E^{\rm uu}$ implies that there are invariant foliations $\mathcal{F}^{\rm ss}$ and $\mathcal{F}^{\rm uu}$ tangent to $E^{\rm ss}$ and $E^{\rm uu}$ called the *strong stable* and *strong unstable foliations,* respectively, see [29]. By [7, 42], because by assumption $E^{\rm c}$ is one-dimensional and there is a compact central leaf, there is a C^1-open and -dense subset $\mathbf{ORTPH}^1(M)$ of $\mathbf{RTPH}^1(M)$ consisting of diffeomorphisms for which both foliations are *minimal* (i.e., every leaf of the foliation is dense in the whole space). A special case occurs when there is a center foliation (tangent to $E^{\rm c}$) whose leaves are *all* compact. Such systems are topologically of skew-product type. An important example which still inspires many open questions is the example in [44].[4] We will further discuss this topic in Section 4.1.

The above geometric features are in the realm of the large family of step skew-products we introduce and discuss in Section 3. On the other hand, the properties of this family seem to capture the essential dynamical properties of diffeomorphisms in $\mathbf{RTPH}^1(M)$ that allow to study the level sets above and several ergodic properties as well as to analyze the topology of the space of invariant measures. The latter we further discuss in the following subsection.

2.3. **Topology of the space of measures: Framework.** Let us now have a look at the topology of the space of Borel probability measures $\mathcal{M}$ invariant under a continuous map of a compact metric space. Equipped with the weak$*$ topology it is a compact metrizable topological space [49, Chapter 6.1]. We denote by $\mathcal{M}_{\rm erg} \subset \mathcal{M}$

[4]For example, [25] asks if a general ergodic volume preserving diffeomorphism sufficiently C^1 close to a partially hyperbolic linear automorphism $L \colon \mathbb{T}^3 \to \mathbb{T}^3$ given by $L(x, y, z) = (A(x, y), z)$, A a linear Anosov diffeomorphism, has dense ergodic measures. This would include the systems in [44].

the subset of ergodic measures. Recall that $\mathcal{M}$ is a nonempty Choquet simplex (see [49, Chapter 6.2]). In particular, it is convex and compact. The extreme points of $\mathcal{M}$ are the ergodic measures.

In general, when studying the topology of $\mathcal{M}$, there are many properties that can be of interest such as density and entropy density of ergodic measures (in $\mathcal{M}$) as well as connectedness of the set of ergodic measures. The density of ergodic measures has immediately very strong consequences. Indeed, then $\mathcal{M}$ is either a singleton (the map is uniquely ergodic) or a nontrivial Choquet simplex in which extreme points are dense, and one calls $\mathcal{M}$ then a *Poulsen simplex*. Poulsen [41] was the first who constructed an example of a space with such properties; by [35] any two metrizable nontrivial simplices with dense extreme points are equivalent up to affine homeomorphisms and hence one can consider $\mathcal{M}$ as *the* Poulsen simplex. Note that, for example, $\mathcal{M}_{\rm erg}$ is then arcwise connected (see [35, 4. in Section 3]). To conclude, recall that one says that ergodic measures are *entropy dense* if for any $\mu \in \mathcal{M}$ and any $\varepsilon > 0$, any neighborhood of μ contains an ergodic measure ν such that $h_\nu(F) > h_\mu(F) - \varepsilon$.

Density of ergodic (even periodic) measures was first shown in [45, 46] under the assumption that it the map satisfies the periodic specification property (in [47] connectedness was concluded for shift spaces, which is however an immediate consequence of density by [35] as explained above). Recall that for smooth dynamical systems, periodic specification holds for any basic set of an axiom A diffeomorphism (see [12]). In a more general context, in [1] it was shown that for $\Lambda \subset M$ being an isolated nontrivial transitive set of a C^1-generic diffeomorphism, periodic measures are dense (and also have many further properties, see [24]). Below we will give more details on two more recent results [28, 10].

All known results on properties such as (entropy) density and connectedness involve approximations of hyperbolic ergodic measures by either periodic measures or Markov ergodic measures supported on horseshoes. We will see that this can be achieved also in some nonhyperbolic context, in particular when the set of ergodic measures contains measures of different index as well as nonhyperbolic measures.

Let us observe that connectedness and (entropy) density of ergodic measures is not always guaranteed. Note that [24] provides a number of counterexamples in shift spaces, though, in the following we would like to focus on partially hyperbolic systems. For that, we point out the porcupine-like examples of compact invariant sets of partially hyperbolic transitive C^1 diffeomorphism studied in [22, 34, 17, 18] which have a spectrum of central Lyapunov exponents with at least two disjoint components, and at least two connected components of ergodic measures. In particular, ergodic measures are not dense.

2.4. **Topology of the space of measures: Intersection and homoclinic classes.** To understand the topology of the space $\mathcal{M}$, it turns out useful to consider so-called intersection and homoclinic classes of hyperbolic periodic points. To state the results more precisely, let us briefly define them. We say that two hyperbolic periodic points of the same index are *homoclinically related* if the invariant sets of their orbits meet cyclically (note that in our partially hyperbolic setting with one dimensional center transversality is not involved, see Section 5 for details). Note that this is an equivalence relation on the set of hyperbolic periodic points and we call *intersection classes* the equivalence classes for the homoclinic relation. Given a hyperbolic periodic point P, we denote by $\mathrm{Int}(P)$ its intersection class.

The *intersection class* of a hyperbolic periodic point was first considered in [37] and called *h-class* to obtain the so-called spectral decomposition of axiom A diffeomorphisms. With this terminology, a *homoclinic class* (called *h-closure* in [37]) is the closure of the intersection class it contains. Note that a homoclinic class is always a transitive invariant set. Moreover, a homoclinic class of a hyperbolic periodic point may contain periodic points which are not homoclinically related with it and, consequently, may contain several distinct intersection classes. Indeed, a homoclinic class which is not hyperbolic may support ergodic measures of different indices and/or nonhyperbolic ergodic measures ([27, 21, 4]). Moreover, there are examples of homoclinic classes which are not hyperbolic whose ergodic measures are all hyperbolic but that simultaneously support ergodic measures with negative and positive central Lyapunov exponent, see [22, 34].

We call a (not necessarily ergodic) measure $\mu \in \mathcal{M}$ *hyperbolic with negative central Lyapunov exponent* if μ-almost every point has a negative central Lyapunov exponent and denote by $\mathcal{M}_{<0}$ the set of all such measures. Similarly, we define a measure to be *hyperbolic with positive central Lyapunov exponent* and define $\mathcal{M}_{>0}$. By *weak$*$ and in entropy convergence* we mean that the sequence of measures converges in the weak$*$ topology and their entropies also converge to the entropy of the limit measure.

By [6, Theorem E], the C^1-open and -dense subset $\mathbf{ORTPH}^1(M)$ of $\mathbf{RTPH}^1(M)$ above can be chosen to consist of diffeomorphisms such that any pair of saddles with the same index are homoclinically related, that is, share the same intersection class. Thus, for $F \in \mathbf{ORTPH}^1(M)$ there are precisely two intersection classes which we will denote by $\mathrm{Int}_{<0}$ and $\mathrm{Int}_{>0}$, where

$$\mathrm{Int}_{<0} \stackrel{\mathrm{def}}{=} \{P \in M \colon P \text{ hyperbolic periodic point}, \chi_{\mathrm{c}}(P) < 0\}$$

and $\mathrm{Int}_{>0}$ is analogously defined. Moreover, each of those two sets is dense in M. Given a hyperbolic (not necessarily ergodic nor periodic) measure $\mu \in \mathcal{M}$, we define its *intersection class*, denoted by $\mathrm{Int}(\mu)$, as the intersection class of hyperbolic periodic orbits such that μ is accumulated by periodic measures from that class. Hence, for $F \in \mathbf{ORTPH}^1(M)$ either $\mathrm{Int}(\mu) = \mathrm{Int}_{<0}$ or $= \mathrm{Int}_{>0}$. By [10], the intersection class of μ is indeed well-defined.

We can finally name two results which address the topics mentioned above. We briefly restate them in our more specific setting of $\mathbf{RTPH}^1(M)$. Note that a key argument to show that an *ergodic* hyperbolic measure μ is accumulated by hyperbolic periodic measures is Katok's horseshoe construction. Note that this method holds for either $C^{1+\alpha}$ diffeomorphisms (see [31]) or C^1 diffeomorphisms which have a dominated splitting (hence, in particular in $\mathbf{RTPH}^1(M)$, see [23]) and note that this construction enables approximation in the weak$*$ topology and entropy.

First, the question of density of ergodic measures in $\mathcal{M}$ is partially answered in [10] by showing that for $F \in \mathbf{RTPH}^1(M)$ every $\mu \in \mathcal{M}_{<0}$ (not necessarily ergodic) is approached in the weak$*$ topology by ergodic measures if, and only if, almost all ergodic measures in the ergodic decomposition of μ (with respect to the corresponding distribution supported on the set of ergodic measures) share one intersection class (necessarily, μ must have the same index as the hyperbolic periodic orbits of that class). As a consequence, for $F \in \mathbf{ORTPH}^1(M)$, we have that ergodic measures are dense in $\mathcal{M}_{<0}$ and $\mathcal{M}_{>0}$, respectively. The structure of $\mathcal{M}_0$, however, seems to be much more complicated.

Second, by [28], given $F \in \mathbf{ORTPH}^1(M)$ and a saddle P of F, the set of ergodic measures supported on $\overline{\mathrm{Int}(P)}$ with the same index such as P is arcwise connected. Observe that in [28] the authors originally assume $C^{1+\alpha}$ to apply Katok's result which can be replaced by C^1 plus partial hyperbolicity as explained above. Another crucial assumption is that $\overline{\mathrm{Int}(P)}$ is isolated (observe that this guarantees that approximating measures stay in the same space of measures such as μ). Note that in our setting this is automatically guaranteed since the entire ambient space M is a homoclinic class. One further assumption is that any two saddles with the same index are homoclinically related, which also holds in our case.

3. Step skew-product model

We now turn towards the step skew-product setting. For simplicity, we restrict to a setting with two symbols only. Let $\sigma\colon \Sigma \to \Sigma$ be the usual shift map on the space $\Sigma = \{0,1\}^{\mathbb{Z}}$ of two-sided sequences, equipped with the standard metric and consider C^1 diffeomorphisms $f_0, f_1\colon \mathbb{S}^1 \to \mathbb{S}^1$ and the associated step skew-product

$$F\colon \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1, \quad F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x)), \quad \text{where } \xi = (\xi_i)_{i\in\mathbb{Z}}. \tag{3.1}$$

We denote by $\mathbf{SP}^1(\Sigma \times \mathbb{S}^1)$ the family of maps F as in (3.1) and call f_0, f_1 also their *fiber maps*.

As motivated in Section 2 (see also the discussions in [26, 30]), a step skew-product can be seen as a model of a particular example of a partially hyperbolic diffeomorphism with compact center leaves (homeomorphic to circles).

Before describing precisely our setting let us introduce some notation. Given a sequence $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \Sigma$ we write $\xi = \xi^-.\xi^+$, where $\xi^- = (\ldots\xi_{-1}.) \in \Sigma^- \stackrel{\text{def}}{=} \{0,1\}^{-\mathbb{N}}$ and $\xi^+ = (\xi_0\xi_1\ldots) \in \Sigma^+ \stackrel{\text{def}}{=} \{0,1\}^{\mathbb{N}_0}$ are the corresponding one-sided infinite sequences. We also consider the cylinders

$$[\eta_k \ldots \eta_{k+r}] \stackrel{\text{def}}{=} \{\xi = (\xi_i)_{i\in\mathbb{Z}}\colon \xi_i = \eta_i,\ i = k, \ldots, k+r\},$$

where $r \geq 0$, and define the cylinders $[\xi^-.\xi_0\ldots\xi_r]$ and $[\xi_{-r}\ldots\xi_{-1}.\xi^+]$ in the natural way.

Given *finite* sequences $(\xi_0\ldots\xi_n)$ and $(\xi_{-m}\ldots\xi_{-1})$, we let

$$f_{[\xi_0\ldots\,\xi_n]} \stackrel{\text{def}}{=} f_{\xi_n} \circ \cdots \circ f_{\xi_1} \circ f_{\xi_0} \quad \text{and}$$
$$f_{[\xi_{-m}\ldots\,\xi_{-1}.]} \stackrel{\text{def}}{=} (f_{\xi_{-1}} \circ \ldots \circ f_{\xi_{-m}})^{-1} = (f_{[\xi_{-m}\ldots\,\xi_{-1}]})^{-1}.$$

We can naturally define the *central Lyapunov exponent* of a point $X = (\xi, x)$ by

$$\chi_{\mathrm{c}}(X) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log|f'_{[\xi_0\ldots\xi_{n-1}]}(x)| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log|f'_{\xi_k}(x)|,$$

whenever the limit exists, which is nothing but a Birkhoff average of a continuous function (with respect to F). Taking the point of view that the step skew-product F is a model of a certain partially hyperbolic diffeomorphism, this exponent would correspond to some Lyapunov exponent in (1.1).

We will require that the fiber maps f_0, f_1 of the map F *satisfy the axioms CEC+ and Acc+*, that is, there is a (closed) so-called *forward blending interval* $J^+ \subset \mathbb{S}^1$ having the following properties:

CEC+(J^+) (Controlled Expanding forward Covering relative to J^+). There exist positive constants $K_1, \dots, K_5$ such that for every interval $H \subset \mathbb{S}^1$ intersecting J^+ and satisfying $|H| < K_1$ we have

- (controlled covering) there exists a finite sequence $(\eta_0 \dots \eta_{\ell-1})$ for some positive integer $\ell \le K_2 \,|\log |H|| + K_3$ such that
$$f_{[\eta_0 \dots\, \eta_{\ell-1}]}(H) \supset B(J^+, K_4),$$
where $B(J^+, \delta)$ is the δ-neighborhood of the set J^+.
- (controlled expansion) for every $x \in H$ we have
$$\log |(f_{[\eta_0 \dots\, \eta_{\ell-1}]})'(x)| \ge \ell K_5.$$

We let
$$\mathcal{O}^+(x) \stackrel{\text{def}}{=} \bigcup_{n\ge 0} \bigcup_{(\theta_0 \dots \theta_{n-1})} f_{[\theta_0 \dots\, \theta_{n-1}]}(x).$$

Acc+(J^+) (forward Accessibility relative to J^+).
$$\mathcal{O}^+(\operatorname{int} J^+) = \bigcup_{x \in \operatorname{int} J^+} \mathcal{O}^+(x) = \mathbb{S}^1.$$

Analogously, F *satisfies the axioms CEC− and Acc−* if there is a so-called (closed) *backward blending interval* $J^- \subset \mathbb{S}^1$ such that the inverse maps f_0^{-1}, f_1^{-1} satisfy CEC+ and Acc+ (with J^-).

The overall assumption for our consideration is transitivity.

Axiom T (Transitivity). There is a point $x \in \mathbb{S}^1$ such that the sets $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ are both dense in $\mathbb{S}^1$.

In what follows we denote by $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$ the subset of $\mathbf{SP}^1(\Sigma \times \mathbb{S}^1)$ of skew products satisfying the axioms CEC+(J^+), Acc+(J^+), CEC−(J^-), Acc−(J^-), and T for some intervals J^+ and J^-. Under these assumptions it is possible to choose a common blending interval, see [19, Section 2.2] for a discussion about relations between the choice of (common) blending intervals and transitivity.

Lemma 3.1 (Common blending interval, [19, Lemma 2.3])**.** *Let $F \in \mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$. Then for every $x \in \mathbb{S}^1$ and every δ sufficiently small the interval $J = \overline{B(x, \delta)}$ satisfies Axioms CEC+(J), Acc+(J), CEC−(J), and Acc−(J).*

Returning to our setting, for maps in $\mathbf{SP}^1_{\mathrm{nh}}(\Sigma \times \mathbb{S}^1)$ one can define homoclinic relations and intersection classes as in Section 2.4 (for details see Section 5). By Proposition 5.1, there are precisely two intersection classes:
$$\mathrm{Int}_{<0} \stackrel{\text{def}}{=} \{P \in \Sigma \times \mathbb{S}^1 \colon P \text{ hyperbolic periodic point}, \chi_{\mathrm{c}}(P) < 0\}$$
and the analogously defined set $\mathrm{Int}_{>0}$. Moreover, we have
$$\overline{\mathrm{Int}_{<0}} = \overline{\mathrm{Int}_{>0}} = \Sigma \times \mathbb{S}^1 = \text{homoclinic class}(Q), \tag{3.2}$$
where Q is any hyperbolic periodic point in $\Sigma \times \mathbb{S}^1$. Below we will conclude that each intersection class corresponds to one connected component of ergodic measures. Given a hyperbolic ergodic (not necessarily periodic) measure $\mu \in \mathcal{M}$, we define its *intersection class*, denoted by $\mathrm{Int}(\mu)$, as in Section 2.4. By Proposition 5.1, either $\mathrm{Int}(\mu) = \mathrm{Int}_{<0}$ or $= \mathrm{Int}_{>0}$.

3.1. **Hyperbolic measures.** Following the discussion in Section 2.4 which can be adapted also to $\mathbf{SP}^1(\Sigma\times\mathbb{S}^1)$ (see [19, Section 3]), the following result is a version of [10, Theorem 2] and [28, Theorems 1.1 and 1.4] for $\mathbf{SP}^1_{\rm nh}(\Sigma\times\mathbb{S}^1)$.

Theorem 3.2. *Let $F\in\mathbf{SP}^1_{\rm nh}(\Sigma\times\mathbb{S}^1)$. Every measure $\mu\in\mathcal{M}_{<0}$ is accumulated by measures $\nu_n\in\mathcal{M}_{\rm erg,<0}$ in the weak$*$ topology and $\mathrm{Int}(\mu)=\mathrm{Int}_{<0}$. Moreover, every measure $\mu\in\mathcal{M}_{\rm erg,<0}$ is accumulated by measures $\nu_n\in\mathcal{M}_{\rm erg,<0}$ in the weak$*$ topology and in entropy. The analogous result is true for $\mathcal{M}_{>0}$ and $\mathcal{M}_{\rm erg,>0}$, respectively.*

In particular each of the sets $\mathcal{M}_{\rm erg,<0}$ and $\mathcal{M}_{\rm erg,>0}$ are arcwise connected.

The first of the above stated results was shown in the case of a C^1 diffeomorphism with a dominated splitting $E\oplus F$ in [10] and the above statement is a translation to our setting (we recall again that the isolation condition is satisfied since we study the dynamics on the entire ambient space $\Sigma\times\mathbb{S}^1$). We point out that the essential ingredients in [10] are the C^1 dominated Pesin theory as well as the Kingman subadditive and the maximal ergodic theorems, which all have their natural translation to our setting. We refrain from repeating this part of the proof. To see that $\mathcal{M}_{\rm erg,<0}$ is arcwise connected, we follow the steps of proof in [28]. In this paper the $C^{1+\alpha}$-regularity of the diffeomorphisms is used only to apply Katok's result for approximating an ergodic hyperbolic measure by hyperbolic periodic measures. As mentioned before, this also holds true for a C^1 diffeomorphism with a dominated splitting $E\oplus F$ ([23]), and has its natural translation to our setting. The two main properties assumed in [28] are (i) every pair of hyperbolic periodic points of the same stable index $\Sigma\times\mathbb{S}^1$ are homoclinically related (true by Proposition 5.1) and (ii) the homoclinic class is isolated (true because of (3.2)), but we prefer to sketch the argument. Assume that $\mu^0,\mu^1\in\mathcal{M}_{\rm erg,<0}$. Then μ^i is accumulated by a sequence of hyperbolic periodic measures $\nu_n^i\in\mathcal{M}_{\rm erg,<0}$ supported on the orbits of hyperbolic periodic points P_n^i, $i=0,1$. Since P_1^0 and P_1^1 are homoclinically related, there exists a continuous path $\mu_0\colon[1/3,2/3]\to\mathcal{M}_{\rm erg,<0}$ joining the measures ν_1^0 and ν_1^1. For any pair of measures ν_n^0,ν_{n+1}^0 and any neighborhood U of their convex combination $\{s\nu_n^0+(1-s)\nu_{n+1}^0, s\in[0,1]\}$, one can choose a basic set Γ_n^0 such that all measures supported on it are contained in U. Hence, in particular, there exists a continuous path $\mu_n^0\colon[1/3^{n+1},1/3^n]\to\mathcal{M}_{\rm erg,<0}\cap U$ joining the measure ν_n^0 with ν_{n+1}^0. The same applies to the measures ν_n^1, defining paths $\mu_n^1\colon[1-1/3^n,1-1/3^{n+1}]\to\mathcal{M}_{\rm erg,<0}$ which stay in neighborhoods converging to μ^1. Defining $\mu_\infty|_{(0,1)}\colon(0,1)\to\mathcal{M}_{\rm erg,<0}$ by concatenating the domains of those paths, we complete the definition of the path μ_∞ by letting $\mu_\infty(0)=\lim_{n\to\infty}\mu_n^0(1/3^n)$ and $\mu_\infty(1)=\lim_{n\to\infty}\mu_n^1(1-1/3^n)$.

3.2. **Nonhyperbolic ergodic measures.** For nonhyperbolic ergodic measures, the above ergodic approximation methods in general do not apply. However, in our setting, for a step skew-product in $\mathbf{SP}^1_{\rm nh}(\Sigma\times\mathbb{S}^1)$, the special orbit structure enables us to somehow extend these methods. The following was shown in [19].

Theorem 3.3. *Let $F\in\mathbf{SP}^1_{\rm nh}(\Sigma\times\mathbb{S}^1)$. Every measure in $\mathcal{M}_{\rm erg,0}$ is accumulated by measures in $\mathcal{M}_{\rm erg,<0}$ in the weak$*$ topology and in entropy and by measures in $\mathcal{M}_{\rm erg,>0}$ in the weak$*$ topology and in entropy.*

In particular, any such measure is arcwise connected with any ergodic measure in $\mathcal{M}_{\rm erg,<0}$ and in $\mathcal{M}_{\rm erg,>0}$, respectively.

The rough idea of the proof of the above result is to first follow the essential ingredients in Katok's horseshoe construction (see [31, 23] and [19, Section 3]) that given an *ergodic* measure to find sufficiently many orbits (whose number growths roughly exponentially with a factor given by the entropy $h_\mu(F)$) whose (noninvariant) orbital measures roughly approach μ in the weak$*$ topology. The second ingredient is, based on the special orbit structure of our maps, to choose so-called *skeletons* which connect the before obtained orbit pieces to almost recurrent orbits, which can be shadowed by periodic ones. The main issue is to carefully control distortion (recall that we only assume that the maps are C^1 and that we shadow orbits which roughly have a central Lyapunov exponent equal to zero). This collection of hyperbolic (with exponent close to zero) periodic orbits allows to construct horseshoes, the only obstacle in this final step is that they have periods which can vary in between some numbers which is unavoidable because the "orbit-gluing" step is only achieved by the topological constraints (in particular guaranteed by axioms Acc$\pm$). To bypass this problem, we construct so-called *multi-variable-time horseshoes*. This way we construct by hand horseshoes which support only measures which are weak$*$ close to μ and whose entropy is close to μ.

Theorems 3.2 and 3.3 immediately imply the following result.

Corollary 3.4. *Let $F \in \mathbf{SP}^1_{\rm nh}(\Sigma\times\mathbb{S}^1)$. Then $\mathcal{M}_{\rm erg}$ is arcwise connected.*

3.3. **Entropies of the spectrum of Lyapunov exponents.** We will now answer the question raised in the introduction about what type of behavior (negative, zero, or positive central exponent) predominates? We will do this in terms of entropy.

Taking first an orbitwise point of view studying the sets $\mathcal{L}(\alpha)$ as in (2.3), we obtain the following *multifractal decomposition*

$$\Sigma\times\mathbb{S}^1 = \bigcup_{\alpha\in\mathbb{R}} \mathcal{L}(\alpha)\cup\mathcal{L}_{\rm irr},$$

where $\mathcal{L}_{\rm irr}$ is the set of points where the central Lyapunov exponent is not well-defined (the limit does not exist). Note that each level set $\mathcal{L}(\alpha)$ is nonempty in some range of α which decomposes into three natural nonempty parts

$$\{\alpha\colon \mathcal{L}(\alpha)\ne\varnothing\} = [\alpha_{\min},0)\cup\{0\}\cup(0,\alpha_{\max}].$$

It is easy to verify that max and min are indeed attained. It is an immediate consequence of the fact that any pair of hyperbolic periodic orbits with the same stable index are homoclinically related (and hence are contained in a common horseshoe) that for every $\alpha\in(\alpha_{\min},0)\cup(0,\alpha_{\max})$ there exists an ergodic measure with positive entropy and central Lyapunov exponent equal to α. The corresponding result for $\alpha=0$ is a consequence of [4]. We will determine the "size" of those level sets in terms of *topological entropy.* Since these sets are invariant but in general noncompact, we will rely on the general concept of topological entropy $h_{\rm top}$ introduced by Bowen [13].

The following result from [20] now makes a connection to the above via restricted variational principles (2.2), Figure 1 provides a picture theorem.

Denote by $h_\mu(F)$ the *entropy* of a measure μ. Recall that the system of fiber maps $\{f_0,f_1\}$ is *proximal* if for every $x,y\in\mathbb{S}^1$ there exists at least one sequence $\xi\in\Sigma$ such that $|f^n_\xi(x)-f^n_\xi(y)|\to 0$ as $|n|\to\infty$. Note that the system is proximal if, for example, f_0 is a Morse-Smale map whose nonwandering set contains only on attractor and one repeller and f_1 is an irrational rotation (see also Proposition 4.2).

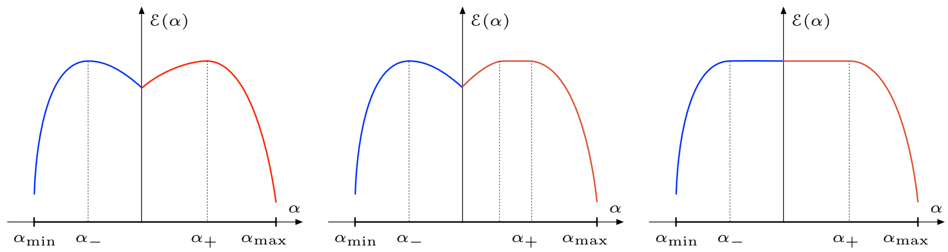

FIGURE 1. Possible shapes of entropies. Left figure: assuming proximality

Theorem 3.5. *For every $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ we have $\mathcal{L}(\alpha) \neq \varnothing$. Moreover, for every $\alpha \in (\alpha_{\min}, 0) \cup (0, \alpha_{\max})$ we have*

$$h_{\rm top}(\mathcal{L}(\alpha)) = \sup\left\{h_\mu(F) \colon \mu \in \mathcal{M}_{\rm erg}, \chi_{\rm c}(\mu) = \alpha\right\}, \tag{3.3}$$

the function $\alpha \mapsto h_{\rm top}(\mathcal{L}(\alpha))$ is continuous on $[\alpha_{\min}, \alpha_{\max}]$ and satisfies $h_{\rm top}(\mathcal{L}(0)) > 0$.

There exist (finitely many) ergodic measures μ_+, μ_- of maximal entropy $h_{\mu_\pm}(F) = \log 2$ and with $\chi_{\rm c}(\mu_-) < 0 < \chi_{\rm c}(\mu_+)$. Moreover, assuming also proximality, there exist unique ergodic F-invariant probability measures $\mu_- \in \mathcal{M}_{{\rm erg},<0}$ and $\mu_+ \in \mathcal{M}_{{\rm erg},>0}$ of maximal entropy $h_{\mu_\pm}(F) = \log 2$ and we have

$$h_{\rm top}(\mathcal{L}(\alpha_-)) = h_{\rm top}(\mathcal{L}(\alpha_+)) = \log 2, \quad h_{\rm top}(\mathcal{L}(\alpha)) < \log 2$$

for all $\alpha \neq \alpha_-, \alpha_+$, where $\alpha_- = \chi_{\rm c}(\mu_-)$ and $\alpha_+ = \chi_{\rm c}(\mu_+)$.

Assuming proximality, one can also show (see [20]) that no measure which is a nontrivial convex combination of the two ergodic measures of maximal entropy is (simultaneously) a weak$*$ and in entropy limit of ergodic measures.

We note that [43] in a somewhat similar setting obtains the finiteness of entropy-maximizing measures and properties about them. Note that this is related to the so-called invariance principle which is also studied in [48] where a similar phenomenon as in Theorem 3.5 (entropy achieving its maximum away from zero exponent) was observed (for certain ergodic measures of C^2 partially hyperbolic diffeomorphisms).

To show Theorem 3.5, our main approach is to treat positive, negative, and zero spectra separately. First, it is an immediate consequence of [13] that the restricted variational entropy (3.3) provides a lower bound for $h_{\rm top}(\mathcal{L}(\alpha))$. Using the fact that by Proposition 5.1 for any pair of uniformly hyperbolic sets with negative (positive) fiber exponents we can find a larger one containing them both, one can conclude that these values can be expressed via the Legendre-Fenchel transform of a certain restricted pressure function (treating negative and positive values separately). Finally, for any α with a level set of given entropy h one can choose so-called *skeletons* established in [19, Section 4] to construct hyperbolic sets with entropy close to h with almost homogeneous exponents close to α. Hence, $h_{\rm top}(\mathcal{L}(\alpha))$ is limited from above by entropies of ergodic measures with exponents close to α. Concavity of the Legendre-Fenchel transform implies its continuity, which concludes the main argument.

4. EXAMPLES

In this section, we first return to the differentiable setting and provide some details about the objects that characterize the diffeomorphisms in $\mathbf{ORTPH}^1(M)$.

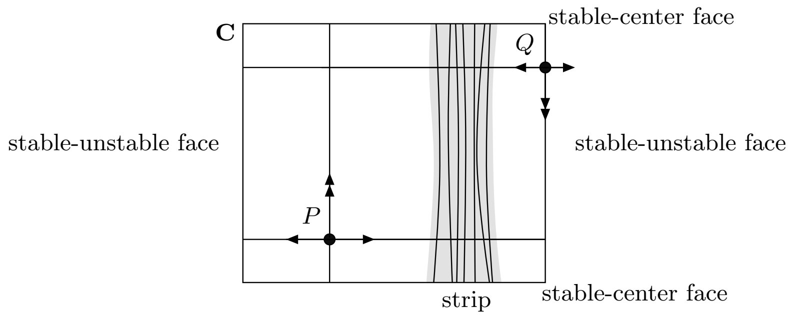

FIGURE 2. Blender-horseshoe: strong unstable curves and strips (projection along the strong stable direction)

Further, we also will see how they motivate the axioms for the skew-product in Section 3.

4.1. **Blender-horseshoes and minimal foliations.** The following discussion does not aim for generality and it is first done for diffeomorphisms in $\mathbf{PH}^1(M^3)$, where the manifold is three-dimensional, and adapted to this setting where there are globally defined strong unstable and stable foliations.

An *unstable blender-horseshoe*, see [5, Section 3], is a hyperbolic and partially hyperbolic set Λ of a diffeomorphism F conjugate to a complete shift of two symbols and having a splitting $E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$ (where E^{ss} is its stable bundle and $E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$ its unstable one) and being isolated in an open neighborhood (a "cube") $\mathbf{C}$, that is, $\Lambda = \bigcap_{n\in\mathbb{Z}} F^n(\mathbf{C})$. Moreover, the splitting is defined in whole neighborhood $\mathbf{C}$ and there is $\lambda > 1$ such that for every $x \in \mathbf{C}$ and every vector $v \in E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$ it holds $\|dF_x(v)\| \geq \lambda \|v\|$. A *stable blender-horseshoe* is an *unstable blender-horseshoe* for F^{-1}.

For the next discussion see Figure 2. First consider the local stable set of Λ defined by $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda) \stackrel{\mathrm{def}}{=} \bigcap_{n\geq 0} F^n(\mathbf{C})$. Naively, the cube $\mathbf{C}$ has six faces: two opposed stable-center faces "tangent" to $E^{\mathrm{ss}} \oplus E^{\mathrm{c}}$, two opposed center-unstable faces "tangent" to $E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$, and two opposed stable-unstable faces "tangent" to $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}}$. A *strong unstable curve* is a closed curve contained in some strong unstable leaf of $\mathcal{F}^{\mathrm{uu}}$ whose boundary is contained in the stable-center faces of the cube (thus the curve joins these two faces). Similarly, a *strong stable curve* is a closed curve contained in some strong stable leaf of $\mathcal{F}^{\mathrm{ss}}$ whose boundary is contained in the center-unstable faces of the cube. An *unstable strip* (or shortly *strip*) S is a "rectangle" that is foliated by strong unstable curves. The *width* of a strip S, denoted by $w(S)$, is the supremum of the numbers w such that there is some curve η tangent to E^{c} and contained in S with length w. Note that there is a number $\kappa > 0$ such that every strip contained in $\mathbf{C}$ has width at most κ.

By hypothesis, the hyperbolic set Λ has two *fixed points* P and Q and we consider their local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ (the connected component of $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ containing P and Q, respectively). There are now two isotopy classes of strong unstable curves disjoint from $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, say to the right and to the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. Similarly for $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. One has that the local strong unstable leaves of points of $\Lambda \setminus \{P, Q\}$ are strong unstable curves which are to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and to the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. A strip S foliated by strong unstable curves to the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ is called a *strip to the right of* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$.

The key property of a blender-horseshoe is the following: for every strip S to the right of $W^{\rm s}_{\rm loc}(P)$ there are two possibilities (that may occur simultaneously): either (1) the set $F(S)$ contains a strip S' (called the *successor* of S) to the right of $W^{\rm s}_{\rm loc}(P)$ and such that $w(S') > \lambda\, w(S)$ (where $\lambda > 1$ is as above), or (2) $F(S)$ contains a strip S' that intersects $W^{\rm s}_{\rm loc}(P)$ in a point at some uniform distance ρ from the boundary of S'. Note that case (1) can occur at most $\ell(S)$ consecutive times where $\ell = \ell(S)$ is the first number with $\lambda^\ell\, w(S) > \kappa$ (with κ as above). Here by "consecutive" we mean that the successor of S also satisfies case (1) and so on. In case (2) let $k = k(\rho)$ the first number with $\lambda^k\,\rho > \kappa$, considering now k additional iterates we have that $F^k(S')$ contains a strip that crosses the two stable-unstable faces of the cube $\mathbf{C}$. Summarizing, given any unstable strip S to the right of $W^{\rm s}_{\rm loc}(P)$, we have that $F^m(S)$ contains a strip that crosses the two stable-unstable faces of the cube $\mathbf{C}$ for some $m \le \ell(S) + k$.

The blender-horseshoe satisfies the following geometric property that we state using the approach in [4]. The family of strong unstable curves $\mathcal{D}$ defined as the ones to the right of $W^{\rm s}_{\rm loc}(P)$ and to the left of $W^{\rm s}_{\rm loc}(Q)$ satisfies the following invariance and covering properties: every $D \in \mathcal{D}$ contains a subset D_0 such that $F(D_0) \in \mathcal{D}$. This implies that the local stable set of Λ, $W^{\rm s}_{\rm loc}(\Lambda)$, intersects every curve of the family $\mathcal{D}$, see [5, Remark 3.10] and [4, Lemma 3.13]. We call $\mathcal{D}$ the *distinctive family of curves of the blender*. Finally, let us also observe that blender-horseshoes have well-defined continuations: if Λ is a blender-horseshoe for F then for every G close enough to F the continuation Λ_G of Λ is also a blender-horseshoe, see [5, Lemma 3.9].

As mentioned above, there is a C^1-open and -dense subset $\mathbf{ORTPH}^1(M^3)$ of $\mathbf{RTPH}^1(M^3)$ whose strong stable and strong unstable foliations are both minimal. The main step of this proof is the following (see [7]). There are an unstable blender-horseshoe Λ^+ with associated cube $\mathbf{C}^+$, a stable blender-horseshoe Λ^- with associated cube $\mathbf{C}^-$, and a constant $\varrho > 0$ such that:

- Every curve α contained in some leaf of $\mathcal{F}^{\rm uu}$ with length $\ell(\alpha) \ge \varrho$ contains strong unstable curves $\alpha^+ \subset \mathbf{C}^+$ and $\alpha^- \subset \mathbf{C}^-$. Moreover, α^+ is in the distinctive family of curves of Λ^+.
- Every curve β contained in some leaf of $\mathcal{F}^{\rm ss}$ with length $\ell(\beta) \ge \varrho$ contains strong stable curves $\beta^+ \subset \mathbf{C}^+$ and $\beta^- \subset \mathbf{C}^-$. Moreover, β^- is in the distinctive family of curves of Λ^-.

The uniform expansion of the bundle $E^{\rm uu}$ implies that for every curve α contained in some leaf of $\mathcal{F}^{\rm uu}$ there is $n = n(\alpha)$ such that $\ell(F^n(\alpha)) > \varrho$. Therefore $\alpha_0 = F^n(\alpha)$ contains curves $\alpha_0^\pm$ as above. Similarly, the uniform contraction of the bundle $E^{\rm ss}$ implies that for every curve β contained in some leaf of $\mathcal{F}^{\rm ss}$ there is $m = m(\beta)$ such that $\beta_0 = F^{-m}(\beta)$ contains curves $\beta_0^\pm$ as above. In particular this implies that there is a number n_0 such that for every $n \ge n_0$ and every strong unstable curve α contained in either $\mathbf{C}^+$ or $\mathbf{C}^-$ the curve $F^n(\alpha)$ contains a strong unstable curve in $\mathbf{C}^+$ and a strong unstable curve in $\mathbf{C}^-$. Similarly for strong stable curves β in $\mathbf{C}^+$ or $\mathbf{C}^-$ and F^{-n}. This means that there are *transitions along the strong unstable and strong stable foliations in finite time n_0 between the cubes of the blenders.*

4.2. **Blender-horseshoes in step skew-products.** We now reformulate the ingredients from above for step skew-products, see [30, Section 5] and [19, Section 8.3] for dictionaries stabilizing the relations between step skew-products and partially

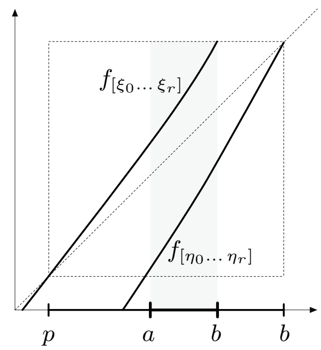

FIGURE 3. Unstable blender-horseshoe

hyperbolic diffeomorphisms. We will state the blender-horseshoes in terms of the underlying one-dimensional dynamics. We consider only the strong unstable foliation, the translation for the strong stable foliation is straightforward and follows considering negative iterations.

Note that the "local strong unstable leaf" $\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(\xi,x)$ of a point (ξ,x) is the set $\{x\}\times[\xi^-.]$, where $\xi=\xi^-.\xi^+$, and the iteration of this leaf is completely governed by the fiber dynamics

$$F^k(\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(\xi,x))\subset\mathcal{F}^{\mathrm{uu}}_{\mathrm{loc}}(F^k(\xi,x))=\{f_{[\xi_0\dots\xi_{k-1}]}(x)\}\times[\xi^-\xi_0\dots\xi_{k-1}.].$$

Therefore the equivalent of a *curve contained in some strong unstable leaf* is a set of the form $\{x\}\times[\xi^-.\xi_0\dots\xi_{k-1}]$. Note that

$$F^k(\{x\}\times[\xi^-.\xi_0\dots\xi_{k-1}])=\{f_{[\xi_0\dots\xi_{k-1}]}(x)\}\times[\xi^-\xi_0\dots\xi_{k-1}.]$$

is a local strong unstable leaf. Thus, for studying the dynamics of a local strong unstable leaf it is enough to consider the forward orbit of the central coordinate for the iterated function system generated by the fiber maps f_0 and f_1.This also means that for obtaining blenders in step skew-products it is enough to consider the dynamics in the fiber coordinate. We now define a blender-horseshoe for a step skew-product map F as in (3.1) with fiber maps f_0 and f_1 using the terminology commonly used for blenders (see [8, Chapter 6.2]), see Figure 3.

Definition 4.1 (Unstable blender-horseshoe for a step skew-product)**.** The skew product map F in (3.1) has an *unstable blender-horseshoe* if there are $\beta>1$, an interval $[p,q]\subset\mathbb{S}^1$, points $a,b\in[p,q]$, $a<b$, finite sequences $(\xi_0\dots\xi_r)$ and $(\eta_0\dots\eta_r)$, $\xi_i,\eta_j\in\{0,1\}$, such that the maps $f_{[\xi_0\dots\xi_r]}$ and $f_{[\eta_0\dots\eta_r]}$ satisfy the following properties:

- (uniform expansion) $(f_{[\xi_0\dots\xi_r]})'(x)\ge\beta$ for all $x\in[p,b]$ and $(f_{[\eta_0\dots\eta_r]})'(x)\ge\beta$ for all $x\in[a,q]$,
- (fixed points) $f_{[\xi_0\dots\xi_r]}(p)=p$ and $f_{[\eta_0\dots\eta_r]}(q)=q$,
- (covering and invariance) $f_{[\xi_0\dots\xi_r]}([p,b])=f_{[\eta_0\dots\eta_r]}([a,q])=[p,q]$.

We say that $[p,q]$ is the *domain of definition* of the blender and that $[a,b]$ is the *superposition interval* of the blender.

The step skew-product map F has a *stable blender-horseshoe* provided F^{-1} has an unstable blender-horseshoe.

To consider the corresponding set for the cube $\mathbf{C}^+$ in the skew-product setting we consider the union $\widehat{\mathbf{C}}^+$ of the sets $[p-\varepsilon,b]\times[.\xi_0\dots\xi_r]$ (for some small $\varepsilon>0$)

and $[b,q]\times[.\eta_0\dots\eta_r]$ and define Λ^+ as the maximal invariant set of F^{r+1} in $\widehat{\mathbf{C}}^+$. In this case, the fixed points of the blender are $P=\big(a,(\xi_0\dots\xi_r)^{\mathbb{Z}}\big)$ and $Q=\big(b,(\eta_0\dots\eta_r)^{\mathbb{Z}}\big)$, and the strong unstable "curves" to the right (of the local stable set) of P are of the form $\{x\}\times[.\xi_0\dots\xi_r]$ if $x\in[p,b]$ or $\{x\}\times[.\eta_0\dots\eta_r]$ if $x\in[a,q]$. With this definition it is immediate that the image by F^{r+1} of any strong unstable curve to the right of P contains a strong unstable curve to the right of P. The stable blender-horseshoe has an associated "cube" $\widehat{\mathbf{C}}^-$ given by the union of the sets $[p'-\varepsilon,b']\times[\xi_{-k}\dots\xi_{-1}.]$ (for some small $\varepsilon>0$) and $[b',q']\times[\eta_{-k}\dots\eta_{-1}.]$.

The forward transition from $\widehat{\mathbf{C}}^+$ to $\widehat{\mathbf{C}}^-$ means that for each $x\in[p,b]$ there is a finite sequence of the form $(\xi_0\dots\xi_r\dots\xi_{r+m})$, $m\ge 0$, such that $f_{[\xi_0\dots\xi_r\dots\xi_{r+m}]}(x)\in(p',q')$ and for each $x\in[a,q]$ there is a finite sequence of the form $(\eta_0\dots\eta_r\dots\eta_{r+n})$, $n\ge 0$ such that $f_{[\eta_0\dots\eta_r\dots\eta_{r+n}]}(x)\in(p',q')$. The forward transition from $\widehat{\mathbf{C}}^-$ to $\widehat{\mathbf{C}}^+$ is defined similarly. The two backward transitions are the corresponding reformulation for backward iterates.

Finally, to get the conditions in Section 3, the two blenders must capture all the dynamics of the map F (e.g. Acc$\pm$). For this we require that every point $x\in\mathbb{S}^1$ has some forward and backward iterate by the iteration of the fiber maps in the intervals (p,q) and (p',q') associated to the blenders. For a complete discussion of these constructions we refer to [19, Section 8.1].

4.3. **Contraction-expansion-rotation in step skew-products.** The next result does not aim for generality and is just a reformulation of the constructions in [27, Theorem 2], where the assumption of forward minimality is replaced by a density-like hypothesis. It also restates [19, Proposition 8.8] in a slightly different way.

Proposition 4.2. *Consider a step skew-product map F as in* (3.1) *with fiber maps $f_0,f_1\colon\mathbb{S}^1\to\mathbb{S}^1$. Suppose that:*

- *There are $\delta>0$ and finite sequences $(\xi_0\dots\xi_r)$ and $(\eta_0\dots\eta_s)$ such that $f_{[\xi_0\dots\xi_r]}$ has an attracting fixed point p and is uniformly contracting in $[p-\delta,p+\delta]$ and $f_{[\eta_0\dots\eta_s]}$ has a repelling fixed point q and is uniformly expanding in $[q-\delta,q+\delta]$.*
- *Every point $x\in\mathbb{S}^1$ has some forward and some backward iterates in $(p-\delta,p+\delta)$ and some forward and some backward iterates in $(q-\delta,q+\delta)$.*

Then there are intervals $J^+,J^-\subset\mathbb{S}^1$ such that the fiber maps of F satisfy Axioms CEC+(J^+) and Acc$\pm(J^+)$ and Axioms CEC$-(J^-)$ and Acc$\pm(J^-)$.

To get the hypothesis of the orbits visiting the neighborhoods of p and q it is enough to have a finite sequence $(\zeta_0\dots\zeta_t)$ such that $f_{[\zeta_0\dots\zeta_t]}$ is an irrational rotation or such that every orbit of the system is "sufficiently dense" in $\mathbb{S}^1$. In particular, if some map $f_{[\zeta_0\dots\zeta_t]}$ is an irrational rotation then small perturbations of the skew-product satisfy the hypotheses of Proposition 4.2.

5. Homoclinic and intersection classes

We briefly discuss the homoclinic relations in the setting of skew-products, for details see, for instance, [16, Section 2.1]. For skew-product maps F as in (3.1) that are only differentiable in the fiber direction we call a periodic point $P=\big((\xi_0\dots\xi_{\pi-1})^{\mathbb{Z}},p\big)$ *hyperbolic* if

$$(f_{[\xi_0\dots\xi_{\pi-1}]})'(p)\ \neq\pm1$$

and call it *contracting* if this derivative has modulus less than one and *expanding* otherwise. As in the hyperbolic case, these points have well-defined and uniquely defined *continuations* for maps G close to F, that is, for $G(\xi,x)=(\sigma(\xi),g_{\xi_0}(x))$ where each g_i is close to f_i.

Given a hyperbolic fixed point p of $f_{[\xi_0\ldots\xi_{\pi-1}]}$ consider its local invariant manifolds $W^{\mathrm{s/u}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\,\xi_{\pi-1}]})$. If p is contracting then $W^{\mathrm{u}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\,\xi_m]})=\{p\}$ and $W^{\mathrm{s}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\,\xi_m]})$ is an open interval containing p. Similarly when p is expanding.

In what follows let $P=((\xi_0\ldots\xi_{\pi-1})^{\mathbb{Z}},p)$ be a hyperbolic periodic point of F. Note that the stable and unstable sets of orbit $\mathcal{O}(P)$ of P are defined, respectively, by

$$\begin{aligned}W^{\mathrm{s}}(\mathcal{O}(P),F)=\Big\{&(\eta,x)\colon \eta=(\ldots\eta_{-1}.\eta_0\ldots\eta_k(\xi_0\cdots\xi_{\pi-1})^{\mathbb{N}}),k\geq 0,\\ &\text{and } f_{[\eta_0\ldots\,\eta_k]}(x)\in W^{\mathrm{s}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\xi_{\pi-1}]})\Big\},\\ W^{\mathrm{u}}(\mathcal{O}(P),F)=\Big\{&(\eta,x)\colon \eta=((\xi_0\ldots\xi_{\pi-1})^{\mathbb{N}}\,\eta_{-k}\ldots\eta_{-1}.\eta_0\ldots),k\geq 0\\ &\text{and } f^{-1}_{[\eta_{-1}\cdots\,\eta_{-k}]}(x)\in W^{\mathrm{u}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\,\xi_{\pi-1}]})\Big\}.\end{aligned}$$

We now adapt the definitions of a *homoclinic class* and *homoclinic relations* of differentiable dynamics to the skew-product setting.

First two hyperbolic periodic points P and Q of the same index are *homoclinically related* if the invariant manifolds of their orbits intersect cyclically, $W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(Q),F)\neq\varnothing$ and $W^{\mathrm{u}}(\mathcal{O}(Q),F)\cap W^{\mathrm{s}}(\mathcal{O}(P),F)\neq\varnothing$. The *intersection class* of P is the set of all hyperbolic periodic points homoclinically related to P. A point $X\in W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(P),F)$ is called a *homoclinic point* of P. As the transverse ones in the differentiable case, these points have well defined continuations. The *homoclinic class* of P is the closure of the homoclinic points of the orbit of P. Note that this definitions does not involve transversality. As in the case of differentiable dynamics, the homoclinic class of P is a transitive set that coincides with the closure of its intersection class.

Proposition 5.1. *Suppose that the skew-product map F in* (3.1) *satisfies the conditions in Section 3. Then*

- *Every pair of hyperbolic periodic points of the same index are homoclinically related.*
- *Every homoclinic class is the whole set $\Sigma\times\mathbb{S}^1$.*

We sketch the proof of this proposition. Consider two hyperbolic periodic points $P=\big((\xi_0\ldots\xi_{\pi_P-1})^{\mathbb{Z}},p\big)$ and $Q=\big((\zeta_0\ldots\zeta_{\pi_Q-1})^{\mathbb{Z}},q\big)$ such that there are a point $c\in W^{\mathrm{u}}_{\mathrm{loc}}(p,f_{[\xi_0\ldots\xi_{\pi_P-1}]})$ and a finite sequence $(\beta_0\ldots\beta_r)$ with

$$f_{[\beta_0\ldots\beta_r]}(c)\in W^{\mathrm{s}}_{\mathrm{loc}}(q,f_{[\eta_0\ldots\eta_{\pi_Q-1}]}).$$

Then

$$C=\big((\xi_0\cdots\xi_{\pi_P-1})^{\mathbb{N}}\,.\,\beta_0\cdots\beta_r\,(\eta_0\cdots\eta_{\pi_Q-1})^{\mathbb{N}},c\big)\in W^{\mathrm{s}}(\mathcal{O}(Q),F)\cap W^{\mathrm{u}}(\mathcal{O}(P),F).$$

This fact implies that, under the conditions in Section 3, any homoclinic class (of a hyperbolic periodic point) is the whole $\Sigma\times\mathbb{S}^1$ and that any pair of hyperbolic periodic points of the same index are homoclinically related. To see why this is so assume that P and Q are both expanding and we show that $W^{\mathrm{u}}(\mathcal{O}(P),F)\cap W^{\mathrm{s}}(\mathcal{O}(Q),F)\neq\varnothing$. Consider now a blending interval J containing q in its interior, as in Lemma 3.1. By condition Acc$-(J)$ there are small $\delta>0$ and a finite sequence $(\tau_0\dots\tau_k)$ such that

$$(p-\delta,p+\delta)\subset W^{\rm u}_{\rm loc}(p,f_{[\xi_0\cdots\xi_{\pi_P-1}]})\quad\text{and}\quad f_{[\tau_0\dots\tau_k]}(p-\delta,p+\delta)\subset J.$$

Now property CEC$+(J)$ provides a finite sequence $(\eta_0\dots\eta_\ell)$ such that

$$J\subset f_{[\tau_0\dots\tau_k\eta_0\dots\eta_\ell]}(p-\delta,p+\delta).$$

Hence there is $c\in(p-\delta,p+\delta)$ such that

$$q=f_{[\tau_0\dots\tau_k\eta_0\dots\eta_\ell]}(c).$$

Taking $C=\big((\xi_0\cdots\xi_{\pi_P-1})^{\mathbb N}.\tau_0\dots\tau_k\eta_0\dots\eta_\ell(\eta_0\cdots\eta_{\pi_Q-1})^{\mathbb N},c\big)$ we get

$$C\in W^{\rm s}(\mathcal O(Q),F)\cap W^{\rm u}(\mathcal O(P),F).$$

The intersection $W^{\rm s}(\mathcal O(P),F)\cap W^{\rm u}(\mathcal O(Q),F)\neq\varnothing$ is obtained reversing the roles of P and Q. The fact that the homoclinic class in the whole set $\Sigma\times\mathbb S^1$ follows using similar arguments.

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