INHOMOGENEOUS DIOPHANTINE APPROXIMATION WITH GENERAL ERROR FUNCTIONS

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Abstract. Let α be an irrational and $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a function decreasing to zero. Let $\omega(\alpha) := \sup\{\theta \geq 1 : \liminf_{n\to\infty} n^\theta \|n\alpha\| = 0\}$. For any α with a given $\omega(\alpha)$, we show some sharp estimations for the Hausdorff dimension of the set

$$E_\varphi(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n\},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

1. Introduction

Let α be an irrational real number. Denote by $\|\cdot\|$ the distance to the nearest integer. A famous result of Minkowski ([Min57]) in 1907 showed that if $y \notin \mathbb{Z} + \alpha\mathbb{Z}$, then for infinitely many $n \in \mathbb{Z}$, we have

$$\|n\alpha - y\| < \frac{1}{4|n|}.$$

If n is restricted to positive integers only, Khintchine ([Khi26]) in 1926 proved that for any real number y, there exist infinitely many $n \in \mathbb{N}$ satisfying the Diophantine inequalities:

$$\|n\alpha - y\| < \frac{1}{\sqrt{5}n}. \tag{1.1}$$

We shall always restrict n to positive integers. Khintchine's result is equivalent to say that the set

$$E(\alpha, c) := \left\{y \in \mathbb{R} : \|n\alpha - y\| < \frac{c}{n} \text{ for infinitely many } n\right\},$$

is the whole space $\mathbb{R}$ when the constant c equals to $1/\sqrt{5}$. It is showed by Cassels [Cas50] in 1950 that the set $E(\alpha, c)$ is of full measure for any constant $c > 0$.

The generalization of this question for more general error functions was first considered by Bernik and Dodson [BD99] in 1999. Define

$$\omega(\alpha) := \sup\{\theta \geq 1 : \liminf_{n\to\infty} n^\theta \|n\alpha\| = 0\}.$$

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(Remark that α is a Liouville number if $\omega(\alpha)=\infty$.) Bernik and Dodson proved that the Hausdorff dimension, denoted by $\dim_H$, of the set

$$E_\gamma(\alpha) := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{1}{n^\gamma} \text{ for infinitely many } n \right\} \quad (\gamma \geq 1),$$

satisfies

$$\frac{1}{\omega(\alpha)\cdot\gamma} \leq \dim_H E_\gamma(\alpha) \leq \frac{1}{\gamma}.$$

In 2003, Bugeaud [Bug03], and independently Schmeling and Troubetzkoy [TS03] improved the above result. They showed that for any irrational α,

$$\dim_H E_\gamma(\alpha) = \frac{1}{\gamma}.$$

Now let $\varphi : \mathbb{N} \to \mathbb{R}^+$ be a function decreasing to zero. Consider the set

$$E_\varphi(\alpha) := \{ y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n \}.$$

This is the set of well-approximated numbers with a general error function φ. It easily follows from the Borel-Cantelli lemma that the Lebesgue measure of $E_\varphi(\alpha)$ is zero whenever the series $\sum_{n=1}^\infty \varphi(n)$ converges. But on the other hand, it seems hard to obtain a lower bound of the Lebesgue measure of $E_\varphi(\alpha)\cap[0,1]$ when the series $\sum_{n=1}^\infty \varphi(n)$ diverges. For the results on the Lebesgue measure, we refer the readers to [Kur55], [LN12], [Kim12], and the references therein.

In this paper, we are concerned with the Hausdorff dimension of the set $E_\varphi(\alpha)$. As in [Dod92] and [Dic94], for an increasing function $\psi : \mathbb{N} \to \mathbb{R}^+$, we define the lower and upper orders at infinity by

$$\lambda(\psi) := \liminf_{n\to\infty} \frac{\log \psi(n)}{\log n} \qquad \text{and} \qquad \kappa(\psi) := \limsup_{n\to\infty} \frac{\log \psi(n)}{\log n}.$$

For simplicity, let us denote

$$u_\varphi := \frac{1}{\lambda(1/\varphi)} \qquad l_\varphi := \frac{1}{\kappa(1/\varphi)}.$$

The results of Bugeaud and Schmeling, Troubetzkoy imply the inequality

$$l_\varphi \leq \dim_H(E_\varphi(\alpha)) \leq u_\varphi.$$

The upper bound was considered a good candidate to be the precise formula for dimension of $E_\varphi(\alpha)$. It would not need to be sharp without the monotonicity of φ: for any irrational α one can easily find a function φ with $u_\varphi = 1$ but with $E_\varphi(\alpha)=\emptyset$. But when φ is nonincreasing, we actually have

$$\dim_H(E_\varphi(\alpha)) = u_\varphi$$

for all α of bounded type, see [FW06]. This result was further strengthened by Xu [Xu10], see below.

However, in [FW06], Fan and Wu constructed an example which shows that the equality is not always true. In fact, they found a Liouville number α and constructed an error function φ such that

$$\dim_H E_\varphi(\alpha) = l_\varphi < u_\varphi.$$

In general case, the dimension formula seems mystery.

Recently, Xu [Xu10] made a progress, he proved the following theorem.

Theorem 1.1 (Xu). *For any α, we have the following estimation*

$$\limsup_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} \le \dim_H(E_\varphi(\alpha)) \le u_\varphi,$$

where q_n denotes the denominator of the n-th convergent of the continued fraction of α.

As a corollary, Xu proved that

$$\dim_H(E_\varphi(\alpha)) = u_\varphi$$

for any irrational number α with $\omega(\alpha) = 1$.

In this paper, we prove the following results.

Theorem 1.2. *For any α with $\omega(\alpha) = w \in [1, \infty]$, we have*

$$\min\left\{u_\varphi,\ \max\left\{l_\varphi, \frac{1+u_\varphi}{1+w}\right\}\right\} \le \dim_H(E_\varphi(\alpha)) \le u_\varphi.$$

Corollary 1.3. *If $w \le 1/u_\varphi$, then*

$$\dim_H(E_\varphi(\alpha)) = u_\varphi.$$

Example 1.4. Take $w = 2$, $u = 1/2$ and $l = 1/3$. We can construct an irrational α such that for all n, $q_n^2 \le q_{n+1} \le 2q_n^2$. Define

$$\varphi(n) = \max\left\{n^{-1/l}, q_k^{-1/l}\right\} \quad \text{if} \quad q_{k-1}^{u/l} < n \le q_k^{u/l}.$$

Then by Corollary 1.3, we have

$$\lim_{n\to\infty} \frac{\log q_n}{-\log \varphi(q_n)} = l < u = \dim_H(E_\varphi(\alpha)).$$

Thus the lower bound of Xu (Theorem 1.1) is not optimal.

The next two theorems show that the estimations in Theorem 1.2 are sharp.

Theorem 1.5. *For any irrational α and for any $0 \le l < u \le 1$, with $u > 1/w$, there exists a decreasing function $\varphi : \mathbb{N} \to \mathbb{R}^+$, with $l_\varphi = l$ and $u_\varphi = u$, such that*

$$\dim_H(E_\varphi(\alpha)) = \max\left\{l, \frac{1+u}{1+w}\right\} < u.$$

Theorem 1.6. *Suppose* $0 \leq l < u \leq 1$*. There exists a decreasing function* $\varphi : \mathbb{N} \to \mathbb{R}^+$*, with* $l_\varphi = l$ *and* $u_\varphi = u$*, such that for any* α *which is not a Liouville number,*

$$\dim_H(E_\varphi(\alpha)) = u.$$

2. Three steps dimension

The goal of this section is to prove Proposition 2.3 which will be the base of our dimension estimation (compare [Xu10, Section 3]).

Let us start with a technical lemma.

Lemma 2.1. *Let* $1 > a > b$ *and* $1 > c > d$*. Then for any* $\delta \in [0, 1]$ *we have*

$$\frac{\log(\delta a + (1-\delta)c)}{\log(\delta b + (1-\delta)d)} \geq \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

Proof. Denote

$$s := \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

Then

$$\frac{\log(\delta a + (1-\delta)c)}{\log(\delta b + (1-\delta)d)} \geq \frac{\log(\delta b^s + (1-\delta)d^s)}{\log(\delta b + (1-\delta)d)}.$$

By concavity of the function $x \to x^s$, we have

$$\delta b^s + (1-\delta)d^s \leq (\delta b + (1-\delta)d)^s$$

and the assertion follows. □

We will also need the following result, which can be found in any standard textbook on geometric measure theory, see for example [F95][Proposition 4.9]. Given a probabilistic measure μ on $\mathbb{R}$, the *lower density* of μ at a point $x \in \mathbb{R}$ is defined by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Lemma 2.2 (Frostman)**.** *Let* $E \subset \mathbb{R}$*. Assume we can find a measure* μ *supported on* E *such that*

$$\underline{d}_\mu(x) \geq s$$

for μ*-almost every* $x \in E$*. Then*

$$\dim_H E \geq s.$$

Let α be an irrational number with $\omega(\alpha) > 1$. Recall that q_n is the denominator of the n-th convergent of the continued fraction of α. Let $B \geq 1$ and suppose there exists a sequence of natural numbers $\{n_i\}$ such that

$$\frac{\log q_{n_i+1}}{\log q_{n_i}} \to B. \tag{2.1}$$

Let $\{m_i\}$ be a sequence of natural numbers such that $q_{n_i} < m_i \leq q_{n_i+1}$. By passing to subsequences, we suppose the limit

$$N := \lim_{i\to\infty} \frac{\log m_i}{\log q_{n_i}}$$

exists. Then obviously, $1 \leq N \leq B$.

Let $K > 1$. Denote

$$E_i := \left\{ y \in \mathbb{R} : ||n\alpha - y|| < \frac{1}{2} q_{n_i}^{-K} \text{ for some } n \in (m_{i-1}, m_i] \right\}.$$

Let

$$E := \bigcap_{i=1}^{\infty} E_i \qquad \text{and} \qquad F := \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i.$$

Proposition 2.3. *Suppose $N > 1$. If $\{n_i\}$ is increasing sufficiently fast then*

$$\dim_H E = \dim_H F = S,$$

where

$$S = S(N, B, K) := \min\left(\frac{N}{K},\ \max\left(\frac{1}{K}, \frac{1}{1 + B - N} \right) \right).$$

We remark that Proposition 2.3 is a generalization of Xu's result, so it could be also used to prove Xu's theorem (Theorem 1.1). However, we need to do some modifications to include the case $N = 1$ in Proposition 2.3, which we skip.

Proof. As $F \supset E$, we only need to get the lower bound for $\dim_H E$ and the upper bound for $\dim_H F$. For the former, we will use the Frostman Lemma, and for the latter, we will use a natural cover.

We will distinguish two cases: $B \geq K$ and $B < K$. Notice the following fact.

Fact: If $B \geq K$ then

$$\frac{N}{K} > \frac{1}{1 + B - N}, \quad \text{and} \quad S = \max\left(\frac{1}{K}, \frac{1}{1 + B - N} \right).$$

If $B < K$, then

$$\frac{1}{K} < \frac{1}{1 + B - N}, \quad \text{and} \quad S = \min\left(\frac{N}{K},\ \frac{1}{1 + B - N} \right).$$

Indeed, the second statement follows by noting $1/K < 1/B$. For the first statement, if $N \geq K$ then it is obviously true because the right hand side is smaller than 1. Otherwise, we have

$$\frac{K-N}{N} < K-N,$$

hence

$$\frac{K}{N} < 1+K-N.$$

Since $B \geq K$, we have

$$1+B-N \geq 1+K-N > K/N.$$

Distribution of the points.
Now, let us study the distribution of the points $\{n\alpha \pmod 1\}$. Let $\{n_i\}$ be a fast increasing sequence satisfying (2.1). By passing to a subsequence, we can always assume that $\{n_i\}$ grows as fast as we wish; the exact conditions on the rate of growth will be clear later. Denote

$$N_i := m_i - m_{i-1}.$$

Since $N > 1$, by passing to a subsequence, we can suppose that $N_i \geq q_{n_i}$. As N will not change when we modify m_i to closest greater or closest smaller multiple of q_{n_i}, it is enough to prove the statement for $q_{n_i}|N_i$.

The three steps theorem tells us how the points $\{n\alpha \pmod 1\}_{n=m_{i-1}+1}^{m_i}$ are distributed on the unit circle: there are q_{n_i} groups of points, each consisting of N_i/q_{n_i} points, the distances between points inside each group are equal to $\xi_i := \|q_{n_i}\alpha\|$ and the distances between groups are $\zeta_i := \|q_{n_i-1}\alpha\| - (N_i/q_{n_i} - 1)\|q_{n_i}\alpha\|$ (except for one distance which is equal to $\zeta_i + \xi_i$).

Let ε be fixed and small. In the first case, i.e., $B \geq K$, we have $\xi_i \leq q_{n_i}^{-K}$ for all i big enough, hence the intervals $[n\alpha - q_{n_i}^{-K}/2, n\alpha + q_{n_i}^{-K}/2]$ intersect each other (inside each group). So E_i consists of $M_i := q_{n_i}$ intervals of length $y_i := (N_i/q_{n_i} - 1)\xi_i + q_{n_i}^{-K}$. By noting that $\|q_n\alpha\|$ is comparable with q_{n+1}^{-1}, we have

$$y_i = (N_i/q_{n_i} - 1)\xi_i + q_{n_i}^{-K} = q_{n_i}^{-\min(K,1+B-N)+\varepsilon}$$

for i large enough.

In the second case, i.e., $B < K$, for big i, E_i consists of N_i intervals of length $z_i := q_{n_i}^{-K}$.

As $q_{n_{i+1}} \gg q_{n_i+1}$, we can freely assume that for i large enough each component of E_i contains at least $M_{i+1}^{1-\varepsilon}$ (in the first case) or $N_{i+1}^{1-\varepsilon}$ (in the second case) components of E_{i+1}.

Calculations.
We will distribute a probability measure μ in the most natural way: the measure attributed to each component of $F_i = E_1 \cap \ldots \cap E_i$ is the same. Here we distribute the measure only on those components of F_i that are components of E_i, i.e., at all stages we count only components completely contained in previous generation sets.

Case 1: $B \geq K$. At level i we have at least $M_i^{1-\varepsilon}$ components of F_i, each of size y_i and inside each component of F_{i-1}, the components of F_i are in equal distance $c_i := \zeta_i - q_{n_i}^{-K}$.

Let $x \in E$. For $y_i \leq r < y_{i-1}$, consider

$$f(r) = \frac{\log \mu(B_r(x))}{\log r}. \tag{2.2}$$

Notice that the convex hull of components of F_i intersecting $B_r(x)$ has measure at most $3\mu(B_r(x))$ and length at most $6r$. Hence, it is enough to consider the case when the interval $B_r(x)$ is a convex hull of some components of F_i contained in one component of F_{i-1}. Denoting the number of components of F_i contained in one component of F_{i-1} by n, we get

$$f(ny_i + (n-1)c_i) \geq \frac{\log(nM_i^{-(1-\varepsilon)})}{\log(ny_i + (n-1)c_i)}. \tag{2.3}$$

As the right hand side of equation (2.3) is the ratio of logarithms of two functions, both linear in n and smaller than 1, by Lemma 2.1 the minimum of $f(r)$ in range (y_i, y_{i-1}) is achieved at one of endpoints. We have

$$f(y_i) \geq (1-\varepsilon)\frac{-\log M_i}{\log y_i} = \max\left(\frac{1}{K}, \frac{1}{1+B-N}\right) + O(\varepsilon) \tag{2.4}$$

and the same holds for $f(y_{i-1})$. Recalling the fact at the beginning of the proof, we get the lower bound by Lemma 2.2.

The upper bound is simpler: for any i, F is contained in $\bigcup_{n>i} E_n$. Hence, we can use the components of all $E_n, n > i$ as a cover for F. For any s the sum of s-th powers of diameters of components of E_n is bounded by $M_n y_n^s$, and for $s > \max(\frac{1}{K}, \frac{1}{1+B-N}) + O(\varepsilon)$ it is exponentially decreasing with n. The upper bound then follows by the definition of Hausdorff dimension.

Case 2: $B < K$. Once again to obtain the lower bound we will consider the function $f(r)$ given by (2.2). However, in this case the components of F_i are not uniformly distributed inside a component of F_{i-1} but they are in groups. There are at least s_i groups in distance c_i from each other, each group is of size y_i and contains at least $N_i^{1-\varepsilon}$ components. Inside each group the components of size z_i are in distance $d_i := \xi_i - q_{n_i}^{-K}$ from each other.

We need to consider $z_i \le r < z_{i-1}$. This range can be divided into two subranges. The equation (2.3) works for $y_i \le r < z_{i-1}$, while for $z_i \le r < y_i$ the same reasoning gives

$$f(nz_i + (n-1)d_i) \ge \frac{\log(nN_i^{-(1-\varepsilon)})}{\log(nz_i + (n-1)d_i)}. \tag{2.5}$$

Like in the first case, Lemma 2.1 implies that the minimum of $f(r)$ in each subrange is achieved at one of endpoints. We have

$$f(z_i) \ge (1-\varepsilon)\frac{-\log N_i}{\log z_i} = \frac{N}{K} + O(\varepsilon)$$

and the same for $f(z_{i-1})$, while $f(y_i)$ is still given by (2.4). Together with Lemma 2.2 and the fact at the beginning of the proof, this gives the lower bound.

To get the upper bound for the dimension of F we can use two covers. One is given by using the convex hulls of groups of components of F_n with $n > i$. As in the first case (taking into account the fact that $1/K < 1/(1+B-N)$), this cover gives

$$\dim_H F \le \frac{1}{1+B-N} + O(\varepsilon).$$

The other cover consists of components of E_n with $n > i$. For any s the sum of s-th powers of diameters of components of E_n is bounded by $N_n z_n^s$, and for $s > \frac{N}{K} + O(\varepsilon)$ it is exponentially decreasing with n. We will choose one of the two covers that gives us the smaller Hausdorff dimension. □

The statement of Proposition 2.3 could be also written in the following way, fixing B and N and varying K:

$$S(N,B,K) = \begin{cases} 1/K & K < 1+B-N \\ 1/(1+B-N) & 1+B-N \le K \le N(1+B-N) \\ N/K & K > N(1+B-N). \end{cases}$$

3. Proof of Theorem 1.2

Note that the lower bound in Theorem 1.2 can be written as

$$\max\left\{ l_\varphi,\ \min\left\{ u_\varphi, \frac{1+u_\varphi}{1+w} \right\} \right\}.$$

We remind the reader that by the result of Bugeaud [Bug03] and Schmeling and Troubetzkoy [TS03], the Hausdorff dimension of E_φ is between l_φ and u_φ. So for the case of Liouville numbers, i.e., $\omega(\alpha) = \infty$, the result trivially holds and we can assume $w < \infty$. With the same reason, we only need to prove the lower bound. Moreover, we just need to show it is not smaller than $\min(u_\varphi, (1+u_\varphi)/(1+w))$ and we can assume that $l_\varphi < u_\varphi$. We shall suppose that $l_\varphi > 0$, the case $l_\varphi = 0$ can be done by a limit argument.

For any irrational number α, $\omega(\alpha)$ can be defined alternatively by

$$\omega(\alpha) = \limsup_{n\to\infty} \frac{\log q_{n+1}(\alpha)}{\log q_n(\alpha)}.$$

Choose a sequence m_i of natural numbers such that

$$\lim_{i\to\infty} \frac{\log m_i}{-\log\varphi(m_i)} = u_\varphi.$$

Let n_i be such that $q_{n_i} < m_i \le q_{n_i+1}$. By passing to a subsequence we can assume that

– the sequence $\log q_{n_i+1}/\log q_{n_i}$ has some limit $B \in [1, w]$,
– the sequence $\log m_i/\log q_{n_i}$ has some limit $N \in [1, B]$,
– the sequence $\{n_i\}$ grows fast enough for Proposition 2.3.

Moreover, we can freely assume that $N > 1$: otherwise, by the monotonicity of φ, we would have

$$\lim_{i\to\infty} \frac{\log q_{n_i}}{-\log\varphi(q_{n_i})} = u_\varphi$$

and the assertion would follow from Theorem 1.1.

Take $K = N/u_\varphi$. By the definition of m_i, for any small $\delta > 0$, we have for all large i

$$\varphi(m_i) \ge (m_i)^{-1/u_\varphi - \delta} \ge q_{n_i}^{-K}.$$

Thus by monotonicity of φ,

$$\varphi(n) \ge q_{n_i}^{-K} \quad \forall n \le m_i. \tag{3.1}$$

The assumptions of Proposition 2.3 are satisfied, so we can calculate the Hausdorff dimension of the set E defined in the previous section. By (3.1), $E \subset E_\varphi$, so this gives the lower bound for the Hausdorff dimension of E_φ:

$$\dim_H E_\varphi \ge M(N,B) := \min\left(u_\varphi, \max\left(\frac{u_\varphi}{N}, \frac{1}{1+B-N}\right)\right)$$

and we want to estimate the minimal value of M for $B \in [1, w], N \in [1, B]$.

First thing to note is that increasing B not only decreases $M(B,N)$ for a fixed N but also increases the range of possible N's. Hence, the minimum of $M(N,B)$ is achieved for $B = w$. Denote $M(N) = M(N,w)$.

We are then left with a simple optimization problem of a function of one variable. We can write

$$M(N) = \min\left(u_\varphi, \max\left(\frac{u_\varphi}{N}, \frac{1}{1+w-N}\right)\right).$$

If $wu_\varphi \le 1$ then $u_\varphi \le 1/(1+w-N)$ for all N, hence
$$\min_N M(N) = u_\varphi \le \frac{1+u_\varphi}{1+w}.$$
Otherwise, as u_φ/N is a decreasing and $1/(1+w-N)$ an increasing function of N, the global minimum over N of the maximum of the two is achieved at the point N_0 where they are equal: $u_\varphi/N_0 = 1/(1+w-N_0)$, that is for
$$N_0 = \frac{u_\varphi(1+w)}{1+u_\varphi}.$$
As $wu_\varphi > 1$ implies $1 < N_0 < wu_\varphi \le w$, N_0 is inside the interval $[1, w]$, hence this global minimum is the local minimum we are looking for. Thus, in this case
$$\min_N M(N) = M(N_0) = \frac{1+u_\varphi}{1+w} < u_\varphi.$$
We are done.

4. Proof of Theorems 1.5 and 1.6

Proof of Theorem 1.5: Let α be of Diophantine type $w > 1/u$. Let q_{n_i} be a sparse subsequence of denominators of convergents such that
$$w = \lim_{i\to\infty} \frac{\log q_{n_i+1}}{\log q_{n_i}}.$$
For any $0 \le l < u \le 1$, define
$$z = \max\left(l, \frac{1+u}{1+w}\right).$$
Note that $z < u$.

Define also a function $\varphi : \mathbb{N} \to \mathbb{R}$ as follows:
$$\varphi(n) := \max\{n^{-1/l},\ k_{n_i}^{-1/u}\}, \quad \text{if } k_{n_{i-1}} < n \le k_{n_i},$$
where
$$k_{n_i} = q_{n_i}^{u/z}.$$
Let D_1 be the set
$$\{y \in \mathbb{R} :\ \text{for infinitely many } i,\ ||n\alpha - y|| < k_{n_i}^{-\frac{1}{u}} \text{ for some } n \in (k_{n_{i-1}}, k_{n_i}]\}$$
and D_2 be the set
$$\{y \in \mathbb{R} : ||n\alpha - y|| < n^{-\frac{1}{l}} \text{ for infinitely many } n\}.$$
Clearly, $E_\varphi(\alpha) = D_1 \cup D_2$. The Hausdorff dimension of D_1 is given by Proposition 2.3 (with $B = w, K = 1/z, N = u/z$):
$$\dim_H D_1 = \min\left(u,\ \max\left(z, \frac{z}{(1+w)z-u}\right)\right) = z$$
(the equality is valid both when $z = l$ and $z = (1+u)/(1+w)$).

By [Bug03] and [TS03] we have

$$\dim_H(D_2) = l.$$

Then the proof is completed.

Proof of Theorem 1.6: Construct a sequence $\{n_i\}_{i\geq 1}$ by recurrence:

$$n_1 = 2, \quad n_{i+1} = 2^{n_i} \ (i \geq 1).$$

Define a function $\varphi : \mathbb{N} \to \mathbb{R}$ as $\varphi(n) = n_i^{-1/l}$ for $n \in (n_i, n_i^{u/l})$ and $\varphi(n) = n^{-1/u}$ elsewhere.

Now we show that for this φ, $\dim_H(E_\varphi(\alpha)) = u$ if α is not a Liouville number. Suppose for contradiction that $\dim_H(E_\varphi(\alpha)) < u$. By Theorem 1.1, no q_m could be between n_i and $n_{i+1}^{l/u}$. Since n_i go to infinity very fast, α cannot be of finite type. Hence, α must be a Liouville number which is a contradiction.

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