Normal numbers with given limits of multiple ergodic averages

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Abstract

We are interested in the set of normal sequences in the space $\{0,1\}^{\mathbb{N}}$ with a given frequency of the pattern 11 in the positions $k, 2k$. The topological entropy of such sets is determined.

1 Introduction and statement of results

Let $\Sigma = \{0,1\}^{\mathbb{N}}$. In [K12, FLM12], the authors proposed to calculate the topological entropy spectrum of level sets of multiple ergodic averages. Here, the topological entropy means Bowen's topological entropy (in the sense of [B73]) which can be defined for any subset, not necessarily invariant. Among other questions, they asked for the topological entropy of

$$A_\alpha := \left\{ (\omega_k)_1^\infty \in \Sigma : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k \omega_{2k} = \alpha \right\} \qquad (\alpha \in [0,1]).$$

As a first step to solve the question, they also suggested to study a subset of A_0:

$$A := \left\{ (\omega_k)_1^\infty \in \Sigma : \omega_k \omega_{2k} = 0 \quad \text{for all } k \geq 1 \right\}.$$

The topological entropy of A was later given by Kenyon, Peres and Solomyak [KPS12].

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Theorem 1.1 (Kenyon-Peres-Solomyak)**.** *We have*

$$h_{\text{top}}(A) = -\log(1-p) = 0.562399...,$$

where $p \in [0,1]$ *is the unique solution of*

$$p^2 = (1-p)^3.$$

Enlightened by the idea of [KPS12], the question about A_α was finally answered by Peres and Solomyak [PS12], and then in higher generality by Fan, Schmeling and Wu [FSW16].

Theorem 1.2 (Peres-Solomyak, Fan-Schmeling-Wu)**.** *For any* $\alpha \in [0,1]$*, we have*

$$h_{\text{top}}(A_\alpha) = -\log(1-p) - \frac{\alpha}{2}\log\frac{q(1-p)}{p(1-q)},$$

where $(p,q) \in [0,1]^2$ *is the unique solution of the system*

$$\begin{cases} p^2(1-q) = (1-p)^3, \\ 2pq = \alpha(2+p-q). \end{cases}$$

In particular, $h_{\text{top}}(A_0) = h_{\text{top}}(A)$*.*

Another, interesting, related set is

$$B := \Big\{(\omega_k)_1^\infty \in \Sigma : \omega_k = \omega_{2k} \quad \text{for all } k \geq 1\Big\}.$$

The sequence $x \in \{0,1\}^{\mathbb{N}}$ is said to be simple normal if the frequency of the digit 0 in the sequence is 1/2. It is said to be normal if for all $n \in \mathbb{N}$, each word in $\{0,1\}^{\mathbb{N}}$ of length n has frequency $1/2^n$. We denote the set of normal sequences by $\mathcal{N}$.

We are interested in the intersection of $\mathcal{N}$ with the set A_α of given frequency of the pattern 11 in $w_k w_{2k}$. For the usual ergodic (Birkhoff) averages the normal numbers all belong to one set in the multifractal decomposition – the situation for multiple ergodic averages turns out to be very different.

Our results are as follows:

Theorem 1.3. *For* $\alpha \leq 1/2$ *we have*

$$h_{\text{top}}(\mathcal{N} \cap A_\alpha) = \frac{1}{2}\log 2 + \frac{1}{2}H(2\alpha),$$

where $H(x) = -x\log x - (1-x)\log(1-x)$*. For* $\alpha > 1/2$ *the set* $\mathcal{N} \cap A_\alpha$ *is empty.*

Further,

$$h_{\text{top}}(\mathcal{N} \cap A) = h_{\text{top}}(\mathcal{N} \cap A_0) = \frac{1}{2}\log 2.$$

Moreover, $\mathcal{N} \cap B \subset A_{1/2}$ *and*

$$h_{\text{top}}(\mathcal{N} \cap B) = h_{\text{top}}(\mathcal{N} \cap A_{1/2}) = h_{\text{top}}(B) = \frac{1}{2}\log 2.$$

The last statement of Theorem 1.3 was recently proved, in higher generality, in [ABC].

Let us now define the set of sequences with prescribed frequency of 0's and 1's:

$$E_\theta := \{x \in [0,1] : \lim_{n\to\infty} \frac{\omega_1(x) + \cdots + \omega_n(x)}{n} = \theta\}.$$

In particular, $E_{1/2}$ is the set of simple normal sequences.

Theorem 1.4. *We have*

$$h_{\text{top}}(E_\theta \cap A_\alpha) = (1 - \frac{\theta}{2})H(\frac{2\theta - \alpha}{2 - \theta}) + \frac{\theta}{2}H(\frac{\theta - \alpha}{\theta})$$

for $\alpha \leq \theta \leq (2+\alpha)/3$, otherwise $E_\theta \cap A_\alpha = \emptyset$. Further,

$$h_{\text{top}}(E_\theta \cap A) = h_{\text{top}}(E_\theta \cap A_0) = \frac{2-\theta}{2}H(\frac{2\theta}{2-\theta}).$$

Note that

$$h_{\text{top}}(E_{1/2} \cap A) = \frac{3}{4}H(\frac{2}{3}) > h_{\text{top}}(\mathcal{N} \cap A).$$

Remark. Applying the results of [PS12] one can show that

$$h_{\text{top}}(E_\theta \cap A_\alpha) = h_{\text{top}}(A_\alpha)$$

if and only if α, θ satisfy the relation

$$(2\theta - \alpha)^2(\theta - \alpha)(2 - \theta) = \theta(2 - 3\theta + \alpha)^3.$$

In particular, when

$$\theta = \frac{2}{3}\left(1 + \left(\frac{2}{23}\right)^{2/3}\sqrt[3]{3\sqrt{69} - 23} - \left(\frac{2}{23}\right)^{2/3}\sqrt[3]{3\sqrt{69} + 23}\right) = 0.354...,$$

i.e., the unique real solution of the equation $4\theta^2(2-\theta) = (2-3\theta)^3$,we have

$$\dim_H E_\theta \cap A = \dim_H A.$$

We omit the details.

2 Proof of Theorem 1.3

Given $\alpha \in [0,1]$, let μ_α be a probability measure on Σ given by

- if k is odd then $\omega_k = 1$ with probability $1/2$,
- if k is even and $\omega_{k/2} = 1$ then $\omega_k = 1$ with probability 2α,
- if k is even and $\omega_{k/2} = 0$ then $\omega_k = 1$ with probability $1 - 2\alpha$,

with the events $\{\omega_k = 1\}$ and $\{\omega_\ell = 1\}$ independent except when k/ℓ is a power of 2. Precisely, let $(p_0, p_1) := (1/2, 1/2)$ and let

$$\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} := \begin{pmatrix} 2\alpha & 1-2\alpha \\ 1-2\alpha & 2\alpha \end{pmatrix}.$$

Let $C_n(\omega_1, \cdots, \omega_n)$ be the set of sequences beginning with the word $\omega_1 \cdots \omega_n \in \{0,1\}^n$. Such sets are called cylinders of order n. The measure μ_α of a cylinder is given by

$$\mu_\alpha([\omega_1 \cdots \omega_n]) = \prod_{k=1}^{\lceil n/2 \rceil} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_k \omega_{2k}} = \frac{1}{2^{\lceil n/2 \rceil}} \cdot \prod_{k=1}^{\lfloor n/2 \rfloor} p_{\omega_k \omega_{2k}}.$$

where $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ denote the ceiling function and the integer part function correspondingly.

We will prove that the measure μ_α is suppored on the set $\mathcal{N} \cap A_\alpha$.

Lemma 2.1. *We have*

$$\mu_\alpha(\mathcal{N} \cap A_\alpha) = 1.$$

Proof. Denote

$$x_n(\omega) = \frac{2}{n} \sum_{k=n/2+1}^{n} \omega_k.$$

For a μ_α-typical ω, the Law of Large Numbers implies

$$x_{2n}(\omega) = \frac{1}{4} + \frac{x_n(\omega)}{2} 2\alpha + \frac{1 - x_n(\omega)}{2}(1 - 2\alpha) + o(1).$$

Noting that $|\frac{4\alpha - 1}{2}| < 1$, we have as $k \to \infty$,

$$x_{2^k n}(\omega) \to \frac{1}{2}.$$

By [PS12, Lemma 5], this implies that μ_α-almost surely

$$\lim_{n \to \infty} x_n(\omega) = \frac{1}{2}. \tag{2.1}$$

Then, for μ_α-a.e. ω,

$$\frac{2}{n}\sum_{k=n/2+1}^{n}\omega_k\omega_{2k} = x_n(\omega)(2\alpha + o(1)) \to \alpha.$$

Thus $\mu_\alpha(A_\alpha) = 1$.

Now, we show $\mu(\mathcal{N}) = 1$. We can divide the set of natural numbers into infinitely many subsets of the form $A_k = \{2k-1, 4k-2, \ldots, 2^\ell(2k-1), \ldots\}$ $(k \geq 1)$. Let $\mathcal{B}_k$ be the σ-field generated by events $\{\omega_{2^\ell(2k-1)} = 1\}$, $\ell \in \mathbb{N}$. Observe that for the measure μ the σ-fields $\mathcal{B}_k$ are independent.

Observe further that $\mu(\omega_{2^\ell(2k-1)} = 1) = 1/2$ for every k, ℓ. Indeed, for $\ell = 0$ it follows from the definition of μ, and then it is proved by induction:

$$\begin{aligned}
&\mu(\omega_{2^\ell(2k-1)} = 1) \\
=&\mu(\omega_{2^\ell(2k-1)} = 1 \wedge \omega_{2^{\ell-1}(2k-1)} = 1) + \mu(\omega_{2^\ell(2k-1)} = 1 \wedge \omega_{2^{\ell-1}(2k-1)} = 0) \\
=&2\alpha \cdot 1/2 + (1-2\alpha)\cdot 1/2 = 1/2.
\end{aligned}$$

Consider now, for any n, the sequence $\omega_{m+1}, \ldots, \omega_{m+n}$. If $m \geq n$ then positions $m+1, \ldots, m+n$ come all from different A_k's, thus $\omega_{m+1}, \ldots, \omega_{m+n}$ are independent and each of them takes values $0, 1$ with probability $1/2$ respectively. That is, the measure μ restricted to such subset of positions is $(\frac{1}{2}, \frac{1}{2})$-Bernoulli, and for any word $\eta \in \{0,1\}^n$ with $n \leq m$, the probability that we have $\omega_{m+i} = \eta_i$ for $i = 1, \ldots, n$ equals 2^{-n}. Thus, for a given word $\eta \in \{0,1\}^n$ we can divide $\mathbb{N}$ into intervals $[2^j+1, 2^{j+1}]$, inside all except initial finitely many of them (with $j < \log_2 n$) for any μ-generic sequence ω the frequency of appearance of η equals $2^{-n} + O(2^{-j/2} j \log j)$, and this means that the μ-generic sequence ω is normal.

Next, we will calculate the local dimension of the measure μ_α with the help of Mass Distribution Principle, [**?**, ?]. We denote for $x \in [0,1]$

$$H(x) = -x\log x - (1-x)\log(1-x)$$

with convention $H(0) = H(1) = 0$.

Lemma 2.2. *We have*

$$h_{\mu_\alpha} = \frac{1}{2}\log 2 + H(2\alpha).$$

Proof. For $\omega \in \Sigma$ denote

$$C_n(\omega) = \{\tau \in \Sigma; \tau_k = \omega_k\ \forall k \leq n\}.$$

Let

$$h_n(\omega) := \log \mu_\alpha(C_{2n}(\omega)) - \log \mu_\alpha(C_n(\omega)).$$

By the Law of Large Numbers, for μ_α-typical ω and for big enough n we have

$$\frac{2}{n}h_n(\omega) = -\log 2 + (1 - x_n(\omega))(2\alpha\log(2\alpha) + (1-2\alpha)\log(2\alpha)) \\ + x_n(\omega)((1-2\alpha)\log(1-2\alpha)) + (2\alpha)\log(2\alpha)) + o(1).$$

Thus,

$$\lim_{n\to\infty} -\frac{1}{n}h_n(\omega) = \frac{1}{2}\log 2 + \frac{1}{2}H(2\alpha) \qquad \mu_\alpha - a.e..$$

Note that for all $k, n \in \mathbb{N}$

$$\frac{1}{k2^n}\log\mu_\alpha(C_{k2^n}(\omega)) = \frac{1}{k2^n}\sum_{i=1}^{n-1} h_{2^i}.$$

Then for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} -\frac{1}{k2^n}\log\mu_\alpha(C_{k2^n}(\omega)) = \frac{1}{2}\log 2 + \frac{1}{2}H(2\alpha) \qquad \mu_\alpha - a.e..$$

Hence, by [PS12, Lemma 5],

$$h_{\mu_\alpha} = \liminf_{n\to\infty} -\frac{1}{n}\mu_\alpha(C_n(\omega)) = \frac{1}{2}\log 2 + \frac{1}{2}H(2\alpha) \qquad \mu_\alpha - a.e..$$

Applying the Mass Distribution Principle ends the proof. $\square$

To finish the proof of the lower bound we note that $A \subset A_0$ but the measure μ_0 is actually supported on A, that the measure $\mu_{1/2}$ is supported on B, and that the relation $\mathcal{N} \cap B \subset A_{1/2}$ follows from

$$\frac{1}{n}\sharp\{n+1 \le j \le 2n : \omega_j = \omega_{2j} = 1\} = \frac{1}{n}\sharp\{n+1 \le j \le 2n : \omega_j = 1\} \to \frac{1}{2}$$

being satisfied for every $\omega \in \mathcal{N} \cap B$.

For the upper bound, let us first observe that

$$\frac{1}{n}\sum_{k=1}^{n}\omega_k\omega_{2k} \le \frac{1}{n}\sum_{k=1}^{n}\omega_k$$

and the right hand side converges to $1/2$ for every normal sequence ω. Thus, the set $\mathcal{N} \cap A_\alpha$ is empty for all $\alpha > 1/2$.

We will now need the following lemma

Lemma 2.3. *Let ω be a normal sequence and let $(n_k = \ell_1 + k\ell_2)$ be an arithmetic subsequence of $\mathbb{N}$. Then ω restricted to the positions (n_k) is normal.*

Proof. This is a well-known result of Kamae [K73]. $\square$

Let us fix some $m > 0$. For $N > m$ and $i = 0, 1, \ldots, m$ denote by $R(N,i)$ the set $\{2^i(2k-1), k \le 2^{N-i-1}\}$ (for example, $R(N,0)$ is the set of odd numbers smaller than 2^N). Further, let $R(N,i,I) = R(N-2,i)$, $R(N,i,II) = R(N-1,i) \setminus R(N-2,i)$, and $R(N,i,III) = R(N,i) \setminus R(N-1,i)$. Note here obvious relations

$$
\begin{aligned}
&2R(N,i,I) = R(N,i+1,I) \cup R(N,i+1,II),\\
&2R(N,i,II) = R(N,i+1,III),\\
&2R(N,i,III) \cap R(N,i+1) = \emptyset.
\end{aligned}
$$

We denote by $\mathcal{N}(N,m,\varepsilon)$ the set of sequences ω such that for all $n \ge N$ in each $R(n,i,*)$, $i = 0, \ldots, m$, $* \in \{I, II, III\}$ the frequency of 1's is between $1/2 - \varepsilon$ and $1/2 + \varepsilon$. By Lemma 2.3,

$$
\mathcal{N} \subset \bigcap_{\varepsilon>0} \bigcap_{m=1}^{\infty} \bigcup_{N=m+1}^{\infty} \mathcal{N}(N,m,\varepsilon).
$$

Similarly, let us denote by $A(\alpha, N, \varepsilon)$ the set of sequences ω such that for all $n \ge N$ we have

$$
\alpha - \varepsilon < 2^{-n+1} \sum_{j=1}^{2^{n-1}} \omega_j \omega_{2j} < \alpha + \varepsilon.
$$

We have

$$
A_\alpha = \bigcap_{\varepsilon>0} \bigcup_{N=1}^{\infty} A(\alpha, N, \varepsilon).
$$

To obtain the upper bound, we will estimate from above the number of cylinders $[\omega_1, \ldots, \omega_{2^N}]$ needed to cover the set $\mathcal{N}(N,m,\varepsilon) \cap A(\alpha,N,\varepsilon)$. Let us fix N, m, ε. For $i = 1, \ldots, m$, $k_1, k_2 \in \{0,1\}$, and $* \in \{I, II\}$ we denote

$$
X^i_{k_1k_2,*}(\omega) = \sharp\{n \in R(N,i-1,*); \omega_n = k_1, \omega_{2n} = k_2\}.
$$

For example, $X^1_{01,I}(\omega)$ denotes the number of odd positions smaller than 2^{N-2} such that $\omega_n = 0, \omega_{2n} = 1$. Similarly, let

$$
X^i_{k_1,*}(\omega) = \sharp\{n \in R(N,i,*); \omega_n = k_1\}.
$$

We have obvious relations: for any i

$$
X^i_{10,I} + X^i_{11,I} = X^{i-1}_{1,I}
$$

$$
X^i_{00,I} + X^i_{01,I} = X^{i-1}_{0,I}
$$

$$
X^i_{10,II} + X^i_{11,II} = X^{i-1}_{1,II}
$$

$$
\begin{aligned}
X^i_{00,II} + X^i_{01,II} &= X^{i-1}_{0,II}\\
X^i_{01,I} + X^i_{11,I} &= X^i_{1,I} + X^i_{1,II}\\
X^i_{00,I} + X^i_{10,I} &= X^i_{0,I} + X^i_{0,II}\\
X^i_{01,II} + X^i_{11,II} &= X^i_{1,III}\\
X^i_{00,II} + X^i_{10,II} &= X^i_{0,III}.
\end{aligned}
$$

Note that for a sequence $\omega \in N(N,m,\varepsilon)$ the right hand sides in all those relations is in range $2^{N-3-i} \cdot (1-\varepsilon, 1+\varepsilon)$. In particular,

$$|X^i_{11,I} - X^i_{00,I}| \leq \varepsilon \cdot 2^{N-2-i}.$$

We can now start the counting. The values of $\{\omega_n; n \in R(N,0)\}$ can be chosen in no more than $2^{2^{N-1}}$ ways. After we have chosen $\{\omega_n; n \in R(N, i-1)\}$, we can choose $\{\omega_n; n \in R(N,i)\}$ in no more than

$$\binom{X^{i-1}_{1,I}}{X^i_{11,I}} \cdot \binom{X^{i-1}_{0,I}}{X^i_{00,I}} \cdot \binom{X^{i-1}_{1,II}}{X^i_{11,II}} \cdot \binom{X^{i-1}_{0,II}}{X^i_{00,II}}$$

ways. Finally, after we have chosen $\{\omega_n; n \in R(N,i)\}$ for all $i \leq m$, we will still have 2^{N-m-1} positions left, which we can cover in no more than $2^{2^{N-m-1}}$ ways. Thus, for any choice of $(X^i_{00,I}, X^i_{11,I}, X^i_{00,II}, X^i_{11,II})_i$ the logarithm of total number of cylinders needed Z is not larger than

$$
\begin{aligned}
&\log Z((X^i_{00,I}, X^i_{11,I}, X^i_{00,II}, X^i_{11,II})_i)\\
\leq&(2^{N-1} + 2^{N-m-1}) \log 2\\
&+ \sum_{i=1}^{m} \left(2 \log \binom{2^{N-3-i}}{X^i_{11,I}} + 2 \log \binom{2^{N-3-i}}{X^i_{11,II}} + 2^{N-3-i} O(\varepsilon) \right)
\end{aligned}
$$

and there are no more than $\prod_{i=1}^m 2^{4(N-i-3)} < 2^{4mN} \ll 2^{2^N}$ such choices.

We estimate

$$\log \binom{n}{k} \approx n \left(-\frac{k}{n} \log \frac{k}{n} - \frac{n-k}{n} \log \frac{n-k}{n} \right) = nH(\frac{k}{n})$$

and observe that H is a concave function, thus we can apply Jensen inequality. We get

$$
\begin{aligned}
&\log Z((X^i_{00,I}, X^i_{11,I}, X^i_{00,II}, X^i_{11,II})_i)\\
\leq&(2^{N-1} + 2^{N-m-1}) \log 2\\
&+ \sum_{i=1}^{m} 2^{N-i-1} \cdot H \left(\frac{1}{\sum_{i=1}^m 2^{N-i-1}} \cdot \sum_{i=1}^{m} 2^{N-i-2} \frac{X^i_{11,I} + X^i_{11,II}}{2^{N-i-3}} \right)\\
&+ \sum_{i=1}^{m} 2^{N-i-3} \cdot O(\varepsilon).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\log Z((X^i_{00,I}, X^i_{11,I}, X^i_{00,II}, X^i_{11,II})_i)\\
\leq& 2^{N-1}\log 2 + 2^{N-1} H\left(2^{-N+1}\sum_{i=1}^{m}(X^i_{11,I}+X^i_{11,II})\right) + 2^N\cdot(O(\varepsilon+2^{-m})).
\end{aligned}
$$

On the other hand, for all $\omega \in A(\alpha, N, \varepsilon)$,

$$
\left|2^{-N+1}\sum_{i=1}^{m}(X^i_{11,I}+X^i_{11,II}) - 2\alpha\right| < \varepsilon.
$$

Passing with m, N to infinity and with ε to 0, we finish the proof of the upper bound. $\square$

3 Proof of Theorem 1.4

Given $p, q \in [0,1]$, let $\mu_{p,q}$ be a probability measure on S given by

- if k is odd then $\omega_k = 1$ with probability p,
- if k is even and $\omega_{k/2} = 0$ then $\omega_k = 1$ with probability p,
- if k is even and $\omega_{k/2} = 1$ then $\omega_k = 1$ with probability q,

with events $(\omega_k = 1)$ and $(\omega_\ell = 1)$ independent except when k/ℓ is a power of 2. Precisely, let $(p_0, p_1) := (1-p, p)$ and let

$$
\begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} := \begin{pmatrix} 1-p & p \\ 1-q & q \end{pmatrix}.
$$

Then the measure $\mu_{p,q}$ of a cylinder is given by

$$
\mu_{p,q}([\omega_1\cdots\omega_n]) = \prod_{k=1}^{\lceil n/2\rceil} p_{\omega_{2k-1}} \cdot \prod_{k=1}^{\lfloor n/2\rfloor} p_{\omega_k\omega_{2k}}.
$$

where $\lceil\cdot\rceil$, $\lfloor\cdot\rfloor$ denote the ceiling function and the integer part function correspondingly.

For positive integers $m < n$, write ω_m^n for the word $\omega_m\omega_{m+1}\cdots\omega_n$. For $i, j \in \{0,1\}$ and $\omega \in \Sigma$, denote

$$
N_i(\omega_m^n) = \sharp\{m \leq k \leq n : \omega_k = i\},
$$

and

$$
N_{ij}(\omega_m^n) = \sharp\{m \leq k \leq n : \omega_k\omega_{2k} = ij\}.
$$

We also denote

$$
N_{i,\mathrm{odd}}(\omega_m^n) = \sharp\{m \leq k \leq n : k \text{ odd},\ \omega_k = i\}.
$$

Then we have

$$\mu_{p,q}(C_n(\omega)) = (1-p)^{N_{0,\mathrm{odd}}} p^{N_{1,\mathrm{odd}}} (1-p)^{N_{00}} p^{N_{01}} (1-q)^{N_{10}} q^{N_{11}},$$

with $N_{i,\mathrm{odd}} = N_{i,\mathrm{odd}}(\omega_1^n)$, and $N_{ij} = N_{ij}(\omega_1^{n/2})$. Thus

$$\begin{aligned}\frac{-\log \mu_{p,q}(C_n(\omega))}{n} =& -\frac{1}{2}\Big(\frac{N_{0,\mathrm{odd}}}{n/2}\log(1-p) + \frac{N_{1,\mathrm{odd}}}{n/2}\log p \\ &+ \frac{N_{00}}{n/2}\log(1-p) + \frac{N_{01}}{n/2}\log p \\ &+ \frac{N_{10}}{n/2}\log(1-q) + \frac{N_{11}}{n/2}\log q\Big).\end{aligned} \tag{3.1}$$

Lemma 3.1. *If $p = (2\theta - \alpha)/(2-\theta)$ and $q = \alpha/\theta$, then*

$$\mu_{p,q}(E_\theta \cap A_\alpha) = 1.$$

Proof. Denote

$$x_n(\omega) = \frac{N_1(\omega_{n/2+1}^n)}{n/2} = \frac{2}{n}\sum_{k=n/2+1}^{n} \omega_k.$$

By the Law of Large Numbers, for $\mu_{p,q}$-almost all ω

$$x_{2n}(\omega) = \frac{p}{2} + \frac{x_n(\omega)}{2}q + \frac{1-x_n(\omega)}{2}p + o(1) = p + \frac{x_n(\omega)}{2}\cdot\frac{q-p}{2} + o(1).$$

Note that $|\frac{q-p}{2}| < 1$. Then, as $k \to \infty$,

$$x_{2^k n}(\omega) \to \frac{2p}{2+p-q}.$$

By [PS12, Lemma 5], it implies that $\mu_{p,q}$-almost surely

$$\lim_{n\to\infty} x_n(\omega) = \frac{2p}{2+p-q} = \theta, \tag{3.2}$$

where the last equality comes from the choices of p and q. Thus $\mu_{p,q}(E_\theta) = 1$.

On the other hand, by applying the Law of Large Numbers again, for $\mu_{p,q}$-a.e. ω,

$$\frac{2}{n}\sum_{k=n/2+1}^{n} \omega_k \omega_{2k} = x_n(\omega)(q + o(1)) \to q\theta = \alpha.$$

By [PS12, Lemma 5], we conclude $\mu_{p,q}(A_\alpha) = 1$.

□

Lemma 3.2. *For $p = (2\theta - \alpha)/(2-\theta)$ and $q = \alpha/\theta$, we have*

$$h_{\mu_{p,q}} = (1 - \frac{\theta}{2})H(\frac{2\theta-\alpha}{2-\theta}) + \frac{\theta}{2}H(\frac{\theta-\alpha}{\theta}).$$

Proof. By (3.1), we have for $\mu_{p,q}$ almost all $w \in \Sigma$,

$$\begin{aligned}
h_{\mu_{p,q}} =& \lim_{n\to\infty} \frac{-\log \mu_{p,q}(C_n(\omega))}{n} \\
=& -\frac{1}{2}\Big((1-p)\log(1-p) + p\log p + (1-\theta)(1-p)\log(1-p) \\
& + (1-\theta)p\log p + \theta(1-q)\log(1-q) + \theta q\log q\Big) \\
=& \frac{1}{2}\Big((2-\theta)H(p) + \theta H(q)\Big) \\
=& (1-\frac{\theta}{2})H(\frac{2\theta-\alpha}{2-\theta}) + \frac{\theta}{2}H(\frac{\theta-\alpha}{\theta}).
\end{aligned}$$

□

Lemma 3.3. *If $\theta \notin [\alpha, (2+\alpha)/3]$ we have $E_\theta \cap A_\alpha = \emptyset$, otherwise for $p = (2\theta-\alpha)/(2-\theta)$ and $q = \alpha/\theta$, we have for all $x \in E_\theta \cap A_\alpha$,*

$$\lim_{n\to\infty} \frac{-\log \mu_{p,q}(C_n(\omega))}{n} = (1-\frac{\theta}{2})H(\frac{2\theta-\alpha}{2-\theta}) + \frac{\theta}{2}H(\frac{\theta-\alpha}{\theta}).$$

Proof. Observe that for any $x \in E_\theta \cap A_\alpha$, for any small $\varepsilon > 0$, for n large enough, we have

$$\begin{aligned}
N_1(\omega_{n/2}^n) &\in [\frac{\theta n}{2}(1-\varepsilon), \frac{\theta n}{2}(1+\varepsilon)] \\
N_1(\omega_n^{2n}) &\in [\theta n(1-\varepsilon), \theta n(1+\varepsilon)] \\
N_{11}(\omega_n^{2n}) &\in [\frac{\alpha n}{2}(1-\varepsilon), \frac{\alpha n}{2}(1+\varepsilon)].
\end{aligned}$$

The obvious inequalities $N_{11}(\omega_n^{2n}) \le N_1(\omega_{n/2}^n)$ and $N_1(\omega_n^{2n}) - N_{11}(\omega_n^{2n}) \le n/2 + N_0(\omega_{n/2}^n) = n - N_1(\omega_{n/2}^n)$ imply $\theta \in [\alpha, (2+\alpha)/3]$. Furthermore, we have

$$\begin{aligned}
& \log \mu_{p,q}(C_{2n}(\omega)) - \log \mu_{p,q}(C_n(\omega)) \\
=& N_{11}(\omega_n^{2n})\log q + \big(N_1(\omega_{n/2}^n) - N_{11}(\omega_n^{2n})\big)\log(1-q) \\
& + \big(N_1(\omega_n^{2n}) - N_{11}(\omega_n^{2n})\big)\log p \\
& + \big(n - N_1(\omega_{n/2}^n) - N_1(\omega_n^{2n}) + N_{11}(\omega_n^{2n})\big)\log(1-p) \\
=& n\left((1-\frac{\theta}{2})H(\frac{2\theta-\alpha}{2-\theta}) + \frac{\theta}{2}H(\frac{\theta-\alpha}{\theta}) + \varepsilon\cdot O(1)\right).
\end{aligned}$$

Hence by the same argument of the proof of Lemma 2.2, we have for all $x \in E_\theta \cap A_\alpha$,

$$\lim_{n\to\infty} \frac{-\log \mu_{p,q}(C_n(\omega))}{n} = (1-\frac{\theta}{2})H(\frac{2\theta-\alpha}{2-\theta}) + \frac{\theta}{2}H(\frac{\theta-\alpha}{\theta}).$$

□

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