THE LYAPUNOV SPECTRUM OF SOME PARABOLIC SYSTEMS

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Abstract. We study the Hausdorff dimension for Lyapunov exponents for a class of interval maps which includes several non-hyperbolic situations. We also analyze the level sets of points with given lower and upper Lyapunov exponents and, in particular, with zero lower Lyapunov exponent. We prove that the level set of points with zero exponent has full Hausdorff dimension, but carries no topological entropy.

1. Introduction

Our goal is to present results on the Lyapunov spectrum of interval maps with parabolic periodic points. We are going to work in the following setting. Let $f\colon I\to I$ be a map on some interval $I\subset\mathbb{R}$ for which there is a partition $I=I_1\cup\ldots\cup I_\ell$ into sub-intervals such that $f|I_i$ is monotone and continuously differentiable for every i. Let $\Lambda\subset I$ be a compact f-invariant set such that $f|\Lambda$ is topologically conjugate to a topologically mixing subshift of finite type, modulo at most countable number of points. Assume that $f|\Lambda$ satisfies the tempered distortion property (see Definition 1). Let $(\Lambda_m)_m$ be an increasing family of compact f-invariant sets having the property that $f|\Lambda_m$ has bounded distortion and is uniformly expanding and topologically conjugate to subshifts of finite type, and that Λ_m converges to Λ in the Hausdorff topology.

We will study the spectrum of Lyapunov exponents of such systems. Given $x\in\Lambda$ we denote by $\underline{\chi}(x)$ and $\overline{\chi}(x)$ the *lower* and *upper Lyapunov exponent* at x, respectively,

$$\underline{\chi}(x)\stackrel{\text{def}}{=}\liminf_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|,\quad \overline{\chi}(x)\stackrel{\text{def}}{=}\limsup_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|,$$

and if both values coincide then we call the common value the *Lyapunov exponent* at x and denote it by $\chi(x)$. For given numbers $0\le\alpha\le\beta$ we

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consider the following level sets

$$\mathcal{L}(\alpha,\beta) \stackrel{\text{def}}{=} \{x \in \Lambda \colon \underline{\chi}(x) = \alpha,\, \overline{\chi}(x) = \beta\}.$$

If $\alpha < \beta$ then $\mathcal{L}(\alpha,\beta)$ is contained in the set of so-called *irregular points*

$$\mathcal{L}_{\text{irr}} \stackrel{\text{def}}{=} \left\{x \in \Lambda \colon \underline{\chi}(x) < \overline{\chi}(x)\right\}.$$

It follows from the Birkhoff ergodic theorem that then we have $\mu(\mathcal{L}_{\text{irr}}) = 0$ for any f-invariant probability measure μ supported on Λ. We denote by $\mathcal{L}(\alpha) \stackrel{\text{def}}{=} \mathcal{L}(\alpha,\alpha)$ the set of *regular points* with exponent α. Similarly, given $0 \le \alpha \le \beta$, $\beta > 0$ we will study

$$\widehat{\mathcal{L}}(\alpha,\beta) \stackrel{\text{def}}{=} \{x \in \Lambda \colon 0 < \underline{\chi}(x) \le \beta,\, \overline{\chi}(x) \ge \alpha\}$$

and $\widehat{\mathcal{L}}(\alpha) \stackrel{\text{def}}{=} \widehat{\mathcal{L}}(\alpha,\alpha)$. Recall that the continuous function $\log|f'| \colon \Lambda \to \mathbb{R}$ is said to be *cohomologous* to a constant if there exist a continuous function $\psi \colon \Lambda \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $\log|f'| = \psi - \psi\circ f + c$ on Λ, which immediately implies that $\mathcal{L}(c) = \Lambda$. From the following considerations we will exclude this trivial case.

We want to determine the complexity of these sets in terms of their Hausdorff dimension $\dim_{\text{H}}$. The multifractal analysis of dynamical systems, including level sets of more general local quantities than Lyapunov exponents, are so far well understood mainly in the uniformly hyperbolic case only (see [11] for main results and further references). Nevertheless, we can mention several results beyond the hyperbolic setting. Nakaishi [10] studied Manneville-Pomeau-like maps and derived the Hausdorff dimension of the level sets $\mathcal{L}(\alpha)$ for Lyapunov exponents α in the interior of the spectrum. Similar results for a different map were obtained by Kesseböhmer and Stratmann [8]. See also [1] and [14], [12] for an analysis of the topological entropy on various level sets in a more general setting.

In many approaches to a multifractal analysis of such level sets one characterizes their dimension (or their entropy) in terms of a conditional variational principle of dimension (or entropy) of measures. We prefer instead a description which involves the Legendre-Fenchel transform of the pressure function. To start with our general scheme, it would be desirable to obtain in the above setting a formula for the dimension spectrum of Birkhoff averages of a *general* continuous (or Hölder continuous) potential $\varphi \colon \Lambda \to \mathbb{R}$. In the present paper we will investigate the particular case of the potential $\varphi = -\log|f'|$. Given $\alpha \ne 0$, let

$$F(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha} \inf_{d\in\mathbb{R}} \big(P(-d\log|f'|) + \alpha d\big) \tag{1}$$

and let

$$F(0) \stackrel{\text{def}}{=} \lim_{\alpha\to 0+} F(\alpha) = d_0, \tag{2}$$

where

$$d_0 \stackrel{\text{def}}{=} \inf\big\{d \colon P(-d\log|f'|) = 0\big\} \tag{3}$$

(see Section 2.2 for fundamental properties of F).

The following is our first main result.

Theorem 1. *Under the conditions above, for all* $0 \le \alpha \le \beta$, $\beta > 0$, *for which* $\widehat{\mathcal{L}}(\alpha,\beta)$, $\mathcal{L}(\alpha,\beta)$ *are nonempty we have*

$$\dim_{\mathrm{H}} \widehat{\mathcal{L}}(\alpha,\beta) = \max_{\alpha\le q\le\beta} F(q) \quad \text{and} \quad \dim_{\mathrm{H}} \mathcal{L}(\alpha,\beta) = \min_{\alpha\le q\le\beta} F(q).$$

The above formulas extend what is known in the hyperbolic setting in several aspects. First of all, it applies to several non-hyperbolic situations. Second, we are able to cover the boundary points of the Lyapunov spectrum (see [14, 12] for related results in the case of the topological entropy of level sets). Finally, we give a description of the dimension of level sets containing irregular points with zero lower Lyapunov exponent. It generalizes results of Barreira and Schmeling [2].

Of particular interest is the set $\mathcal{L}(0)$. If $f|\Lambda$ satisfies the specification property, then the entropy spectrum of Birkhoff averages of general continuous potentials have been studied in [14] using a different approach, see also [12]. For such a system, the vanishing of the entropy $h(f|\mathcal{L}(0))$ as stated below follows in fact from [14, Theorem 3.5] and the Ruelle inequality. (Here we note that $\mathcal{L}(0)$ need not to be compact, and we are using the notion of topological entropy on non-compact sets introduced by Bowen, see Section 6). On the other hand, in terms of Hausdorff dimension $\mathcal{L}(0)$ is a rather large set.

Theorem 2. *Under the conditions above, if* $\mathcal{L}(0)$ *is nonempty then we have*

$$\dim_{\mathrm{H}} \mathcal{L}(0) = \dim_{\mathrm{H}} \Lambda \ge F(0) \quad \text{and} \quad h(f|\mathcal{L}(0)) = 0.$$

We now sketch the exposition of our paper. In Section 2 we present our setting in details and review several concepts and results from ergodic theory. In Section 3 we analyze the main properties of the hyperbolic subsystems which we are going to consider. Upper bounds for the dimension are studied in Section 4. Section 5 is devoted to the analysis of lower dimension bounds: for the set of regular points with an exponent from the interior of the spectrum such bounds simply follow from the maximal lower bound for the corresponding hyperbolic sub-systems. In order to handle exponents at the boundary of the spectrum as well as a set of irregular points, we introduce the concept of a w-measure as the main tool of our analysis. The proofs of Theorems 1 and 2 are given at the end of Section 5 and in Section 6.

2. Preliminaries

2.1. **Assumptions and examples.** Consider the topological Markov chain $\sigma\colon \Sigma \to \Sigma$ defined by the shift map $\sigma(i_1 i_2 \ldots) = (i_2 i_3 \ldots)$ on the set

$$\Sigma \stackrel{\text{def}}{=} \{1,\ldots,p\}^{\mathbb{N}}$$

The corresponding inverse branches of σ we will denote by σ_i. Given a sequence $\omega \in \Sigma$ we will denote by ω_n n-th element and by ω^n the sequence of first n elements of ω.

Let I be a compact interval and $I_1, \ldots, I_p$ be a family of compact subintervals with pairwise disjoint interiors. Let $f_1, \ldots, f_p$ be a family of C^1 diffeomorphisms such that $\mathrm{Dom}(f_i) \subset I_i$, $f_i(\mathrm{Dom}(f_i)) \subset I$, and that we have the following Markov property

$$I_j \subset f_i(\mathrm{Dom}(f_i)) \text{ whenever } \operatorname{int} I_j \cap f_i(\mathrm{Dom}(f_i)) \neq \varnothing.$$

Let $A = (a_{ij})$ be a $p \times p$ matrix with entries $a_{ij} = 1$ if $I_j \subset f_i(\mathrm{Dom}(f_i))$ and $a_{ij} = 0$ otherwise. Let $\Sigma_A \subset \Sigma$ be the subset containing only those sequences ω for which $a_{\omega_i \omega_{i+1}} = 1$ for all i. Let g_i be the inverse branch of f_i. Let $\Sigma_{A,n} = \{(\omega_1, \ldots, \omega_n) \colon \omega \in \Sigma_A\}$ and $\Sigma_{A,*} = \bigcup_{n=0}^\infty \Sigma_{A,n}$, where we use the convention $\Sigma_{A,0} = \{\varnothing\}$.

For each $(i_1 \ldots i_n) \in \Sigma_{A,n}$ we define

$$\widetilde{\Delta}_{i_1 \ldots i_n} \stackrel{\text{def}}{=} g_{i_1 \ldots i_{n-1}}(I_{i_n})$$

and $\widetilde{\Delta}_\varnothing = I$. For $\omega \in \Sigma_A$ let

$$\widetilde{\pi}(\omega) = \bigcap_{n=1}^\infty \widetilde{\Delta}_{\omega^n}.$$

Let $\widetilde{\Lambda} = \widetilde{\pi}(\Sigma_A)$. If $\widetilde{\pi}$ is a bijection, then every $x \in \widetilde{\Lambda}$ belongs to only one interval I_i. In this situation we can set $f = f_i$ on each $\widetilde{\Lambda} \cap I_i$ and the resulting system is a subshift of finite type (and $f|\widetilde{\Lambda}$ is conjugate to the shift $\sigma \colon \Sigma_A \to \Sigma_A$ by $\widetilde{\pi}$). However, we want to deal also with systems which are only semi-conjugate to a subshift of finite type. The so-called Manneville-Pomeau map [13] $f \colon [0,1] \to [0,1]$ defined by $x \mapsto x(1 + x^s) \mod 1$, $s > 0$ being some parameter, is one example [13] (note that $\widetilde{\pi}$ is not a bijection at the endpoints of some cylinders). Therefore we need a slightly more general construction which leads to exposition problems.

An easy way of dealing with that would be working entirely in the symbolic space. This would lead to rather awkward looking formulas and to the dilatation of the geometric parts of the paper. Hence we decided to use another approach, while exchanging simplicity for clear notation.

Notice that there are at most countably many points x_i at which $\widetilde{\pi}^{-1}$ is not uniquely defined, each of them has precisely two preimages under $\widetilde{\pi}^{-1}$. Loosely speaking, we will replace each of them by a pair of points, each of which has a uniquely defined symbolic expansion (and hence a unique image under f, a unique Lyapunov exponent, etc.). Formally speaking, let $\Lambda \stackrel{\text{def}}{=} \bigcup\{x_i^+, x_i^-\} \cup \widetilde{\Lambda} \setminus \bigcup\{x_i\}$. We define a semi-metric ρ on Λ as follows. Let $\rho(x_i^\pm, x_j^\pm) \stackrel{\text{def}}{=} d(x_i, x_j)$ and $\rho(x_i^\pm, y) \stackrel{\text{def}}{=} d(x_i, y)$, where d denotes the Euclidean metric. We define a topology on Λ by requiring that all open balls $B(y, r)$ in semi-metric ρ are open. And we further require that for any $y < x_i$ the set $\Lambda \cap (B(y, d(y, x_i)) \cup \{x_i^-\})$ is open and for any $y > x_i$ the set

$\Lambda \cap (B(y, d(y, x_i)) \cup \{x_i^+\})$ is open. Let $\pi^{-1} = \widetilde{\pi}^{-1}$ on $\Lambda \cap \widetilde{\Lambda}$, we need to extend π^{-1} to the points $x_i^{\pm}$. For any $x_i = \widetilde{\pi}(\omega) = \widetilde{\pi}(\tau)$ there exists n such that $\omega^n \neq \tau^n$ and in this case $x_i = \widetilde{\Delta}_{\omega^n} \cap \widetilde{\Delta}_{\tau^n}$. Assuming that $\widetilde{\Delta}_{\omega^n}$ is the cylinder lying to the left of x_i and $\widetilde{\Delta}_{\tau^n}$ the one to the right of x_i, we define $\pi(\omega) \stackrel{\text{def}}{=} x_i^-$ and $\pi(\tau) \stackrel{\text{def}}{=} x_i^+$.

Now, let $f(\pi(\omega)) = \pi(\sigma\omega)$ and $\Delta_{\omega^n} = \pi(\{\tau \in \Sigma_A \colon \tau^n = \omega^n\})$. Clearly, (Λ, f) is conjugate to (Σ_A, σ) by π, and π is a homeomorphism. Comparing Λ with $\widetilde{\Lambda}$ we see that both sets differ by at most a countable set of points and that they share the metric except at those points. So from the point of view of Hausdorff dimension they differ by a set of dimension 0. In particular, as in this paper we are concerned with results on the Hausdorff dimension, it will be sufficient to prove them for Λ to conclude them for $\widetilde{\Lambda}$ as well.

Given $x \in \Lambda$, we denote a cylinder $\Delta_{i_1 \ldots i_n}$ containing x by $\Delta_n(x)$. We now define the tempered distortion property which is one of our standing assumptions.

Definition 1. The map f is said to have *tempered distortion* on Λ if there exists a positive sequence $(\rho_n)_n$ decreasing to 0 such that for every n we have

$$\sup_{(i_1 \ldots i_n)} \sup_{x,y \in \Delta_{i_1 \ldots i_n}} \frac{|(f^n)'(x)|}{|(f^n)'(y)|} \leq e^{n\rho_n}. \tag{4}$$

Recall that f is said to have *bounded distortion* on Λ if the terms in (4) are uniformly bounded in n by some positive constant.

We say that f is *uniformly expanding* or *uniformly hyperbolic* on an f-invariant compact set $K \subset \Lambda$ if there exists $c > 0$ and $\lambda > 1$ such that $|(f^n)'| \geq c\lambda^n$ everywhere on K.

There are two main classes of (nonuniformly hyperbolic) examples we can work with. The first class is closely related to parabolic Cantor sets, introduced in [15].

Example 1 (parabolic IFS)**.** Assume that $|f'| > 1$ everywhere except a finite set of fixed points p_i where $|f'(p_i)| = 1$. Assume also that f is C^{1+s} for some positive s. One can construct subsystems Λ_m by removing some small cylinder neighborhoods of parabolic points and all their pre-images. Those subsystems are uniformly hyperbolic and have bounded distortion.

The second class is related to the one introduced in [5].

Example 2 (expansive Markov systems)**.** Assume that the family of maps $\{g_i\}_{i=1}^p$ considered above is a collection of bijective $C^{1+\text{Lip}}$ diffeomorphisms with range $R(g_i) \subset \text{int}(I_i)$ and domain $D(g_i) = \bigcup_{j=1}^p I_j$, and let $f(x) \stackrel{\text{def}}{=} g_i^{-1}(x)$ when $x \in R(g_i)$. Consider less restrictive assumptions about f, demanding only that

$$\lim_{n\to\infty} \max_{i_1 \ldots i_n} |\Delta_{i_1 \ldots i_n}| = 0.$$

The subsystems Λ_m are constructed like in the previous case. Their hyperbolicity and bounded distortion property follows from the Mañé hyperbolicity theorem, see [9] for the reference.

Remark 1. For both above-mentioned classes of examples we have equality in the assertion of Theorem 2.

2.2. **Topological pressure.** Let φ be a continuous function on Λ and let $K\subset\Lambda$ be an f-invariant compact set. The *topological pressure* of φ (with respect to $f|K$) is defined by

$$P_{f|K}(\varphi)\stackrel{\text{def}}{=}\lim_{n\to\infty}\frac{1}{n}\log\sum_{(i_1\dots i_n)}\exp\max_{x\in\Delta_{i_1\dots i_n}}S_n\varphi(x),\tag{5}$$

where here and in the sequel the sum is taken over the cylinders with non-empty intersection with the set K. In the case $K=\Lambda$ we simply write $P(\varphi)=P_{f|\Lambda}(\varphi)$. The existence of the limit follows easily from the fact that the sum constitutes a sub-multiplicative sequence. Moreover, the value $P(\varphi)$ does not depend on the particular Markov partition that we use in its definition.

Denote by $\mathcal{M}(\Lambda)$ the family of f-invariant Borel probability measures on Λ. We simply write $\mathcal{M}=\mathcal{M}(\Lambda)$ if there is no confusion about the system. By the variational principle we have

$$P(\varphi)=\max_{\mu\in\mathcal{M}}\left(h_\mu(f)+\int_\Lambda\varphi\,d\mu\right),\tag{6}$$

where $h_\mu(f)$ denotes the entropy of f with respect to μ (see [17]). A measure $\mu\in\mathcal{M}$ is called *equilibrium state* for the potential φ if

$$P(\varphi)=h_\mu(f)+\int_\Lambda\varphi\,d\mu.$$

Given $d\in\mathbb{R}$, we define the function $\varphi_d\colon\Lambda_Q\to\mathbb{R}$ by

$$\varphi_d(x)\stackrel{\text{def}}{=}-d\log|f'(x)|.\tag{7}$$

The tempered distortion property (4) ensures in particular that in the definition of $P(\varphi_d)$ in (5) one can replace the maximum by the minimum or, in fact, by any intermediate value.

Proposition 1. *The function $d\mapsto P(\varphi_d)$ is a continuous, convex, and non-increasing function of $\mathbb{R}$. $P(\varphi_d)$ is negative for large d if and only if there exist no f-invariant probability measures with zero Lyapunov exponent.*

Proof. The claimed properties follow immediately from general facts about the pressure together with the variational principle (6). □

Recall the definition of d_0 in (3).

Lemma 1. *We have $d_0\le\dim_{\mathrm{H}}\Lambda$.*

Proof. It follows immediately from the generator condition that every ergodic f-invariant measure ν has non-negative Lyapunov exponent. Suppose now that $P(-t\log|f'|)>0$ for some $t\ge 0$. Then, by the variational principle, there exists an ergodic f-invariant measure ν such that $h_\nu(f)-t\chi(\nu)>0$ and thus $h_\nu(f)>0$. It follows then from [6] that

$$t<\frac{h_\nu(f)}{\chi(\nu)}=\dim_{\mathrm H}\nu\le\dim_{\mathrm H}\Lambda.$$

This proves the inequality. □

We now give a geometric description of the function F defined in (1), (2), when α is positive. In general, F is always a concave function with range $\{-\infty\}\cup[0,\infty)$. Let us write

$$P(d)\stackrel{\text{def}}{=}P(-d\log|f'|).$$

Note that, by continuity and convexity, the pressure function $d\mapsto P(d)$ may fail to be differentiable on an at most countable set. We sketch below the particular case that we may have at most one point of non-differentiability. If $P'(d)=-\alpha$ then

$$F(\alpha)=\frac{P(d)+\alpha d}{\alpha}.$$

If $P(d)\ge 0$ for all d, that is, the system is parabolic, then we have

$$F(0)=\inf\{s\ge 0\colon P(s)=0\}.$$

In this situation we have the following two possible cases.
Case I: The pressure function is differentiable at $F(0)$. Then $F(\alpha)$ is strictly decreasing for positive α.
Case II: The pressure function is not differentiable at $F(0)$. Then $F(\alpha)=F(0)$ for every $\alpha\le-\lim_{s\to F(0)_-}P'(s)$, and F is strictly decreasing for greater α.

2.3. **Conformal measures.** The *Ruelle-Perron-Frobenius transfer operator* $\mathcal{L}_\varphi\colon C(\Lambda)\to C(\Lambda)$ defined on the space $C(\Lambda)$ of continuous functions $\psi\colon\Lambda\to\mathbb{R}$ is given by

$$\mathcal{L}_\varphi\psi(x)\stackrel{\text{def}}{=}\sum_{\substack{f(y)=x,\\ y\in\Lambda}}e^{\varphi}(y)\psi(y)$$

if $x\in\Lambda$. Denote by λ_φ the spectral radius of $\mathcal{L}_\varphi$. Let ν be an eigenmeasure of the dual operator $\mathcal{L}_\varphi^*$ with eigenvalue λ_φ. Note that ν is a probability measure but not necessarily f-invariant. However, the dynamical properties of ν with respect to $f|\Lambda$ are captured through its Jacobian. The *Jacobian* of ν with respect to $f|\Lambda$ is the (essentially) unique function $J_\nu f$ determined through

$$\nu(f(A))=\int_A J_\nu f\,d\nu. \tag{8}$$

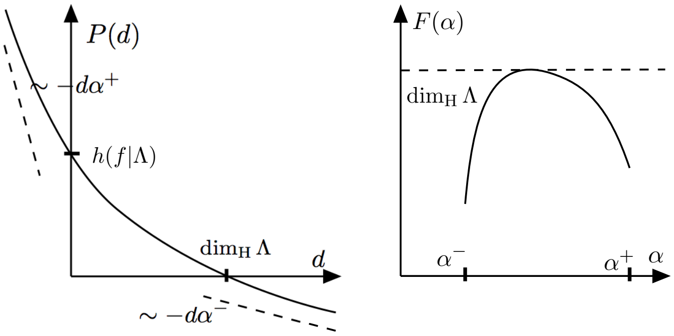

FIGURE 1. Pressure and Lyapunov spectrum for uniformly hyperbolic system

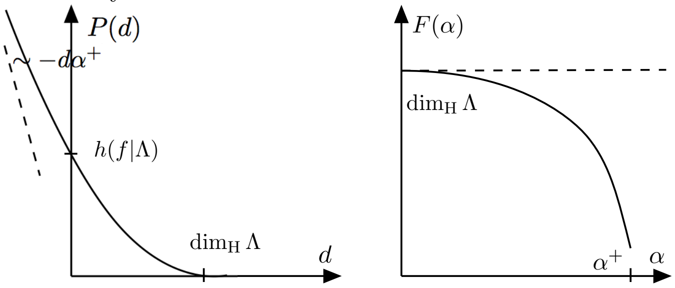

FIGURE 2. Pressure and Lyapunov spectrum for parabolic system, Case I

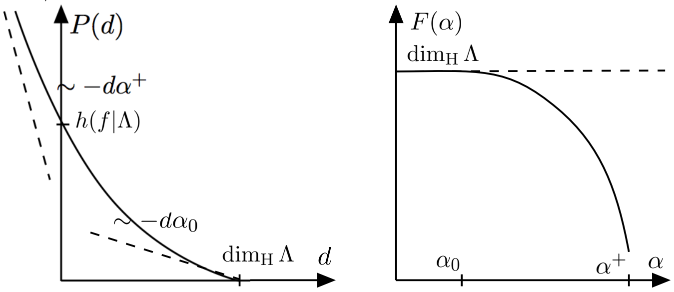

FIGURE 3. Pressure and Lyapunov spectrum for parabolic system, Case II

for every Borel subset A of Λ such that $f|_A$ in injective, and is given by $J_\nu f = \lambda_\varphi e^{-\varphi}$ (see [16]). Moreover, by [19, Theorem 2.1] we have $\log \lambda_\varphi = P(\varphi)$. A measure satisfying (8) is called $e^{P(\varphi)-\varphi}$-*conformal measure*. Such a measure always exists if $f|\Lambda$ is expansive and open ([3, Theorem 3.12]).

3. Hyperbolic sub-systems

For shortness, we will write

$$P_m = P_{f|\Lambda_m}.$$

Given m, the function $d \mapsto P_m(\varphi_d)$ is analytic and strictly decreasing. For fixed $d \in \mathbb{R}$, the sequence $P_m(\varphi_d)$ is non-decreasing.

Proposition 2. *Given $d \in \mathbb{R}$, we have $P(\varphi_d) = \lim_{m\to\infty} P_m(\varphi_d)$.*

Proof. Assume that this is not the case for some $d \in \mathbb{R}$. Clearly, $P_m(\varphi_d)$ form an increasing sequence. Notice that $P^*(d) \stackrel{\text{def}}{=} \lim_{m\to\infty} P_m(\varphi_d) \le P(\varphi_d)$. Let $\delta \stackrel{\text{def}}{=} P(\varphi_d) - P^*(d)$. Let $m_0 \ge 1$ such that $P_m(\varphi_d) \ge P^*(d) - \delta/2$ for every $m \ge m_0$.

There exists a sequence of $\exp(P_m(\varphi_d) - \varphi_d)$-conformal measures (with respect to the sub-system $f|\Lambda_m$) which we denote by ν_d^m. Each such measure satisfies

$$1 = \nu_d^m(f^n(\Delta_n(x))) = \int_{\Delta_n(x)} e^{nP_m(\varphi_d)} |(f^n)'(y)|^d \, d\nu_d^m(y)$$

for every $n \ge 1$ and every $x \in \Lambda_m$. Hence, from the tempered distortion property (4) we can conclude that

$$e^{-n\rho_n} \le \frac{\nu_d^m(\Delta_n(x))}{\exp(-nP_m(\varphi_d))\,|(f^n)'(x)|^{-d}} \le e^{n\rho_n}.$$

Notice that this inequality holds only for cylinders $\Delta_n(x)$ which intersects Λ_m. However, if $\Delta_n(x)$ intersects Λ, then it intersects Λ_m for every m sufficiently big.

Likewise for $f|\Lambda$ we obtain for the $\exp(P(\varphi_d) - \varphi_d)$-conformal measure

$$e^{-n\rho_n} \le \frac{\nu_d(\Delta_n(x))}{\exp(-nP(\varphi_d))\,|(f^n)'(x)|^{-d}} \le e^{n\rho_n}$$

for every cylinder $\Delta_n(x)$ intersecting Λ. Hence, we obtain for every $n \ge 1$ and every cylinder $\Delta_{i_1\ldots i_n}(x)$ which intersects Λ_m.

$$\nu_d(\Delta_{i_1\ldots i_n}) \le \nu_d^m(\Delta_{i_1\ldots i_n}) e^{-n(P(\varphi_d)-P_m(\varphi_d))} e^{2n\rho_n} \le \nu_d^m(\Delta_{i_1\ldots i_n}) e^{-n\delta} e^{2n\rho_n}$$

for every $m \ge 1$. Take a subsequence $(\nu_d^{m_k})_k$ converging to some probability measure ν_d^* in the weak$*$ topology. Then we obtain

$$\nu_d(\Delta_{i_1\ldots i_n}) \le \nu_d^*(\Delta_{i_1\ldots i_n}) e^{n(2\rho_n-\delta)} < \nu_d^*(\Delta_{i_1\ldots i_n})$$

for every $(i_1 \ldots i_n) \in \Sigma_{Q_m,n}$. This contradicts the fact that both measures are probability measures. $\square$

We introduce some further notation. Let

$$\alpha_m^- \stackrel{\text{def}}{=} \inf\{\alpha \ge 0 \colon \chi(x) = \alpha \text{ for some } x \in \Lambda_m\},$$
$$\alpha_m^+ \stackrel{\text{def}}{=} \sup\{\alpha \ge 0 \colon \chi(x) = \alpha \text{ for some } x \in \Lambda_m\}.$$

Similarly, let

$$\alpha^- \stackrel{\text{def}}{=} \inf\{\alpha \ge 0 \colon \chi(x) = \alpha \text{ for some } x \in \Lambda\},$$
$$\alpha^+ \stackrel{\text{def}}{=} \sup\{\alpha \ge 0 \colon \chi(x) = \alpha \text{ for some } x \in \Lambda\}.$$

These are easy to calculate using the pressure, since we have

$$\alpha^- = \lim_{d\to\infty} -\frac{1}{d}P(\varphi_d),$$
$$\alpha^+ = \lim_{d\to-\infty} -\frac{1}{d}P(\varphi_d).$$

Lemma 2. *We have*

$$\lim_{m\to\infty} \alpha_m^- = \inf_{m\ge 1} \alpha_m^- = \alpha^-, \quad \lim_{m\to\infty} \alpha_m^+ = \sup_{m\ge 1} \alpha_m^+ = \alpha^+.$$

Proof. We have for $d<0$

$$P_m(\varphi_d) + d\alpha_m^+ \le P_m(0) \le P(0),$$

hence, by Proposition 2, we obtain

$$\alpha^+ = \lim_{d\to-\infty} -\frac{1}{d}P(\varphi_d) \le \sup_{m\ge 1} \alpha_m^+.$$

Similarly, for $d>0$ and m big enough we have

$$P_m(\varphi_d) \le P_m(0) - d\alpha_m^- \le P(0) - d\alpha_m^-,$$

and thus

$$\alpha^- = \lim_{d\to\infty} -\frac{1}{d}P(\varphi_d) \ge \inf_{m\ge 1} \alpha_m^-.$$

The opposite inequalities follow from the definition of $\alpha^\pm$ and $\alpha_m^\pm$ and from $\Lambda_m \subset \Lambda$. $\square$

Given $\alpha>0$ and $m\ge 1$ let

$$F_m(\alpha) \stackrel{\text{def}}{=} \frac{1}{\alpha} \inf_{d\in\mathbb{R}} \left(P_m(\varphi_d) + \alpha d\right).$$

Lemma 3. *For every $\alpha \in (\alpha^-, \alpha^+)$ we have*

$$\sup_{m\ge 1} F_m(\alpha) = \lim_{m\to\infty} F_m(\alpha) = F(\alpha). \tag{9}$$

Proof. First notice that we can rewrite

$$\begin{aligned} F(\alpha) &= \frac{1}{\alpha} \inf_{d\in\mathbb{R}} \left(P(\varphi_d) + \alpha d\right) \\ &= \sup\{d\colon d\alpha \le P(\varphi_s) + s\alpha \text{ for every } s\}. \end{aligned}$$

The analogous relation holds for F_m with P replaced by P_m. Let us assume that there exists $\varepsilon>0$ such that $F_m(\alpha) < F(\alpha) - \varepsilon$ for every $m\ge 1$. This would imply that for every m the set

$$J_m \stackrel{\text{def}}{=} \{s\colon P_m(\varphi_s) + s\alpha \le (F(\alpha) - \epsilon)\alpha\}$$

is non-empty, closed, and bounded. Moreover, as $P_{m+1} \ge P_m$, we have $J_{m+1} \subset J_m$. Hence, $\bigcap_{m\ge 1} J_m$ is non-empty. For $s \in \bigcap_{m\ge 1} J_m$ we conclude that

$$P_m(\varphi_s) + s\alpha \le (F(\alpha) - \varepsilon)\alpha$$

for every $m \geq 1$. Together with Proposition 2 we hence would obtain

$$P(\varphi_s) + s\alpha \leq (F(\alpha) - \varepsilon)\alpha$$

which is a contradiction.

We mention a second way of proving (9) which is based on the convex conjugate functions. Let

$$T_m(\alpha) \stackrel{\text{def}}{=} \sup_{d \in \mathbb{R}} (\alpha d - P_m(\varphi_{-d}))$$

denote the convex conjugate of $d \mapsto P_m(\varphi_{-d})$. Then (P_m, T_m) form a Legendre-Fenchel pair. Wijsman [18] has shown that for given Legendre-Fenchel pairs (P_m, T_m) and (P, T), the functions P_m converge infimally to P if and only if T_m converges infimally to T (we refer to [18] for the definition of infimal convergence). In general, this kind of convergence does not coincide with the pointwise convergence. However, by monotonicity and continuity of the pressure function we obtain that P_m converges infimally if and only if it converges pointwise. The application of Proposition 2 implies (9). □

For the remainder of this section let $K \subset \Lambda$ be some f-invariant compact set such that $f|K$ is uniformly expanding and conjugate to a topologically mixing subshift of finite type. We have the following result by Jenkinson [7].

Lemma 4 ([7])**.** *For any $\alpha \in (\inf_{\nu \in \mathcal{M}(K)} \chi(\nu), \sup_{\nu \in \mathcal{M}(K)} \chi(\nu))$ there exists a number $q = q(\alpha)$ and some equilibrium state $\nu = \nu(\alpha)$ for the potential $q \log |f'|$ (with respect to $f|K$) such that*

$$\int_K \log |f'| \, d\nu = \alpha.$$

We finally collect results on the Hausdorff dimension of level sets for uniformly hyperbolic systems.

Proposition 3. *For every $\alpha \in (\inf_{\nu \in \mathcal{M}(K)} \chi(\nu), \sup_{\nu \in \mathcal{M}(K)} \chi(\nu))$ we have*

$$\dim_{\mathrm{H}} (K \cap \mathcal{L}(\alpha)) = \frac{1}{\alpha} \inf_{d \in \mathbb{R}} \left(P_{f|K}(\varphi_d) + d\alpha \right). \tag{10}$$

Proof. From [1, Theorem 6] it follows that for arbitrary $d \in \mathbb{R}$

$$\begin{aligned}
\dim_{\mathrm{H}} (K \cap \mathcal{L}(\alpha)) &= \max \left\{ \frac{h_\nu(f)}{\int_K \log |f'| \, d\nu} \colon \nu \in \mathcal{M}(K), \int_K \log |f'| \, d\nu = \alpha \right\} \\
&= \frac{1}{\alpha} \max \left\{ h_\nu(f) + d \int_K \log |f'| \, d\nu \colon \nu \in \mathcal{M}(K), \chi(\nu) = \alpha \right\} - d \\
&\leq \frac{1}{\alpha} \max \left\{ h_\nu(f) + d \int_K \log |f'| \, d\nu \colon \nu \in \mathcal{M}(K) \right\} - d \\
&= \frac{1}{\alpha} \left(P_{f|K}(d \log |f'|) - d\alpha \right),
\end{aligned}$$

where we applied the variational principle for the topological pressure. So we obtain

$$\dim_{\mathrm{H}}(K\cap\mathcal{L}(\alpha))\le\frac{1}{\alpha}\inf_{d\in\mathbb{R}}\left(P_{f|K}(\varphi_d)+d\alpha\right).$$

Lemma 4 implies that

$$\begin{aligned}\max\{h_\nu(f)\colon\nu\in\mathcal{M}(K),\chi(\nu)=\alpha\}&\ge P_{f|K}(-q\log|f'|)+q\alpha\\&\ge\inf_{d\in\mathbb{R}}\left(P_{f|K}(\varphi_d)+d\alpha\right).\end{aligned}$$

This finishes the proof. □

Given any Hölder continuous potential $\psi\colon K\to\mathbb{R}$, there exists a unique ergodic equilibrium state $\mu\in\mathcal{M}(K)$ which moreover has the Gibbs property, that is, for which there exists a constant $c>1$ such that for all $x\in K$ and every $n\ge1$ we have

$$c^{-1}\le\frac{\mu(\Delta_n(x)\cap K)}{\exp\left(-nP_{f|K}(\psi)+S_n\psi(x)\right)}\le c\,. \tag{11}$$

We refer for example to [11] for more details and references of the above results.

4. Upper bound for the dimension

Proposition 4. *We have for every* $0\le\alpha\le\beta$, $\beta>0$

$$\dim_{\mathrm{H}}\widehat{\mathcal{L}}(\alpha,\beta)\le\max_{\alpha\le q\le\beta}F(q)\quad\text{ and }\quad\dim_{\mathrm{H}}\mathcal{L}(\alpha,\beta)\le\min_{\alpha\le q\le\beta}F(q).$$

Proof. Note first that for any $q\in[\alpha,\beta]$, $q>0$, we have $\mathcal{L}(\alpha,\beta)\subset\widehat{\mathcal{L}}(q,q)$. Hence, the second assertion follows from the first one.

We now prove the first assertion. For a point $x\in\widehat{\mathcal{L}}(\alpha,\beta)$ there exists a positive number $q=q(x)\in[\alpha,\beta]$ and a sequence $(n_k)_k$ for which we have

$$\lim_{k\to\infty}\frac{1}{n_k}\log|(f^{n_k})'(x)|=q \tag{12}$$

Let $\delta\in(0,q)$. There exists $N=N(x)\ge1$ such that

$$e^{n_k(q-\delta)}\le|(f^{n_k})'(x)|\le e^{n_k(q+\delta)} \tag{13}$$

for every $n_k\ge N$. By the tempered distortion property (4) we obtain

$$|\Lambda|\,|(f^{n_k})'(x)|^{-1}\,e^{-n_k\rho_{n_k}}\le|\Delta_{n_k}(x)|\le|\Lambda|\,|(f^{n_k})'(x)|^{-1}\,e^{n_k\rho_{n_k}}. \tag{14}$$

Using again the tempered distortion property (4) we can conclude that the $\exp(P(\varphi_d)-\varphi_d)$-conformal measure ν_d satisfies

$$e^{-nP(\varphi_d)}|(f^n)'(x)|^{-d}e^{-n\rho_n}\le\nu_d(\Delta_n(x))\le e^{-nP(\varphi_d)}|(f^n)'(x)|^{-d}e^{n\rho_n}. \tag{15}$$

We obtain

$$P(\varphi_d)+\lim_{k\to\infty}\frac{1}{n_k}\log\nu_d(\Delta_{n_k}(x))=-d\,q, \tag{16}$$

and in particular the limit on the left hand side exists. Hence, possibly after increasing N, we have

$$e^{-n_k(P(\varphi_d)+dq+\delta)} \le \nu_d(\Delta_{n_k}(x))$$

for every $n_k \ge N$. With (14) and (15) we can conclude that for every $n_k \ge N$

$$\nu_d(\Delta_{n_k}(x)) \ge e^{-n_k P(\varphi_d)} |\Delta_{n_k}(x)|^d |\Lambda|^{-d} \, e^{-n_k \rho_{n_k}(1+|d|)}.$$

Case I: Let us first assume that $P(\varphi_d) \ge 0$. Using (13) and (14) we can estimate

$$e^{-n_k P(\varphi_d)} \ge \left(|\Delta_{n_k}(x)| \, |\Lambda|^{-1} e^{-n_k \rho_{n_k}}\right)^{P(\varphi_d)/(q-\delta)}.$$

Thus we obtain

$$\nu_d(\Delta_{n_k}(x)) \ge \left(|\Delta_{n_k}(x)| \, |\Lambda|^{-1}\right)^{d+\frac{P(\varphi_d)}{q-\delta}} \, e^{-n_k \rho_{n_k}\left(\frac{P(\varphi_d)}{q-\delta}+(1+|d|)\right)}.$$

There exists $\varepsilon = \varepsilon(\delta) > 0$ such that, perhaps after increasing N again, we have

$$|\Lambda|^{-d-\frac{P(\varphi_d)}{q-\delta}} e^{-n_k \rho_{n_k}\left(\frac{P(\varphi_d)}{q-\delta}+(1+|d|)\right)} \ge |\Delta_{n_k}(x)|^{\varepsilon}$$

for every $n_k \ge N$. Note that $\Delta_{n_k}(x) \subset B(x, |\Delta_{n_k}(x)|)$. Hence, we obtain the following upper bound for the lower pointwise dimension of the measure ν_d at x

$$\underline{d}_{\nu_d}(x) \le \frac{P(\varphi_d)}{q-\delta} + d + \varepsilon. \tag{17}$$

Case II: Let us now assume that $P(\varphi_d) < 0$. Using (13) and (14) we can estimate

$$e^{-n_k P(\varphi_d)} \ge \left(|\Delta_{n_k}(x)||\Lambda|^{-1}\right)^{\frac{P(\varphi_d)}{q+\delta}} \, e^{n_k \rho_{n_k} \frac{P(\varphi_d)}{q+\delta}}.$$

Thus we obtain

$$\nu_d(\Delta_{n_k}(x)) \ge |\Delta_{n_k}(x)|^{d+\frac{P(\varphi_d)}{q+\delta}+\varepsilon}$$

for every $n_k \ge N$, possibly after increasing N, and hence in this case

$$\underline{d}_{\nu_d}(x) \le \frac{P(\varphi_d)}{q+\delta} + d + \varepsilon. \tag{18}$$

In both cases, continuity of the function $d \mapsto P(\varphi_d)$ implies that for any given sufficiently small interval $(q_1, q_2) \subset (\max\{0, \alpha-\delta\}, \beta+\delta)$ there exist $d \in \mathbb{R}$ such that

$$\frac{1}{q_2} P(\varphi_d) + d \le F(q_2) + \varepsilon.$$

We can then choose a countable family of intervals (q_1^i, q_2^i) which cover $(\max\{0, \alpha-\delta\}, \beta+\delta)$ and consider the corresponding sequence $(d_i)_i$. Defining the measure

$$\nu \overset{\text{def}}{=} \sum_{i=1}^{\infty} 2^{-i} \nu_{d_i}$$

we obtain for the lower pointwise dimension of that measure ν that

$$\underline{d}_{\nu}(x) \le \sup_{i \ge 1} \underline{d}_{\nu_{d_i}}(x) \le \max_{\alpha-\delta \le q \le \beta+\delta} F(q) + 2\varepsilon,$$

where the second inequality follows from (17) and (18). This implies that

$$\dim_{\mathrm{H}} \widehat{\mathfrak{L}}(\alpha,\beta) \le \max_{\alpha-\delta\le q\le\beta+\delta} F(q) + 2\varepsilon.$$

Since δ and ε can be chosen arbitrarily small, this finishes the proof. □

Given $\alpha > 0$ let

$$\widehat{\mathfrak{L}}(\alpha) \stackrel{\text{def}}{=} \left\{x \in \Lambda\colon \underline{\chi}(x) = 0,\, \overline{\chi}(x) \ge \alpha\right\}. \tag{19}$$

The following proposition is proved in a similar way to Proposition 4.

Proposition 5. *For every $\alpha > 0$ we have* $\dim_{\mathrm{H}} \widehat{\mathfrak{L}}(\alpha) \le F(\alpha)$.

5. Lower bound for the dimension

5.1. **The interior of the spectrum – regular points.** We first analyze the set of Lyapunov regular points with an exponent α from the interior of the spectrum. This restriction is essentially due to the fact that Jenkinson's result in Lemma 4 provides us only with an (ergodic) equilibrium state having the corresponding average $\alpha > \alpha^-$.

Proposition 6. *For $\alpha \in (\alpha^-, \alpha^+)$ we have* $\dim_{\mathrm{H}} \mathfrak{L}(\alpha) \ge F(\alpha)$.

Proof. Let $H_m(\alpha) \stackrel{\text{def}}{=} \mathfrak{L}(\alpha) \cap \Lambda_m$. For each exponent $\alpha \in (\alpha^-, \alpha^+)$ there exists $m \ge 1$ such that $\alpha > \alpha_m^-$ and hence $\alpha > \alpha_{m'}^-$ for every $m' \ge m$. By Proposition 3 we have $F_{f|\Lambda_{m'}}(\alpha) = \dim_{\mathrm{H}} H_{m'}(\alpha)$, and we can conclude that

$$\dim_{\mathrm{H}} \mathfrak{L}(\alpha) \ge \sup_{m\ge1} \dim_{\mathrm{H}} H_m(\alpha) = \sup_{m\ge1} F_{f|\Lambda_m}(\alpha).$$

The application of Lemma 3 finishes the proof. □

5.2. **Construction of w-measures and their properties.** The purpose of the present section is to obtain the lower bound for the Hausdorff dimension of the set of points that have a Lyapunov exponent equal to one of the numbers from the boundary $\{\alpha^-, \alpha^+\}$ of the spectrum, or, more generally, for the set of Lyapunov irregular points with certain prescribed upper and lower Lyapunov exponents. Our approach is to construct a Borel probability measure μ supported on Λ (which we will call a *w-measure*) such that the trajectory of a μ-typical point will for some time be distributed according to some measure μ_1, then for some longer time according to some other measure μ_2 and so on. Hence, the asymptotic properties of a μ-typical point can be described by ergodic properties of measures μ_i and we can chose $(\mu_i)_i$ in a way to obtain any specific behavior. Needless to say, the measure μ is in general not invariant.

Recall the notation for hyperbolic sub-systems introduced in Section 3. Given a nondecreasing sequence of positive integers $(n_i)_i$, let $(\mu_i)_i$ be a sequence of certain equilibrium states for potentials ϕ_i with respect to $f|\Lambda_{n_i}$. Let

$$h_i \stackrel{\text{def}}{=} h_{\mu_i}(f), \quad \chi_i \stackrel{\text{def}}{=} \chi(\mu_i), \quad d_i \stackrel{\text{def}}{=} \frac{h_i}{\chi_i} = \dim_{\mathrm{H}} \mu_i.$$

(Note that the last equality uses a result in [6].) We note that the same construction can be performed for an arbitrary, not necessarily non-decreasing, sequence $(n_i)_i$, but this assumption simplifies the exposition. We will in the following assume that

$$P_{f|\Lambda_{n_i}}(\phi_i) = 0 \tag{20}$$

(note that otherwise we can replace ϕ_i by $\phi_i - P_{f|\Lambda_{n_i}}(\phi_i)$ without changing the equilibrium state μ_i).

We now describe the construction of a measure μ, satisfying certain special properties. Let $(m_i)_i$ be a fast increasing sequence of positive integers. We will specify the specific growth speed in the course of this section. We demand that

$$\frac{\rho_{m_i}}{\chi_i} \to 0, \tag{21}$$

where $(\rho_n)_n$ is a positive sequence decreasing to 0 for which (4) is satisfied. We define a probability measure μ on the algebra generated by the cylinders $\Delta_{i_1 \ldots i_n}$. As the beginning of the construction, for cylinders of level m_1 we define

$$\mu(\Delta_{\omega^{m_1}}) \stackrel{\text{def}}{=} \mu_1(\Delta_{\omega^{m_1}}).$$

Given a cylinder of level m_i of positive measure μ, we sub-distribute the measure on its sub-cylinders of level m_{i+1} which intersect $\Lambda_{m_{i+1}}$ in the following way. Let

$$\mu(\Delta_{\omega^{m_i}\tau^{m_{i+1}-m_i}}) \stackrel{\text{def}}{=} c_{i+1}(\omega^{m_i})\, \mu(\Delta_{\omega^{m_i}})\, \mu_{i+1}(\Delta_{\tau^{m_{i+1}-m_i}})$$

where

$$c_{i+1}(\omega^{m_i}) = \left(\sum_{\tau^{m_{i+1}-m_i} \colon \omega^{m_i}\tau^{m_{i+1}-m_i} \in \Sigma_{n_{i+1}}} \mu_{i+1}(\Delta_{\tau^{m_{i+1}-m_i}}) \right)^{-1}$$

is the normalizing constant. For every $m_i < m < m_{i+1}$ let

$$\mu(\Delta_{\pi^m}) \stackrel{\text{def}}{=} \sum_{\tau^{m_{i+1}-m}} \mu(\Delta_{\pi^m \tau^{m_{i+1}-m}}) \,.$$

We extend the measure μ arbitrarily to the Borel σ-algebra of Λ. We call the probability measure μ a *w-measure* with respect to the sequence $(f|\Lambda_{n_i}, \phi_i, \mu_i)_i$. We will in the following analyze the precise way in which the Lyapunov exponents and the local entropies of μ are determined by the asymptotic fluctuations of the Lyapunov exponents and the local entropies of the equilibrium states μ_i.

As each μ_{i+1} is an equilibrium state for a uniformly hyperbolic system, it has the Gibbs property (11) with some constant D_{i+1}. Hence we can conclude that

$$D_{i+1}^{-2} \le \frac{1}{\mu_{i+1}(\Delta_{\omega^{m_i}})}\left[\sum_{\tau^{m_{i+1}-m_i}\colon\, \omega^{m_i}\tau^{m_{i+1}-m_i}\in\Sigma_{n_{i+1}}} \mu_{i+1}(\Delta_{\omega^{m_i}\tau^{m_{i+1}-m_i}})\right] c_{i+1}(\omega^{m_i}) \le D_{i+1}^2.$$

Now observe that $\mu_{i+1}(\Delta_{\omega^{m_i}})^{-1}[\cdots] = 1$ and hence

$$D_{i+1}^{-2} \le c_{i+1}(\omega^{m_i}) \le D_{i+1}^2.$$

Furthermore, the Gibbs property (11) implies that

$$D_{i+1}^{-3} \le \frac{\mu_{i+1}(\Delta_{\omega^{m_i}\tau^{m-m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}})\mu_{i+1}(\Delta_{\tau^{m-m_i}})} \le D_{i+1}^3. \tag{22}$$

Hence we obtain for the constructed measure μ

$$D_{i+1}^{-5}\frac{\mu(\Delta_{\omega^{m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}})} \le \frac{\mu(\Delta_{\omega^{m_i}\tau^{m_{i+1}-m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}\tau^{m_{i+1}-m_i}})} \le D_{i+1}^5\frac{\mu(\Delta_{\omega^{m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}})}\,. \tag{23}$$

Consider now the measure of a cylinder at level m for any $m_i < m < m_{i+1}$. Notice that

$$\mu_{i+1}(\Delta_{\pi^m}) = \sum_{\tau^{m_{i+1}-m}} \mu_{i+1}(\Delta_{\pi^m\tau^{m_{i+1}-m}})\,.$$

Hence, (23) implies that for any $m_i < m < m_{i+1}$

$$D_{i+1}^{-5}\frac{\mu(\Delta_{\omega^{m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}})} \le \frac{\mu(\Delta_{\omega^{m_i}\tau^{m-m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}\tau^{m-m_i}})} \le D_{i+1}^5\frac{\mu(\Delta_{\omega^{m_i}})}{\mu_{i+1}(\Delta_{\omega^{m_i}})}\,. \tag{24}$$

Now (24) and (22) imply

$$D_{i+1}^{-8} \le \frac{\mu(\Delta_{\omega^{m_i}\tau^{m-m_i}})}{\mu(\Delta_{\omega^{m_i}})\mu_{i+1}(\Delta_{\tau^{m-m_i}})} \le D_{i+1}^8 \tag{25}$$

for any $m_i < m \le m_{i+1}$ and any sequence $\omega^{m_i}\tau^{m-m_i}$ such that $\Delta_{\omega^{m_i}\tau^{m-m_i}}$ intersects Λ_{i+1}.

We now introduce a "finite-time Lyapunov exponent"

$$L_m(x) \stackrel{\text{def}}{=} \frac{1}{m}\log|(f^m)'(x)|$$

and similarly we also introduce a "finite-level local entropy"

$$H_m(x) \stackrel{\text{def}}{=} -\frac{1}{m}\log\mu(\Delta_m(x)).$$

While for $m_i < m \le m_{i+1}$ these numbers are simply related to each other by the formula

$$L_m(x) = \frac{m-m_i}{m}L_{m-m_i}(f^{m_i}(x)) + \frac{m_i}{m}L_{m_i}(x), \tag{26}$$

the values of $H_m(x)$ can in general only be approximated, as by (25) for $m_i < m \le m_{i+1}$ we can estimate

$$\left|H_m(x) + \frac{1}{m}\log\mu_{i+1}(\Delta_{m-m_i}(f^{m_i}(x))) - \frac{m_i}{m}H_{m_i}(x)\right| \le \frac{8\log D_{i+1}}{m_i}. \tag{27}$$

Depending on our choice of constants, we can make the set of points which on a level $m_i < m \le m_{i+1}$ have a finite-level entropy approximately equal to h_{i+1} as large as we want. This is made precise in the following large deviation result.

Proposition 7. *Let*

$$\begin{aligned} A_H(\varepsilon, i) &\stackrel{\text{def}}{=} \\ &\Bigg\{x\colon \left|H_m(x) - \frac{m_i}{m}H_{m_i}(x) - \frac{m-m_i}{m}h_{i+1}\right| \le \varepsilon\,|h_{i+1}-h_i| \\ &\qquad\qquad \textit{for every } m_i < m \le m_{i+1}\Bigg\}. \end{aligned} \tag{28}$$

Then for any $\varepsilon > 0$ and $\delta \in (0,1)$ there exists $M = M(\varepsilon, \delta, \phi_{i+1}, \mu_{i+1}) \ge 1$ such that for every $m_i \ge M$ we have

$$\mu\big(A_H(\varepsilon, i)\big) > 1 - \delta.$$

Proof. Fix some $\varepsilon > 0$ and $\delta \in (0,1)$. By (27), for any $m_i < m \le m_{i+1}$, any $x \in \Lambda$, and any $y \in \Delta_{m-m_i}(f^{m_i}(x))$ we can estimate

$$\begin{aligned} &\left|H_m(x) - \frac{m_i}{m}H_{m_i}(x) - \frac{m-m_i}{m}h_{i+1}\right| \\ &\quad\le \left|H_m(x) + \log\mu_{i+1}(\Delta_{m-m_i}(f^{m_i}(x))) - \frac{m_i}{m}H_{m_i}(x)\right| \\ &\qquad + \left|\frac{m-m_i}{m}h_{i+1} + \log\mu_{i+1}(\Delta_{m-m_i}(y))\right| \\ &\quad\le \frac{8\log D_{i+1}}{m_i} + \frac{1}{m}\left|(m-m_i)h_{i+1} + \log\mu_{i+1}(\Delta_{m-m_i}(y))\right|. \end{aligned} \tag{29}$$

From the Gibbs property of the measure μ_{i+1} we obtain that

$$|\log\mu_{i+1}(\Delta_n(y)) - S_n\phi_{i+1}(y)| \le \log D_{i+1}$$

for any $n \ge 1$. Thus, the rightmost term in the inequalities in (29) is not greater than

$$W(y, m, m_i) \stackrel{\text{def}}{=} \frac{9\log D_{i+1}}{m_i} + \frac{1}{m}\left|(m-m_i)h_{i+1} + S_{m-m_i}\phi_{i+1}(y)\right|. \tag{30}$$

The first summand in (30) is arbitrarily small for big m_i. To estimate the second one we note that it is a consequence of the Birkhoff ergodic theorem and the Egorov theorem that for the given numbers $\varepsilon > 0$ and $\delta > 0$ there exists $N_{i+1} = N_{i+1}(\varepsilon, \delta, \phi_{i+1}, \mu_{i+1})$ such that we have

$$\mu_{i+1}\left(\left\{y\colon \left|S_n\phi_{i+1}(y) - n\int \phi_{i+1}\,d\mu_{i+1}\right| \le n\varepsilon \quad \forall n \ge N_{i+1}\right\}\right) \ge 1 - \delta.$$

From (20) and from the fact that μ_{i+1} is an equilibrium state for ϕ_i we conclude that

$$h_{i+1} = -\int \phi_{i+1}\,d\mu_{i+1}.$$

Now let

$$K_{i+1} \stackrel{\text{def}}{=} \max_{1\le n\le N_{i+1}} \max_{y\in\Lambda_{n_{i+1}}} \left|S_n\phi_{i+1}(y) + nh_{i+1}\right|.$$

We obtain

$$\mu_{i+1}\Big(\Big\{y\colon \left|S_n\phi_{i+1}(y) + nh_{i+1}\right| \le n\varepsilon + K_{i+1} \quad \forall n \ge 1\Big\}\Big) \ge 1 - \delta.$$

Hence, if m_i is chosen sufficiently big, the set of points y for which $W(y, m, m_i)$ is smaller than any constant has arbitrarily large μ_{i+1}-measure. Now let $M = N_{i+1}(\varepsilon|h_{i+1} - h_i|, D_{i+1}^{-8}\delta, \phi_{i+1}, \mu_{i+1})$. In particular, for $m_i \ge M$ we have

$$\sum_\tau \mu_{i+1}(\Delta_\tau) \ge 1 - D_{i+1}^{-8}\delta, \tag{31}$$

where the sum is taken over all cylinders τ of level $m_{i+1} - m_i$ containing a point y for which $W(y, m_{i+1}, m_i) \le \varepsilon|h_{i+1} - h_i|$.

Now observe that for every ω^{m_i} the set $f^{m_i}(A_H(\varepsilon, i) \cap \Delta_{\omega^{m_i}})$ is a union of cylinders of level at most $m_{i+1} - m_i$, and with (31) and (25) we can conclude that

$$\mu\big(A_H(\varepsilon, i) \cap \Delta_{\omega^{m_i}}\big) \ge 1 - \mu(\Delta_{\omega^{m_i}})\delta.$$

This proves the proposition. □

Proposition 8. *Let*

$$A_L(\varepsilon, i) \stackrel{\text{def}}{=} \left\{x\colon \left|L_m(x) - \frac{m_i}{m}L_{m_i}(x) - \frac{m - m_i}{m}\chi_{i+1}\right| \le \varepsilon\left|\chi_{i+1} - \chi_i\right| \quad \textit{for every } m_i < m \le m_{i+1}\right\}. \tag{32}$$

Then for any $\varepsilon > 0$ and $\delta \in (0, 1)$ there exists $M = M(\varepsilon, \delta, \phi_{i+1}, \mu_{i+1}) \ge 1$ such that for every $m_i > M$

$$\mu\big(A_L(\varepsilon, i)\big) \ge 1 - \delta.$$

Proof. We note that for every $y \in \Delta_m(x)$ we have $|L_m(x) - L_m(y)| \le \rho_m$, where $(\rho_n)_{n\ge1}$ is the to 0 decreasing sequence from the tempered distortion property (4). Now we can apply (21) and repeat the same arguments as in the proof of Proposition 7. □

For any sequence $(\varepsilon_i)_i$ we can choose a summable sequence $(\delta_i)_i$ and a sequence $(m_i)_i$ such that Propositions 7 and 8 hold for the constructed measure μ. In such a situation, by the Borel-Cantelli lemma for μ-almost every $x \in \Lambda$ both (28) and (32) are satisfied for all except finitely many i. This leads us to the following proposition.

Proposition 9. *If $(m_i)_i$ in the construction above increases sufficiently fast, then for μ-almost every $x \in \Lambda$ we have*

i) $\liminf_{m\to\infty} H_m(x) = \liminf_{i\to\infty} h_i$, $\limsup_{m\to\infty} H_m(x) = \limsup_{i\to\infty} h_i$,
ii) $\liminf_{m\to\infty} L_m(x) = \liminf_{i\to\infty} \chi_i$, $\limsup_{m\to\infty} L_m(x) = \limsup_{i\to\infty} \chi_i$, *and*
iii) $\liminf_{m\to\infty} \dfrac{H_m(x)}{L_m(x)} = \liminf_{i\to\infty} d_i$.

Moreover,

$$\dim_{\mathrm{H}} \mu \geq \liminf_{i\to\infty} d_i.$$

Proof. Given $j \geq 1$, we denote by

$$A_j \stackrel{\text{def}}{=} \bigcap_{i\geq j} (A_H(\varepsilon_i, i) \cap A_L(\varepsilon_i, i))$$

the set of points for which (28) and (32) are satisfied for all $i \geq j$ for some sequence $(\varepsilon_i)_i$ (which will be specified in the following).

Clearly $H_{m_1}(x) \in [H^-, H^+]$ for some $0 < H^-$, $H^+ < \infty$ independent of x (because there are only finitely many cylinders) and $L_{m_1}(x) \in [L^-, L^+]$ for some $0 < L^-$, $L^+ < \infty$ independent of x (because $\log |f'|$ is uniformly bounded on Λ).

Given $x \in A_j$, by (28) we have

$$\begin{aligned}&\left|H_{m_{j+k+1}}(x) - h_{j+k+1}\right| \\ &\quad \leq \left(\frac{m_{j+k}}{m_{j+k+1}} + \varepsilon_{j+k}\right) |h_{j+k+1} - h_{j+k}| + \frac{m_{j+k}}{m_{j+k+1}} \left|H_{m_{j+k}}(x) - h_{j+k}\right|,\end{aligned}$$

and hence for every $\ell \geq j$ we obtain

$$\left|\frac{H_{m_{\ell+1}}(x)}{h_{\ell+1}} - 1\right| \leq \left(\frac{m_\ell}{m_{\ell+1}} + \varepsilon_\ell\right) \left|\frac{h_\ell}{h_{\ell+1}} - 1\right| + \frac{m_\ell}{m_{\ell+1}} \left|\frac{H_{m_\ell}(x)}{h_\ell} - 1\right| \frac{h_\ell}{h_{\ell+1}}.$$

Now let us assume that $\varepsilon_i \to 0$ shrinks and $m_i \to \infty$ grows fast enough that for any $j \geq 1$

$$\frac{H_{m_i}(x)}{h_i} \to 1$$

converges uniformly in $x \in A_j$ as $i \to \infty$.

Note that $\{A_j\}$ is an increasing family of sets and that $\bigcup_{j\geq 1} A_j$ is of full measure μ. In the following let $j \geq 1$ be such that $\mu(A_j) > 0$.

Choose some $\varepsilon > 0$. Let $I = I(j)$ be sufficiently big such that for all $x \in A_j$ and all $i \geq I$ we have

$$\frac{H_{m_i}(x)}{h_i} \in (1-\varepsilon, 1+\varepsilon).$$

By (28), for all $m_i < m \leq m_{i+1}$ we have then

$$\left| H_m(x) - \frac{m_i}{m} h_i - \frac{m-m_i}{m} h_{i+1} \right| \leq \varepsilon_i \left| h_{i+1} - h_i \right| + \varepsilon. \tag{33}$$

Using (32) instead of (28), we can, possibly after changing $(\varepsilon_i)_i$ and $(m_i)_i$, prove in an analogous way that for all $m_i < m \leq m_{i+1}$ we have

$$\left| L_m(x) - \frac{m_i}{m} \chi_i - \frac{m-m_i}{m} \chi_{i+1} \right| \leq \varepsilon_i \left| \chi_{i+1} - \chi_i \right| + \varepsilon. \tag{34}$$

Notice that

$$h(m) \stackrel{\text{def}}{=} \frac{m_i}{m} h_i + \frac{m-m_i}{m} h_{i+1}$$

satisfies $h_i \leq h(m) \leq h_{i+1}$, which implies claim i) of the assertion. Claim ii) follows from (34) in an analogous way together with

$$\chi(m) \stackrel{\text{def}}{=} \frac{m_i}{m} \chi_i + \frac{m-m_i}{m} \chi_{i+1}$$

satisfying $\chi_i \leq \chi(m) \leq \chi_{i+1}$. As

$$\frac{h(m)}{\chi(m)} \geq \min\{d_i, d_{i+1}\}$$

for all $m_i < m \leq m_{i+1}$, claim iii) of the assertion follows from (33) and (34).

We finally prove the lower bound on the Hausdorff dimension of μ. For all $x \in A_j$ and for all $i \geq I$ we have

$$H_{m_i}(x) \geq (1-\varepsilon) h_i$$

and

$$L_{m_i}(x) \leq (1+\varepsilon) \chi_i.$$

Let $\widetilde{\mu}_j$ be the restriction of μ to A_j. For all $x \in A_j$ we have

$$\log \widetilde{\mu}_j(\Delta_m(x)) \leq \log \mu(\Delta_{m_i}(x)) = -m_i H_{m_i}(x) \leq -m_i(1-\varepsilon) h_i$$

and

$$\log |\Delta_{m_i}(x)| \geq -m_i \rho_{m_i} - m_i L_{m_i}(x) \geq -m_i(1+2\varepsilon)\chi_i$$

for m_i big enough, where we use (21) to obtain the second inequality. Let

$$r_i \stackrel{\text{def}}{=} e^{-m_i(1+2\varepsilon)\chi_i}.$$

We obtain

$$\widetilde{\mu}_j(B(x, r_i)) \leq 2 e^{-m_i(1-\varepsilon)h_i},$$

and hence the lower pointwise dimension of $\widetilde{\mu}_j$ at x is bounded by

$$\underline{d}_{\widetilde{\mu}_j}(x) \geq (1-3\varepsilon) \liminf_{i\to\infty} d_i.$$

Since ε was arbitrary, we obtain $\underline{d}_{\widetilde{\mu}_j}(x) \geq \liminf_{i\to\infty} d_i$ for every $x \in A_j$. Thus, we can conclude that $\dim_{\mathrm{H}} \mu \geq \liminf_{i\to\infty} d_i$. □

5.3. **The boundary of the spectrum – regular and irregular points.** The considerations in the previous section prove the following theorem.

Theorem 3. *Let $(n_i)_i$ be a nondecreasing sequence of positive integers and $(\mu_i)_i$ be a sequence of equilibrium states for $f|\Lambda_{n_i}$. Then*

$$\dim_{\mathrm{H}} \left\{ x \colon \underline{\chi}(x) = \liminf_{i\to\infty} \chi(\mu_i), \quad \overline{\chi}(x) = \limsup_{i\to\infty} \chi(\mu_i) \right\} \geq \liminf_{i\to\infty} \dim_{\mathrm{H}} \mu_i.$$

We now are able to bound the dimension of irregular points from below. We also obtain bounds on the dimension of the level sets at the boundary of the spectrum.

Proposition 10. *For $\alpha^- \leq \alpha \leq \beta \leq \alpha^+$ we have*

$$\dim_{\mathrm{H}} \widehat{\mathcal{L}}(\alpha, \beta) \geq \max_{\alpha \leq q \leq \beta} F(q)$$

and

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha, \beta) \geq \min_{\alpha \leq q \leq \beta} F(q).$$

In particular, we have

$$\dim_{\mathrm{H}} \mathcal{L}(\alpha) \geq F(\alpha).$$

Proof. Note that by our assumptions, $\alpha_n^- < \alpha_n^+$ for n big enough. We consider some sequence $(a_n)_n$ of numbers $\alpha_n^- < a_n < \alpha_n^+$ such that $\lim_{n\to\infty} a_n = \alpha$. By Lemma 4, there exists a sequence $(q_n)_n$ of numbers and a sequence $(\nu_n)_n$ of equilibrium states of the potentials $q_n \log|f'|$ such that

$$P_{f|\Lambda_n}\left(q_n \log|f'|\right) = h_{\nu_n}(f) + q_n a_n.$$

Applying [6, Theorem 1] to each of the hyperbolic sub-systems $f|\Lambda_n$ we obtain that

$$\dim_{\mathrm{H}} \nu_n = \frac{1}{a_n}\left(P_{f|\Lambda_n}(q_n \log|f'|) - q_n a_n\right) \geq F_{f|\Lambda_n}(a_n).$$

Note that continuity of the function F and Lemma 3 together imply that $\lim_{n\to\infty} F_{f|\Lambda_n}(a_n) = F(\alpha)$ for properly chosen $(a_n)_n$. The application of Theorem 3 finishes the proof. □

Proof of Theorem 1. The assertions follow from Proposition 10 and Proposition 4. □

Proof of Theorem 2. Recall the definition of $\widehat{\mathcal{L}}(\alpha)$ in (19). To prove the first part of assertion we use the fact that

$$\Lambda \setminus \mathcal{L}(0) = \widehat{\mathcal{L}}(0, \alpha^+) \cup \bigcup_{n\in\mathbb{N}} \widehat{\mathcal{L}}(2^{-n}\alpha^+)$$

If $\mathcal{L}(0) \neq \varnothing$, then $q \mapsto F(q)$ is a non-increasing function and hence Propositions 4 and 5 imply that the Hausdorff dimension of $\Lambda \setminus \mathcal{L}(0)$ is not greater than $F(0)$. At the same time, by Proposition 10 we have $\dim_{\rm H}\mathcal{L}(0)\ge F(0)$. This implies that $\dim_{\rm H}\Lambda=\dim_{\rm H}\mathcal{L}(0)\ge F(0)$.

The second part of the assertion of Theorem 2 we will prove in the following section. $\square$

6. Topological entropy

We first briefly recall one more concept from the thermodynamic formalism (for a detailed account we refer to [11]). Given a set $Z\subset\Lambda$ (not necessarily compact nor f-invariant) and numbers $\varepsilon>0$, $n\in\mathbb{N}$, we denote by $M_\varepsilon(Z,n)$ the maximal cardinality of a set of points in Z which belong to an (n,ε)-separated set in Λ. We define the *upper capacitive topological entropy* of f on Z by

$$\overline{Ch}(f|Z)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log M_\varepsilon(Z,n). \tag{35}$$

Analogously, we define the lower capacitive topological entropy of f on Z, denoted by $\underline{Ch}(f|Z)$, by replacing the limsup in (35) with the liminf. Since $f|\Lambda$ is expansive, it in fact suffices for sufficiently small ε to take in (35) only the limit in n. We denote by $h(f|Z)$ the *topological entropy* of f on Z, but we refer the reader to [11] for its precise definition. The following properties hold:

1. $\underline{Ch}(f|Z_1)\le\underline{Ch}(f|Z_2)$ and $\overline{Ch}(f|Z_1)\le\overline{Ch}(f|Z_2)$ whenever $Z_1\subset Z_2\subset\Lambda$,
2. $h(f|Z)\le\underline{Ch}(f|Z)\le\overline{Ch}(f|Z)$,
3. for a countable union $Z=\bigcup_{k\in\mathcal{I}}Z_i$ we have $h(f|Z)=\sup_{k\in\mathcal{I}}h(f|Z_i)$.

Moreover, when $Z\subset\Lambda$ is f-invariant and compact then we have coincidence with the classical topological entropy with respect to $f|Z$, that is,

$$h(f|Z)=\underline{Ch}(f|Z)=\overline{Ch}(f|Z).$$

Recall that $\Delta_n(x)$ denotes the cylinder $\Delta_{i_1\ldots i_n}$ containing the point x.

Lemma 5. $\mathcal{L}(0)=\left\{x\in\Lambda\colon \liminf_{n\to\infty}\frac{1}{n}\log|\Delta_n(x)|=0\right\}.$

Proof. Given $x\in\Lambda$, we have $f^n(\Delta_n(x))\supset I_i$ for some $i\in\{1,\ldots,p\}$. By the mean value theorem there exists $y=y(n)\in\Delta_n(x)$ such that $|(f^n)'(y)|\,|\Delta_n(x)|\ge I_i$ and hence

$$|I|^{-1}\le|\Delta_n(x)|^{-1}\le|I_i|^{-1}\left|(f^n)'(x)\right|\frac{|(f^n)'(y)|}{|(f^n)'(x)|}.$$

By the tempered distortion property we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log\sup_{y\in\Delta_n(x)}\frac{|(f^n)'(y)|}{|(f^n)'(x)|}=0.$$

from here the statement follows. $\square$

Proposition 11. *We have* $h(f|\mathcal{L}(0))=\underline{Ch}(f|\mathcal{L}(0))=\overline{Ch}(f|\mathcal{L}(0))=0.$

Proof. We are going to show the existence of a decreasing sequence of sets $\mathcal{L}_k$, all containing $\mathcal{L}(0)$, such that $\overline{Ch}(f|\mathcal{L}_k)$ decreases to 0.

Given $\varepsilon > 0$ and $N \in \mathbb{N}$ we define the set

$$L_{\varepsilon,N} = \{x \in \Lambda\colon |\Delta_n(x)| \geq (1+\varepsilon)^{-n} \text{ for every } n \geq N\}\,.$$

Notice that $L_{\varepsilon,N} \subset L_{\varepsilon,N'}$ for $N \leq N'$ and that by Lemma 5

$$\mathcal{L}(0) \subset \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} L_{\varepsilon,N}.$$

For every $x \in L_{\varepsilon,N}$ we have $|\Delta_n(x)| \geq (1+\varepsilon)^{-n}$ for every $n \geq N$. Hence, the number of (n,ε)-separated sets needed to cover the set $L_{\varepsilon,N}$ is at most $I(1+\varepsilon)^n$. From the definition of $\overline{Ch}$ we obtain $\overline{Ch}(f|L_{\varepsilon,N}) \leq \log(1+\varepsilon)$. Now it follows that

$$h(f|\mathcal{L}(0)) \leq h(f|\bigcup_{N\in\mathbb{N}} L_{\varepsilon,N}) = \sup_{N\in\mathbb{N}} h(f|L_{\varepsilon,N}) \leq \sup_{N\in\mathbb{N}} \overline{Ch}(f|L_{\varepsilon,N}) \leq \log(1+\varepsilon)\,.$$

Since ε is arbitrary, we can conclude $h(f|\mathcal{L}(0)) = 0$. $\square$

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