THE DIMENSION OF PROJECTIONS OF FRACTAL PERCOLATIONS

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ABSTRACT. In this paper we consider a very well studied a family of random Cantor sets E, the so called fractal percolation or Mandelbrot percolation. We prove that for almost all realizations and for **all** angle θ, the size (Hausdorff dimension) of the the angle θ projection of the random set E is as big as possible. We apply the same method to prove the existence of some intervals in the algebraic sums and distance sets of random Cantor sets.

1. NOTATION

1.1. **The d-dimensional fractal percolation with parameters M, p.** Mandelbrot in the 1970's introduced a family of random sets to study turbulence. This family is called fractal percolation (or Mandelbrot percolation or canonical curdling). The intuitive definition provided below is given in $\mathbb{R}^d$ for an arbitrary $d \geq 1$ (for the precise definition see [4, Section 2.2]). Given a natural number $M \geq 2$ and a probability $0 < p < 1$. Throughout the construction we define a random, nested sequence E_n which is the union of some randomly chosen level-n cubes. These are the M-adic cubes, that is coordinate-hyperplane parallel cubes of side length M^{-n} with centers chosen from

$$\boldsymbol{\mathcal{N}_n} := \left\{ \mathbf{x} = (x_1, \dots, x_d) : x_i = \left(k_i + \frac{1}{2}\right) M^{-n},\ 0 \leq k_i \leq M^n - 1 \right\}.$$

We denote the level-n cube with center $\mathbf{x} \in \mathcal{N}_n$ by $K_n(\mathbf{x})$.

$$K_n(\mathbf{x}) = \mathbf{x} + \left[-\frac{1}{2M^n}, \frac{1}{2M^n} \right]^d.$$

2000 *Mathematics Subject Classification.* Primary 28A80 Secondary 60J80, 60J85

Key words and phrases. Random fractals, Hausdorff dimension, processes in random environment.

Rams was partially supported by the MNiSW grant N201 607640 (Poland). The research of Simon was supported by OTKA Foundation # K 104745.

The level-n cubes can be identified with their centers. Therefore (slightly abusing the notations) we also write $\mathcal{N}_n$ for the collections of level n cubes.
Now the construction of the fractal percolation is as follows: We retain the cube $K_1(\mathbf{x})$ with probability p and we discard it with probability $1-p$, independently for every $\mathbf{x} \in \mathcal{N}_1$. The union of cubes retained is called E_1. For every retained cube $K_1(\mathbf{x})$ we repeat the process described above in $K_1(\mathbf{x})$ independently. The union of retained level-2 cubes is the random set $E_2 \subset E_1$. We continue this process at infinitum to obtain E_n for every n in each step independently of everything. Clearly, $E_n \subset E_{n-1}$. For an $n \geq 1$ set

$$\boldsymbol{\mathcal{E}_n} := \{\mathbf{x} \in \mathcal{N}_n : K_n(\mathbf{x}) \text{ is retained}\}. \tag{1.1}$$

The *d-dimensional fractal percolation with parameters M, p* is the random set $E = E(d, M, p)$ that remains after infinitely many steps.

$$\boldsymbol{E} := \bigcap_{n=1}^{\infty} E_n.$$

We call E_n the *n-th approximation of the fractal percolation*. The corresponding probability space $\boldsymbol{(\Omega, \mathcal{F}, \mathbb{P})}$ can be described in terms of infinite M^d-ary labeled trees (see e.g. [4, Section 2] for the details.) Further, we write $\boldsymbol{\mathcal{F}_n} \subset \mathcal{F}$ for the σ-algebra generated by the selected level-n cubes.

Remark 1. *A very important feature of the construction is that*

*E is **statistically self-similar** with completely **independent** cylinders.*

That is

- **(a):** *For every $n \geq 1$ and $\mathbf{x} \in \mathcal{E}_n$, an appropriately re-scaled copy of the random set $E \cap K_n(\mathbf{x})$ has the same distribution as E itself.*
- **(b):** *The sets $\{E \cap K_n(\mathbf{x})\}_{\mathbf{x} \in \mathcal{E}_n}$ are independent.*

By construction, $\{\#\mathcal{E}_n\}$ is a branching process with offspring distribution $\texttt{Binomial}(M^d, p)$. Hence

$$\mathbb{P}(E \neq \emptyset) \text{ if and only if } p > \frac{1}{M^d}. \tag{1.2}$$

Falconer [7] and Mauldin, Williams [20] proved that

$$E \neq \emptyset \text{ implies that } \dim_{\mathrm{H}}(E) = \dim_{\mathrm{B}}(E) = \frac{\log p \cdot M^d}{\log M} \text{ a.s.} \tag{1.3}$$

In both parts of the paper the intersection of E with hyperplanes play the most important role so we introduce a notation related to it. Let H be a hyperplane in $\mathbb{R}^d$. Set

$$\boldsymbol{\mathcal{E}_n(H)} := \{\mathbf{x} \in \mathcal{E}_n : \mathtt{int}(K_n(\mathbf{x})) \cap H \neq \emptyset\}.$$

The $d-1$ dimensional Lebesgue measure of $E_n \cap H$ is

$$\boldsymbol{L_n(H)} := \mathcal{L}\mathrm{eb}_{d-1}(E_n \cap H) = \sum_{\mathbf{x} \in \mathcal{E}_n(H)} \mathcal{L}\mathrm{eb}_{d-1}\left(K_n(\mathbf{x}) \cap H\right). \tag{1.4}$$

2. The slices and orthogonal projections of E on the plane. The case of small E.

In this section $d = 2$. The main result of the first part of the paper is

Theorem 2. *Let E be a percolation on the plane. Let E_ℓ be the projection of E to a straight line ℓ. Then for almost every realization of E (conditioned on $E \neq \emptyset$) for* ***all*** *straight lines ℓ which are not parallel to the coordinate axes we have:*

$$\dim_{\mathrm{H}}(E_\ell) = \min\{1, \dim_{\mathrm{H}}(E)\}. \tag{2.1}$$

In [21] we proved that whenever $Mp > 1$ (that is conditioned on $E \neq \emptyset$, a.s. $\dim_{\mathrm{H}}(E) > 1$) for almost all realization ω, for all straight lines ℓ, the orthogonal projection $E_\ell(\omega)$ contains some intervals. This implies that the assertion of our Theorem 2 holds whenever $Mp > 1$. Using this and (1.2) without loss of generality in the rest of the section we may always assume that

Principal Assumption for this Section:

$$M^{-2} < p \leq M^{-1}. \tag{2.2}$$

2.1. **Projection** proj^α**.** The dimension of the orthogonal projection of E to a line having angle $\theta \neq 0, \pi/2$ with the positive half of the x-axis is the same as the the angle $\alpha \perp \theta$ projection to the decreasing or increasing diagonal of K for $\alpha \in (0, \pi/2)$ or $\alpha \in (\pi/2, \pi)$ respectively. If $\alpha \in (0, \pi/2)$ then $\boldsymbol{\Delta^\alpha}$ denotes the decreasing diagonal of K (the diagonal connecting points $(0,1)$ and $(1,0)$). If $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ then Δ^α is the increasing diagonal of K. For an $\alpha \in (0,\pi) \setminus \left\{\frac{\pi}{2}\right\}$ we write $\mathbf{proj}^{\boldsymbol{\alpha}} : K \to \Delta^\alpha$ for the angle α projection to the diagonal Δ^α in K. Without loss of generality we may confine ourselves to the angle

$$0 < \alpha < \pi/2. \tag{2.3}$$

projections to the decreasing diagonal which we denote by Δ.

2.2. **The slices.** The other object of this section is to study the neither vertical nor horizontal slices of E. These are the intersections of E with straight lines non-parallel to the coordinate axes. For a line segment ℓ we write $\mathbf{Arg}(\boldsymbol{\ell}) \in [0,\pi)$ for the angle between ℓ and the positive half of the x-axis. For a line ℓ intersecting Δ we define the line segment $\boldsymbol{\ell^{\alpha}(z)}$ as the intersection of K with ℓ, where $\alpha = \mathrm{Arg}(\ell)$ and $\{z\} = \ell \cap \Delta$. The set of all of these segments is denoted by $\mathfrak{L}$. We will approximate the length of the slices of the level-n approximation E_n:

$$\boldsymbol{L_n(\ell)} := |E_n \cap \ell|, \quad \ell \in \mathfrak{L}. \tag{2.4}$$

It is immediate from the construction of the fractal percolation that for every $\ell \in \mathfrak{L}$, $n \geq 1$,

$$\forall \mathbf{x} \in \mathcal{N}_{n-1},\ \mathbb{E}\left[|E_n \cap \ell \cap K_{n-1}(\mathbf{x})| \middle| \mathbf{x} \in \mathcal{E}_{n-1}\right] = p|\ell \cap K_{n-1}(\mathbf{x})|. \tag{2.5}$$

Clearly, $\mathfrak{L}$ can be presented as a countable union of families of lines segments $\mathfrak{L}^{\theta}$ whose angles $\mathrm{Arg}(\ell)$ are θ-separated from both 0 and $\pi/2$:

$$\boldsymbol{\mathfrak{L}^{\theta}} := \left\{\ell \in \mathfrak{L} : \min\left\{\mathrm{Arg}(\ell), \frac{\pi}{2} - \mathrm{Arg}(\ell)\right\} > \theta\right\}, \quad 0 < \theta < \pi/4.$$

Then $\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}^{1/k}$. We would like to get an upper bound for $\#\mathcal{E}_n(\ell)$ for an arbitrary $\ell \in \mathfrak{L}^{\theta}$. To do so, first we give a uniform upper bound for $L_n(\ell)$ for all $\ell \in \mathfrak{L}^{\theta}$ and then we use the following easy fact:

Fact 3. *Let* $\mathcal{E}_n^{\mathrm{bigg}}(\ell) := \left\{\mathbf{x} \in \mathcal{E}_n(\ell) : |\ell \cap K_n(\mathbf{x})| \geq M^{-n}/\sqrt{2}\right\}$, *analogously set* $\mathcal{E}_n^{\mathrm{small}}(\ell) := \left\{\mathbf{x} \in \mathcal{E}_n(\ell) : |\ell \cap K_n(\mathbf{x})| < M^{-n}/\sqrt{2}\right\}$. *Let* ℓ^u, ℓ^l *be lines which are parallel to* ℓ, *their distance from* ℓ *is in* $\left(M^{-n}/2\sqrt{2}, M^{-n}/\sqrt{2}\right)$ *and they lie on opposite sides of* ℓ. *Then*

$$\mathcal{E}_n^{\mathrm{small}}(\ell) \subset \mathcal{E}_n^{\mathrm{bigg}}(\ell^u) \cup \mathcal{E}_n^{\mathrm{bigg}}(\ell^l) \tag{2.6}$$

That is

$$\#\mathcal{E}_n(\ell) \leq 2M^n \left(L_n(\ell) + L_n(\ell^u) + L_n(\ell^l)\right). \tag{2.7}$$

We will need the following M^{-n}-dense subset of $\mathfrak{L}^{\theta}$

Definition 4. *For every* $0 < \theta < \pi/4$ *pick an arbitrary* M^{-2n}*-dense set subset* $\Delta_n \subset \Delta$ *and also an* M^{-2n}*-dense subset* A_n^{θ} *of* $\left(\theta, \frac{\pi}{2} - \theta\right)$ *such that* $\Delta_n \subset \Delta_{n+1}$ *and* $A_n^{\theta} \subset A_{n+1}^{\theta}$. *Let*

$$\mathfrak{L}_n^{\theta} := \left\{\ell \in \mathfrak{L}^{\theta} : z(\ell) \in \Delta_n \text{ and } \mathrm{Arg}(\ell) \in A_n^{\theta}.\right\}$$

Clearly,

$$\mathfrak{L}_n^{\theta} \subset \mathfrak{L}_{n+1}^{\theta} \text{ and } \#\mathfrak{L}_n^{\theta} = M^{4n}. \tag{2.8}$$

The following fact follows from elementary geometry:

Fact 5. *For every $0 < \theta < \pi/4$ we can find an s_θ such that for every $\ell \in \mathfrak{L}^\theta$ we can choose an $\ell' \in \mathfrak{L}_n^\theta$ satisfying*

$$L_{n-1}(\ell) \leq L_{n-1}(\ell') + s_\theta M^{-(n-1)}. \tag{2.9}$$

We fix such an s_θ for every θ.

2.3. **The length of the slices.** In this Section we prove a theorem which says that for almost all realizations, the length of **all** (non-vertical, non-horizontal) level-n slices of angle α are less than const $\cdot$ nM^{-n} if n is big enough.

Theorem 6. *There exists a C_2 (defined in (2.36)) such that for all $0 < \theta < \frac{\pi}{4}$ the following holds almost surely:*

$$\exists N,\ \forall n \geq N,\ \forall \ell \in \mathfrak{L}^\theta; \quad L_n(\ell) < C_2 M^{-n} n. \tag{2.10}$$

Here the threshold N depends on both θ and the realization.

Using that $\mathfrak{L} = \bigcup_{k=2}^{\infty} \mathfrak{L}^{1/k}$ we obtain that

Corollary 7. *Almost surely, for all $\ell \in \mathfrak{L}$*

$$\exists N,\ \forall n \geq N, \quad L_n(\ell) \leq C_2 \cdot M^{-n} n. \tag{2.11}$$

Using Fact 3 we get

Corollary 8. *The following statement holds almost surely:*

$$\forall \theta \in \left(0, \frac{\pi}{4}\right),\ \exists N,\ \forall n \geq N,\ \forall \ell \in \mathfrak{L}^\theta; \quad \#\mathcal{E}_n(\ell) \leq 6C_2 n. \tag{2.12}$$

To prove Theorem 6 we apply the Azuma-Hoeffding inequality to estimate $L_n^\alpha(x)$.

2.4. **Large deviation estimate for $L_n(\ell)$.** An immediate reformulation of the Azuma-Hoeffding inequality [18, Theorem 2] is

Theorem 9 (Hoeffding)**.** *Let $X_1, \dots, X_m$ be independent bounded random variables with $a_i \leq X_i \leq b_i$, $(i = 1, \dots, m)$. Then for any $t > 0$*

$$\mathbb{P}\left(X_1 + \cdots + X_m - \mathbb{E}\left[X_1 + \cdots + X_m\right] \geq t\right) \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right). \tag{2.13}$$

Using this we obtain that

Lemma 10. *For every $u > 1$ there is a constant $r = r(u) > 0$ such that for every $n \geq 1$, $\ell \in \mathfrak{L}$ and $0 < R < |\ell|$,*

$$\mathbb{P}\left(L_n(\ell) > pL_{n-1}(\ell) \cdot u | L_{n-1}(\ell) \geq R\right) < \exp\left(-rM^{(n-1)}R\right) \tag{2.14}$$

Proof. Fix an arbitrary $u > 1$, n, $\ell \in \mathfrak{L}$ and $0 < R < |\ell|$. Let

$$r = \sqrt{2} \cdot (u-1)^2 p^2. \tag{2.15}$$

We write $\boldsymbol{\mathfrak{N}_{n-1}(\ell)}$ for the collection of all $\mathtt{N} \subset \mathcal{N}_{n-1}$ satisfying

(a): $\forall \mathbf{x} \in \mathtt{N}$, we have $\ell \cap \mathtt{int}(K_{n-1}(\mathbf{x})) \neq \emptyset$ and
(b): $\sum\limits_{\mathbf{x} \in \mathtt{N}} |\ell \cap \mathtt{int}(K_{n-1}(\mathbf{x}))| \geq R$.

For an $\mathtt{N} \in \mathfrak{N}_{n-1}(\ell)$ let

$\widetilde{\mathtt{N}}$ be the event that $\mathtt{N} = \{\mathbf{x} \in \mathcal{E}_{n-1} : \ \mathtt{int}(K_{n-1}(\mathbf{x})) \cap \ell \neq \emptyset\}$.

Note that

$$\{L_{n-1}(\ell) \geq R\} = \bigcup_{\mathtt{N} \in \mathfrak{N}_{n-1}(\ell)} \widetilde{\mathtt{N}}$$

with disjoint union. Hence to verify 2.14 it is enough to prove that

$$\forall \mathtt{N} \in \mathfrak{N}_{n-1}(\ell), \ \mathbb{P}\left(L_n(\ell) > pL_{n-1}(\ell) \cdot u | \widetilde{\mathtt{N}}\right) < \exp\left(-rM^{n-1}R\right). \tag{2.16}$$

Fix an arbitrary $\mathtt{N} \in \mathfrak{N}_{n-1}(\ell)$ and set

$$\widetilde{\mathbb{P}}(\cdot) := \mathbb{P}\left(\cdot | \widetilde{\mathtt{N}}\right).$$

Clearly, $L_{n-1}(\ell)$ is deterministic on $\widetilde{\mathtt{N}}$.

$$L_{n-1}(\ell) = \sum_{\mathbf{x} \in \mathtt{N}} |\ell \cap K_{n-1}(\mathbf{x})| \geq R \text{ on } F^{\mathtt{N}}, \tag{2.17}$$

by definition. With this notation (2.16) is of the form:

$$\widetilde{\mathbb{P}}\left(L_n(\ell) > puL_{n-1}(\ell)\right) < \exp\left(-rM^{n-1}R\right). \tag{2.18}$$

Let $\{X_{\mathbf{x}}\}_{\mathbf{x} \in \mathtt{N}}$ be independent random variables on $\widetilde{\mathtt{N}}$ with $X_{\mathbf{x}} \stackrel{d}{=} |\ell \cap E_n \cap K_{n-1}(\mathbf{x})|$. Then by (2.5)

$$L_n(\ell) - p \cdot L_{n-1}(\ell) = \sum_{\mathbf{x} \in \mathtt{N}} \left(X_{\mathbf{x}} - \mathbb{E}\left[X_{\mathbf{x}}\right]\right).$$

We will apply Theorem 9 for the random variables $X_{\mathbf{x}}$. Using the notation of Theorem 9 , observe that

$$a_{\mathbf{x}} := 0 \leq X_{\mathbf{x}} \leq |\ell \cap K_{n-1}(\mathbf{x})| =: b_{\mathbf{x}}.$$

The sum on the right hand side of the formulae (2.13) satisfies

$$\begin{aligned}(2.19)\ \sum_{\mathbf{x}\in\mathtt{N}}(b_{\mathbf{x}}-a_{\mathbf{x}})^2 &\leq \sum_{\mathbf{x}\in\mathtt{N}}|\ell\cap K_{n-1}(\mathbf{x})|^2 \\ &= 2M^{-2(n-1)}\sum_{\mathbf{x}\in\mathtt{N}}\left(\frac{M^{n-1}}{\sqrt{2}}\,|\ell\cap K_{n-1}(\mathbf{x})|\right)^2 \\ &\leq \sqrt{2}\cdot M^{-(n-1)}\sum_{\mathbf{x}\in\mathtt{N}}|\ell\cap K_{n-1}(\mathbf{x})| \\ &= \sqrt{2}M^{-(n-1)}L_{n-1}(\ell),\end{aligned}$$

where in the one but last step we used that all the summands are less than or equal to 1. Using this and Theorem 9 for $t=(u-1)pL_{n-1}(\ell)$ on the space $(\widetilde{\mathtt{N}},\widetilde{\mathbb{P}})$ we obtain that

$$\begin{aligned}\widetilde{\mathbb{P}}\left(L_n(\ell)>pL_{n-1}(\ell)\cdot u\right) &= \widetilde{\mathbb{P}}\left(\sum_{\mathbf{x}\in O}\left(X_{\mathbf{x}}-\mathbb{E}\left[X_{\mathbf{x}}\right]\right)>t\right) \\ &\leq \exp\left(\frac{-2t^2}{\sum\limits_{i=1}^{m}(b_i-a_i)^2}\right) \\ &\leq \exp\left(-rM^{n-1}L_{n-1}\right) \\ &\leq \exp\left(-rM^{n-1}R\right),\end{aligned}$$

where in the last step we used (2.17). So, (2.18) holds which implies the assertion of the Lemma. □

We use this Lemma for a re-scaled version of $L_n(\ell)$. Namely, let

$$F_n(\ell):=L_n(\ell)\cdot M^n.$$

Then we can reformulate (2.14) as follows:

Corollary 11. *For an $u>1$ let $r=r(u)=\sqrt{2}\cdot(u-1)^2p^2$. For all $\ell\in\mathfrak{L}$, and*

$$0<R_n\leq M^{n-1}|\ell|, \tag{2.20}$$

we have

$$\mathbb{P}\left(F_n(\ell)>pMF_{n-1}(\ell)\cdot u|F_{n-1}(\ell)\geq R_n\right)<\exp\left(-rR_n\right). \tag{2.21}$$

2.5. **The proof of Theorem 6.** Now we prove Theorem 6 using the large deviation estimate of Corollary 11.
Fix an arbitrary $0 < \theta < \frac{\pi}{4}$ for this Section and let $0 < \varepsilon < \min\left\{p, \frac{1}{10}\right\}$ such that $Mp(1+\varepsilon) < 1$. We will use Corollary 11 with

$$u = 1 + \varepsilon/3 \text{ that is } r = \sqrt{2}p^2\varepsilon^2/9. \tag{2.22}$$

Set

$$\boldsymbol{a_n} := \max_{\ell \in \mathfrak{L}_n^\theta} F_{n-1}(\ell), \ \boldsymbol{b_n} := \frac{8 \log M}{r} \cdot n, \tag{2.23}$$

where $\mathcal{L}_n^\theta$ was defined in Definition 3. The reason for this particular choice of b_n was to ensure that

$$M^4 \cdot \exp\left(-r b_n / n\right) < 1 \tag{2.24}$$

which we will need later to apply Borel-Cantelli Lemma. Clearly,

$$a_{k+1} \le M \cdot a_k \text{ and } b_k < b_{k+1}. \tag{2.25}$$

Now we prove that

Lemma 12. *For almost all realization there exists an N_0 (which depends on the realization) such that*

$$\forall n \ge N_0, \quad \textit{either } a_n \le b_n \textit{ or } a_{n+1} \le \lambda a_n, \tag{2.26}$$

where $\lambda = pM(1 + \frac{2\varepsilon}{3}) < 1$.

Proof. We define the events

$$\mathcal{A}_n(\ell) := \{F_n(\ell) > pMu \cdot F_{n-1}(\ell), \ F_{n-1}(\ell) > b_n\}$$

and

$$\mathcal{A}_n := \bigcup_{\ell \in \mathfrak{L}_n^\theta} \mathcal{A}_n(\ell).$$

Note that $\mathcal{A}_n(\ell) = \emptyset$ for those $\ell \in \mathfrak{L}_n^\theta$ satisfying $b_n > M^{n-1}|\ell|$. Otherwise, we can use Corollary 11 to obtain that

$$\mathbb{P}\left(F_n(\ell) > pMu \cdot F_{n-1}(\ell), \ F_{n-1}(\ell) \ge b_n\right) \le \exp\left(-r b_n\right). \tag{2.27}$$

Using this and (2.8) we obtain that

$$\mathbb{P}\left(\mathcal{A}_n\right) \le M^{4n} \mathrm{e}^{-r b_n}, \tag{2.28}$$

which is summable by (2.24). The Borel-Cantelli Lemma yields that for almost all realizations there exists an N' such that for all $n \ge N'$ we have

$$\forall \ell \in \mathfrak{L}_n^\theta \text{ either } F_{n-1}(\ell) \le b_n \text{ or } F_n(\ell) < pM\left(1 + \frac{\varepsilon}{3}\right) F_{n-1}\left(\ell\right). \tag{2.29}$$

Let $n > N'$ and assume that $a_n > b_n$. Choose $\ell' \in \mathfrak{L}_n^\theta$ such that $F_{n-1}(\ell') \geq F_{n-1}(\ell)$ for all $\ell \in \mathfrak{L}_n^\theta$. Then

$$F_{n-1}(\ell') > b_n. \tag{2.30}$$

Fix an arbitrary $\ell \in \mathfrak{L}_n^\theta$. If $F_{n-1}(\ell) \leq b_n$ then

$$F_n(\ell) \leq M^{-1} b_n < M^{-1} F_{n-1}(\ell') < pM\left(1 + \frac{\varepsilon}{3}\right) \cdot F_{n-1}(\ell'), \tag{2.31}$$

since $p > M^{-2}$. On the other hand, if $F_{n-1}(\ell) > b_n$ then by (2.29) and the definition of ℓ' we get

$$F_n(\ell) < pM\left(1 + \frac{\varepsilon}{3}\right) \cdot F_{n-1}(\ell'). \tag{2.32}$$

We choose $N_0 \geq N'$ such that for all $n \geq N_0$ we have $\frac{\varepsilon M p}{3} b_n > s_\theta$. Then by (2.32), (2.31) and (2.30) we obtain

$$\begin{aligned} a_{n+1} &= \max_{\ell \in \mathfrak{L}_{n+1}^\theta} F_n(\ell) \\ &\leq \max_{\ell \in \mathfrak{L}_n^\theta} F_n(\ell) + s_\theta \\ &\leq pM\left(1 + \frac{\varepsilon}{3}\right) \cdot F_{n-1}(\ell') + s_\theta \\ &< \underbrace{pM\left(1 + \frac{2\varepsilon}{3}\right)}_{\lambda} \cdot a_n, \end{aligned}$$

since we assumed that $a_n > b_n$. □

The proof of Theorem 6. Let N_0 and λ be as in Lemma 12. First we show that there is an $N_1 > N_0$ such that $a_{N_1} \leq b_{N_1}$. Namely, if $a_{N_0+l} > b_{N_0+l}$ then by Lemma 12, $a_{N_0+l+1} < \lambda a_{N_0+l}$. Since $\lambda < 1$ and $\{b_n\}$ is increasing we find a $N_1 > N_0$ such that $a_{N_1} \leq b_{N_1}$. Then for all $k > N_1$ we have

$$a_k \leq M b_k. \tag{2.33}$$

Namely, consider the ratio $r_k := \frac{a_k}{b_k}$. If $r_k < 1$ (as it happens for $k = N_1$) and $r_{k+1} > 1$ then $r_{k+1} < M$ since $a_{k+1} \leq M a_k$ and $\{b_k\}$ is increasing. Then by Lemma 12 we have $r_{k+i+1} < \lambda r_{k+i}$ as long as $r_{k+i} > 1$. Then the same cycle is repeated which completes the proof of (2.33). Using (2.33) we obtain that almost surely there is an N_1 such that for $n \geq N_1$

$$F_n(\ell) \leq \frac{8M \log M}{r} \cdot (n+1), \text{ if } \ell \in \mathfrak{L}_{n+1}^\theta. \tag{2.34}$$

Then by Fact 5

$$\forall \ell' \in \mathfrak{L}^{\theta}, \quad F_n(\ell') < \frac{8M\log M}{r} \cdot n + s_\theta < C_2 \cdot n, \tag{2.35}$$

for

$$C_2 := \frac{8M\log M}{r} + 1 \tag{2.36}$$

(We remind that r was defined in (2.22).) □

2.6. **The proof of the main result of the Section.** Now we prove Theorem 2. Using the Mass distribution principle [16], Theorem 2 follows from the combination of Theorem 6 and Frostman's Lemma ([19, Theorem 8.8]) with (1.3). For the convenience of the reader here we cite Frostman's Lemma from [19]. For a $d \geq 1$ and $B \subset \mathbb{R}^d$ let $\mathcal{M}(B)$ be the set of Radon measures μ supported by B with $0 < \mu(\mathbb{R}^d) < \infty$. Then

Lemma 13 (Frostman's Lemma)**.** *Let $B \subset \mathbb{R}^d$ be a Borel set. Then $\mathcal{H}^r(B) > 0$ if and only if there exists a $\mu \in \mathcal{M}(B)$ such that*

$$\forall x \in \mathbb{R}^d, \forall \rho > 0, \quad \mu(B(x,\rho)) \leq \rho^s. \tag{2.37}$$

The other ingredient of the proof is the following very well known lemma [16]

Lemma 14 (Mass distribution principle)**.** *Let $B \subset \mathbb{R}^d$. Assume that there exists a measure $\mu \in \mathcal{M}(B)$ and $\delta > 0$ such that $\mu(A) < \mathrm{const} \cdot |A|^s$. Then $\dim_{\mathrm{H}}(A) \geq s$.*

Proof of Theorem 2. In what follows we always condition on $E \neq \emptyset$. Then by (1.3)

$$\dim_{\mathrm{H}}(E) = \frac{\log(p \cdot M^2)}{\log M} =: s, \text{ almost surely.} \tag{2.38}$$

It is enough to verify that

$$\forall q < s, \forall \theta \in \left(0, \frac{\pi}{4}\right), \forall \alpha \in (\theta, \pi/2 - \theta), \quad \dim_{\mathrm{H}}(\mathrm{proj}^{\alpha}(E)) > q. \tag{2.39}$$

To see this, we fix an r with $q < r < s$. Then $\mathcal{H}^r(E) = \infty$. So, by Frostman's Lemma there exists a random measure $\mu \in \mathcal{M}(E)$ such that (2.37) holds. In particular

$$\forall \mathbf{x} \in \mathcal{E}_n, \quad \mu\left(K_n(\mathbf{x})\right) \leq M^{-nr}. \tag{2.40}$$

Put $\nu_\alpha := \mathrm{proj}^{\alpha}_{*}\mu$. Fix an arbitrary $0 < \varepsilon$ and fix an arbitrary $0 < \rho$ which is so small that

- for $M^{-(n+1)} < \rho \leq M^{-n}$ we have $n \geq N$, for the N defined in (2.12) and
- $nM^{-nr} < M^{-nq}$.

Then using these two properties and (2.40) and 2.12 implies that

$$\nu_\alpha(x-\rho, x+\rho) \leq 10C_2 nM^{-nr} \leq 10C_2 M^{-nq} \leq 10C_2 M^q \rho^{-q}.$$

This completes the proof of (2.39) by the Mass distribution principle. □

In the rest of the section we prove that we can find slices of angle $\frac{\pi}{4}$ which intersect constant times n level n squares almost surely conditioned on $E \neq \emptyset$.

Proposition 15. *There exists a constant* $0 < \lambda < 1$ *such that for almost all realizations, conditioned on* $E \neq \emptyset$*, there exists an* N_6 *such that for all* $n > N_6$ *there exists an* $\ell \in \mathcal{L}$ *with*

$$\#\mathcal{E}_n(\ell) > \lambda n. \tag{2.41}$$

For the proof we need some new notation and an easy Fact.

Let D_k be the event that all the M^k level-k squares of the diagonal of $[0,1]^2$ is retained. That is

$$D_k := \left\{ \forall \underline{\ell}_k = (\ell_1, \dots, \ell_k) \in \{1, \dots, M\}^k, (\underline{\ell}_k, \underline{\ell}_k) \in \mathcal{E}_k \right\}.$$

We get the definition of the event $D_k^{\underline{i}_n, \underline{j}_n}$ if we substitute the diagonal of $[0,1]^2$ above with the diagonal of $K_{\underline{i}_n, \underline{j}_n}$.

$$D_k^{\underline{i}_n, \underline{j}_n} := \left\{ \forall \underline{\ell}_k = (\ell_1, \dots, \ell_k) \in \{1, \dots, M\}^k, (\underline{i}_n \underline{\ell}_k, \underline{j}_n \underline{\ell}_k) \in \mathcal{E}_{n+k} \right\}$$

A simple argument shows that

$$p^{2M^k} < \mathbb{P}(D_k) < p^{M^k}. \tag{2.42}$$

Let $\Omega' \subset \Omega$ be the set of realizations for which (1.3) holds and let $\mathbb{P}'(\cdot) := \mathbb{P}(\cdot | \Omega')$.

Proof of Proposition 15. Since $pM^2 > 1$ we can find a τ satisfying $0 < \tau < \frac{\log M^2 p}{\log 1/p}$. Then $p^{1+\tau} M^2 > 1$. Therefore we can choose a $0 < \gamma < 1$ such that

$$p^{\gamma+\tau} M^{2\gamma} > 1. \tag{2.43}$$

By (1.3) for all $\omega \in \Omega'$ realization we can find an $N_7 = N_7(\omega)$ such that

$$\forall n \geq N_7, \quad \#\mathcal{E}_n > \left(pM^2\right)^{n\gamma}. \tag{2.44}$$

For every k the events $\left\{D_k^{\underline{i}_n,\underline{j}_n}\right\}_{(\underline{i}_n,\underline{j}_n)\in\mathcal{E}_n}$ are independent and each has probability greater than p^{2M^k}. Let A_n be the event that at least one of the events $\left\{D_k^{\underline{i}_n,\underline{j}_n}\right\}_{(\underline{i}_n,\underline{j}_n)\in\mathcal{E}_n}$ holds and A_n^c that non-of them holds. Then

$$\forall n \geq N_7, \quad \mathbb{P}'\left(A_n^c\right) \leq \left(1-p^{2M^k}\right)^{(pM^2)^{n\gamma}}.$$

We choose $k = k(n)$ such that

$$2M^k \leq \tau n < 2M^{k+1}. \tag{2.45}$$

Let $a_n := \left(1-p^{2M^k}\right)^{(pM^2)^{n\gamma}}$ Then $\log a_n < -\left(p^{\gamma+\tau}M^{2\gamma}\right)^n$ which tends to $-\infty$ exponentially fast by (2.43). Hence the series $\sum\limits_n \mathbb{P}'(A_n^c)$ is summable. So, Borel Cantelli Lemma yields that there exists an $N_6 > N_7$ such that for all $n > N_6$ the event A_n holds.
This shows that for $\mathbb{P}'$ almost all realizations ω for all $m = n+k$ big enough there is an $\ell \in \mathcal{L}$ such that $\#\mathcal{E}_m(\ell) \geq \frac{\tau n}{2M}$. Since $m < 2n$ this completes the proof of the proposition. □

3. Sums of random Cantor sets

3.1. **Product of percolations.** In this section we consider d independent fractal percolations $\overline{E}^{(1)}, \dots, \overline{E}^{(d)}$ on the line with possibly different probabilities $p_1, \dots, p_d$ but with the same scale M. The object of this Section is as follows: We fix an $\mathbf{a} := (a_1, \dots, a_d) \in \mathbb{R}^d$ with $a_i \neq 0$ for all $i = 1, \dots, d$ and consider the algebraic sum of coefficients $a_1, \dots, a_d$:

$$\widetilde{\boldsymbol{E}}_{\mathbf{a}} := a_1 \cdot \overline{E}^{(1)} + \cdots + a_d \cdot \overline{E}^{(d)} = \left\{\sum_{k=1}^{d} a_i \cdot e^{(i)} : e^{(i)} \in \overline{E}^{(i)}\right\}.$$

and ask whether such sum contains an interval. It is a generalization of a question solved (in higher generality) by Dekking and Simon in [5] for sums of two independent percolations.
Without loss of generality in the rest of the paper we may assume that

$$p_i > M^{-1}, \ \forall i = 1, \dots, d. \tag{3.1}$$

(otherwise the corresponding $\overline{E}^{(i)}$ would be almost surely empty).
The main result of this Section is as follows:

Theorem 16. *Assume that*

$$\prod_{i=1}^{d} p_i > M^{-d+1}. \tag{3.2}$$

Then for every $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$, $a_i \neq 0$ *for all* $i = 1, \dots, d$ *the sum* $\widetilde{E}_{\mathbf{a}} = \sum_{i=1}^{d} a_i \overline{E}^{(i)}$ *contains an interval almost surely, conditioned on all* $\overline{E}^{(i)}$ *being nonempty.*

Fix an arbitrary $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$, $a_i \neq 0$ for all $i = 1, \dots, d$. Without loss of generality we may assume that

$$\|\mathbf{a}\| = 1 \text{ and } a_i > 0 \text{ for all } i = 1, \dots, d. \tag{3.3}$$

Clearly, $\widetilde{E}_{\mathbf{a}}$ is the orthogonal projection of the random set

$$\widetilde{\boldsymbol{E}} := \overline{E}^{(1)} \times \cdots \times \overline{E}^{(d)} \subset [0,1]^d$$

to the line $\{t \cdot \mathbf{a} | t \in \mathbb{R}\}$. Hence it follows from (2.38) that whenever condition (3.2) does not hold then we have

$$\dim_{\mathrm{H}} \widetilde{E}_{\mathbf{a}} \leq \dim_{\mathrm{H}} \widetilde{E} = \sum_{i=1}^{d} \dim_{\mathrm{H}} E^{(i)} = \sum_{i=1}^{d} \frac{\log M p_i}{\log M} \leq 1.$$

Remark 17. *It is well known that for almost every realization* E *of a fractal percolation in* $\mathbb{R}^d$ $\dim_{\mathrm{H}} E = 1$ *implies that* $\mathcal{H}^1(E) = 0$*, see* [20]*. The same proof goes through for cartesian products of fractal percolations: for typical* $\widetilde{E}$ *if* $\dim_{\mathrm{H}} \widetilde{E} = 1$ *then* $\mathcal{H}^1(\widetilde{E}) = 0$*. Hence, Theorem 16 is sharp.*

3.1.1. *Connection between* $\widetilde{E}$ *and the* d*-dimensional fractal percolation.* Set $p = \prod_{i=1}^{d} p_i$ and we write E for the d-dimensional fractal percolation with parameters M, p. Let $\overline{\mathcal{N}}_n$, $\overline{K}_n(\mathbf{x})$, $\overline{\mathcal{E}}_n^{(i)}$ and $\overline{E}_n^{(i)}$ be the one dimensional analogues of $\mathcal{N}_n, K_n(\mathbf{x})$, $\mathcal{E}_n$ and E_n respectively. That is

$$\overline{\boldsymbol{\mathcal{N}}}_{\boldsymbol{n}} := \left\{ x : x = \left(k + \frac{1}{2} \right) M^{-n},\ k \in \{0, 1, \dots, M^n - 1\} \right\}.$$

We denote the level-n interval with center $x \in \overline{\mathcal{N}}_n$ by $\overline{K}_n(x)$.

$$\overline{\boldsymbol{K}}_{\boldsymbol{n}}(\mathbf{x}) = x + \left[-\frac{1}{2M^n}, \frac{1}{2M^n} \right].$$

Set

$$\overline{\boldsymbol{\mathcal{E}}}_{\boldsymbol{n}}^{(\boldsymbol{i})} := \left\{ x \in \overline{\mathcal{N}}_n : \overline{K}_n(\mathbf{x}) \text{ is retained in the construction of } E_n^{(i)} \right\}$$

Finally, the n-th approximation of $E^{(i)}$ is denoted by $\overline{E}_n^{(i)}$.

$$\boldsymbol{\overline{E}_n^{(i)}} := \bigcup_{x\in\overline{\mathcal{E}}_n^{(i)}} \overline{K}_n(x) \text{ and } \boldsymbol{\widetilde{E}_n} := \overline{E}_n^{(1)} \times \cdots \times \overline{E}_n^{(d)}.$$

Then

$$\widetilde{E} = \bigcap_{n=1}^{\infty} \widetilde{E}_n. \tag{3.4}$$

For an $\mathbf{x} = (x_1, \dots x_d) \in \mathcal{N}_n$ we have

$$K_n(\mathbf{x}) \subset \widetilde{E}_n \Longleftrightarrow \overline{K}_n(x_i) \subset \overline{E}_n^{(i)},\ \forall i \in \{1, \dots, d\}. \tag{3.5}$$

Hence for every $\mathbf{x} \in \mathcal{N}_n$ the events that $K_n(\mathbf{x}) \subset \widetilde{E}_n$ and $K_n(\mathbf{x}) \subset E_n$ share the same probability of p^n. Furthermore, when $\mathbf{x} \in \mathcal{N}_n$ and $\mathbf{y} \in \mathcal{N}_{n-1}$ such that $K_n(\mathbf{x}) \subset K_{n-1}(\mathbf{y})$ then

$$\mathbb{P}\left(\mathbf{x} \in \mathcal{E}_n | \mathbf{y} \in \mathcal{E}_{n-1}\right) = \mathbb{P}\left(\mathbf{x} \in \widetilde{\mathcal{E}}_n | \mathbf{y} \in \widetilde{\mathcal{E}}_{n-1}\right) = p.$$

The difference between E and $\widetilde{E}$ follows from the obvious fact:

Fact 18. *For all distinct* $\mathbf{x}, \mathbf{y} \in \mathcal{N}_n$ *the events:*

- **(a):** $K_n(\mathbf{x}) \cap E$ *and* $K_n(\mathbf{y}) \cap E$ *are always independent.*
- **(b):** $K_n(\mathbf{x}) \cap \widetilde{E}$ *and* $K_n(\mathbf{y}) \cap \widetilde{E}$ *are independent if and only if* $x_i \neq y_i$ *for all* $i = 1, \dots, d$.

3.2. **Sections of codimension 1.** We consider hyperplanes

$$H_t = \{\mathbf{y} \in \mathbb{R}^d : \mathbf{a} \cdot \mathbf{y} = t\}.$$

That is $H_t(\mathbf{a})$ is the set of points $\mathbf{y} \in \mathbb{R}^d$ whose orthogonal projection to the line with direction vector $\mathbf{a}$ is equal to $t \cdot \mathbf{a}$ (since we assumed that $\mathbf{a}$ is a unit vector).

Lemma 19. *Given* $p_1, \dots, p_d > M^{-1}$ *satisfying*

$$\prod_i p_i > M^{-d+1},$$

we can find $q_1, \dots, q_d$ *such that*

$$\prod_i q_i > M^{-d+1},$$

$$\prod_{i\neq j} q_i < M^{-d+2} \forall j, \tag{3.6}$$

and

$$M^{-1} < q_i \leq p_i \forall i.$$

Proof. Without weakening the assumptions we can assume that $p_1 = \min p_i$. Assume that

$$\prod_{i=2}^{d} p_i \geq M^{-d+2}$$

(otherwise we could choose $q_i = p_i$ for all i). There are two cases.
If $p_1 > M^{-1+1/d}$, we can choose any $M^{-1+1/d} < \delta < M^{-1+1/(d-1)}$ and then set $q_i = \min(\delta, p_1)$ for all i.
In the opposite case, let $q_1 = p_1$ and set (for $i \geq 2$ and $0 \leq t \leq 1$)

$$q_i(t) = tp_i + (1-t)p_1.$$

We have

$$\prod_{i=2}^{d} q_i(0) < M^{-d+1} p_1^{-1}$$

and

$$\prod_{i=2}^{d} q_i(1) = \prod_{i=2}^{d} p_i > M^{-d+2}.$$

Hence, we can find $t_0 \in (0,1)$ such that

$$M^{-d+1} p_1^{-1} < \prod_{i=2}^{d} q_i(t_0) < M^{-d+2}$$

and we can just fix $q_i = q_i(t_0)$ for $i \geq 2$. □

To prove the assertion of Theorem 16 for probabilities $\{p_i\}$ it is enough to prove it for $\{q_i\}$ (increasing of probabilities is not going to decrease probability of the algebraic sum of percolation fractals containing an interval). Hence, we might freely assume that (3.6) is satisfied for $\{p_i\}$. The goal of this subsection is to prove the following Proposition:

Proposition 20. *Assume that*

$$\prod_{i=1}^{d} p_i > M^{-d+1} \text{ and } \prod_{i \neq j} p_i < M^{-d+2} \quad \forall j. \tag{3.7}$$

Under assumptions of Theorem 16, there is a constant K such that for almost every nonempty realization of $\widetilde{E}$ there is N such that for all $n > N$ for every t if the hyperplane H_t intersects cube $K_n(x_1, \ldots, x_d)$

then it intersects at most $Kn^{(1+\varepsilon)(d-2)}$ other cubes $K_n(y_1,\dots,y_d)\subset E_n$ such that $x_i=y_i$ for some i.

First we need some auxiliary lemmas

3.2.1. *Auxiliary lemmas to the proof of Proposition 20.*

Lemma 21. *Let $G=(V,E)$ be a graph such that every vertex $v\in V$ has degree not greater than n. Then we can write $V=V_1\cup\ldots\cup V_{n+1}$ in such a way that no edge $e\in E$ connects two vertices from the same V_i.*

Proof. Let $(V_1,E|V_1)$ be a maximal (in V) totally disconnected subgraph. That is, let $V_1\subset V$ such that for any $v_1,v_2\in V_1$, $v_1v_2\notin E$ but if we added to V_1 any additional point, this property would be lost. In particular, it means that any vertex $v\in V\setminus V_1$ is connected to some $v'\in V_1$ (otherwise $(V\cup\{v\},E|V\cup\{v\})$ would be totally disconnected). It implies that in the graph $(V\setminus V_1,E|V\setminus V_1)$ every vertex has degree not greater than $n-1$. The proof proceeds by induction. □

Yet another auxiliary lemma:

Lemma 22. *There exists $K>0$, depending only on d and $\{l_i\}$, such that the following holds. Let F be a union of some M-adic cubes of level n. Assume that the $d-1$-dimensional volume of $F\cap H_t$ is not greater than $ZM^{-n(d-1)}$ for all t. Then every H_t intersects at most KZ cubes from F.*

Proof. Assume that the assertion is not true: for some t H_t intersects $\varepsilon^{-1}Z$ cubes from F. It implies that there are at least $\varepsilon^{-1}Z/2$ cubes in F such that the $d-1$-dimensional volume of the intersection of H_t with each of them is smaller than $\varepsilon M^{-n(d-1)}$. Let us denote the family of those cubes by G.
Consider now the hyperplanes $L_{t+\frac{1}{2}\sum l_i}$ and $L_{t-\frac{1}{2}\sum l_i}$. Each cube from G has empty intersection with one of those hyperplanes but big (with $d-1$-dimensional volume of order $M^{-n(d-1)}$) with the other. Hence, the sum of $d-1$-dimensional volumes of intersecting G with $L_{t+\frac{1}{2}\sum l_i}$ and $L_{t-\frac{1}{2}\sum l_i}$ is of order at least $\varepsilon^{-1}ZM^{-n(d-1)}$, a contradiction. □

We need another useful observation:

Lemma 23. *The $d-1$-dimensional volume of the intersection $H_t\cap E_n$ is lipschitz as a function of t, with the Lipschitz constant at most c_5M^n.*

Proof. There are at most $cM^{n(d-1)}$ cubes H_t can intersect and the volume of intersection of H_t with each of them is Lipschitz with constant $cM^{-n(d-2)}$. □

3.2.2. *The proof of the Proposition 20.* Now we are ready to prove of the main result of this subsection.

The proof of Proposition 20. The proof will be by induction. For $d = 2$ the statement is obvious: the lines $l_1x_1 + l_2x_2 = \text{const}$ and $x_1 = \text{const}$ can both intersect only a bounded number of squares $K_n(y_1, y_2)$ (we do not even check whether $K_n(y_1, y_2) \subset E_n$). Let us assume the assertion is true for $d - 1$ and consider situation for d.
Thanks to Lemma 22, we only need to estimate (for big n) the $d - 1$-dimensional volume of $H_t \cap E_n$ to be not greater than $K'M^{-n(d-1)}n^{d-2}$ for all t. Let us denote this random variable by $M^{-n(d-1)} \cdot g_n(t)$. As it is a lipschitz function of t, it is enough to check that

$$g_n(t) \leq K'n^{(1+\varepsilon)(d-2)}$$

for sufficiently big n, not for all t but only for some M^{-nd}-dense subset T_n. We will choose $\{T_n\}$ in such a way that $T_n \subset T_{n+1}$.
Consider $g_{n+1}(t)$ as a random variable, conditioned on $g_n(t)$. For every cube $K_n(x_1, \ldots, x_d)$ intersecting H_t, the volume of $H_t \cap K_n(x_1, \ldots, x_d)$ equals the sum of volumes of $H_t \cap K_{n+1}(y_1, \ldots, y_d)$ over all $K_{n+1}(y_1, \ldots, y_d) \subset K_n(x_1, \ldots, x_d)$ and each $K_{n+1}(y_1, \ldots, y_d)$ appears in E_{n+1} with probability p. Hence,

$$\mathbb{E}\Big(\sum_{K_{n+1}(y_1,\ldots,y_d)\subset K_n(x_1,\ldots,x_d)} \operatorname{vol}(H_t \cap E_{n+1} \cap K_{n+1}(y_1, \ldots, y_d))\Big) = p \operatorname{vol}(H_t \cap E_n \cap K_n(x_1, \ldots, x_d)) \tag{3.8}$$

If events happening in different $K_n(x_1, \ldots, x_d)$ were independent (as it is for fractal percolations), we would be able to estimate $g_{n+1}(t)$ like in the proof of Theorem 6 because the random variables

$$h_n(x_1, \ldots, x_n)(t) = \sum_{K_{n+1}(y_1,\ldots,y_d)\subset K_n(x_1,\ldots,x_d)} \operatorname{vol}(H_t \cap E_{n+1} \cap K_{n+1}(y_1, \ldots, y_d)) \tag{3.9}$$

would be independent. That this is not the case in our situation is the main difficulty in the proof.
Given $n \in \mathbb{N}$ and $t \in T_n$, we will say the event $B(n, t)$ holds if the random variables $h_n(\cdot)(t)$ can be divided into at most $c_6 n^{(1+\varepsilon)(d-3)}$ subfamilies $H_n^i(t)$ such that inside each $H_n^i(t)$ all the $h_n(\cdot)(t)$ are independent. The constant c_6 will be chosen in the future. We will fix the choice of partition $\{H_n^i(t)\}$ (say, the first in a lexicographical order) if many are possible.

We denote

$$z_n^i(t) = \mathrm{vol}(H_t \cap \bigcup_{(x_1,\ldots,x_d)\in H_n^i(t)} K_n(x_1,\ldots,x_d)) \tag{3.10}$$

and

$$Z_n^i(t) = \sum_{(x_1,\ldots,x_d)\in H_n^i(t)} h_n(x_1,\ldots,x_n)(t) \tag{3.11}$$

We will say $H_n^i(t)$ is *large* if

$$z_n^i(t) > M^{-n(d-1)} n^{1+\varepsilon},$$

otherwise it is *small*.
For any small $H_n^i(t)$ we can write

$$Z_n^i(t) \le z_n^i(t)$$

(as $E_{n+1} \subset E_n$). For any large $H_n^i(t)$ it has at least $n^{1+\varepsilon}$ elements, and we can apply Azuma-Hoeffding inequality to obtain

$$\mathbb{P}(Z_n^i(t) < (1+\varepsilon)p z_n^i(t)) > 1 - \gamma^{n^{1+\varepsilon}} \tag{3.12}$$

for some $\gamma < 1$. We say that event $C(n,t)$ holds if $Z_n^i(t) < (1+\varepsilon)p z_n^i(t)$ holds for all large $H_n^i(t)$. Our main interest is the event

$$A(n,t) = C(n,t) \vee \neg B(n,t).$$

We claim that, almost surely, there are only finitely many (n,t) for which $A(n,t)$ fails (independently of the choice of c_6). Indeed, if $B(n,t)$ fails then $A(n,t)$ is automatically true and if $B(n,t)$ holds the number of $H_n^i(t)$ (large or not) is not greater than $c_6 n^{(1+\varepsilon)(d-3)}$. Hence, (3.12) implies

$$\sum_n \sum_{t\in T_n} (1 - \mathbb{P}(A(n,t))) \le \sum_n M^{nd} c_6 n^{(1+\varepsilon)(d-3)} \gamma^{n^{1+\varepsilon}} < \infty$$

and the claim follows.
Our second claim is that, almost surely, for c_6 large enough there are only finitely many (n,t) for which $B(n,t)$ fails. This claim follows from the induction assumption.
Consider any $K_n(x_1,\ldots,x_d)$ and $K_n(y_1,\ldots,y_d)$ intersecting H_t. If $x_i \ne y_i$ for all i then $h_n(x_1,\ldots,x_d)(t)$ and $h_n(y_1,\ldots,y_d)(t)$ are independent. We need to estimate for any $(x_1,\ldots,x_d)$ the maximal possible number

of different $(y_1,\ldots,y_d)$ such that $h_n(x_1,\ldots,x_d)(t)$ and $h_n(y_1,\ldots,y_d)(t)$ are not independent.
As $K_n(x_1,\ldots,x_d)$ and $K_n(y_1,\ldots,y_d)$ intersect H_t, we have

$$\sum l_j x_j, \sum l_j y_j \in (t - \frac{1}{2}M^{-n}\sum l_j, t + \frac{1}{2}M^{-n}\sum l_j).$$

Assume that for fixed i, $x_i = y_i$. It implies that

$$\sum_{j\neq i} l_j x_j \approx \sum_{j\neq i} l_j y_j \approx t - l_i x_i.$$

More precisely,

$$\left|\sum_{j\neq i} l_j(x_j - y_j)\right| \leq M^{-n}\sum l_j.$$

Hence, the number of such $(y_1,\ldots,y_d)$ is at most as big as the number of $d-1$-dimensional cubes

$$K_n(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_d) \in E_n^{(1)} \times \ldots \times E_n^{(i-1)} \times E_n^{(i+1)} \times \ldots \times E_n^{(d)}$$

intersecting one of at most $2(\sum l_j)/l_i$ hyperplanes

$$\sum_{j\neq i} l_j y_j = \sum_{j\neq i} l_j x_j + k \min_j l_j,$$

k varying between $-(\sum l_j)/l_i$ and $(\sum l_j)/l_i$. Let us denote

$$c_7 = \frac{\sum l_j}{\min_j l_j}.$$

Our assumption is $M^{-1} < p_i \leq 1$ for all i and $p = \prod p_i < M^{-d+1}$. This implies

$$\prod_{j\in\{1,\ldots,d\}\setminus\{i\}} p_j < M^{-d+2}$$

for all $i \in \{1,\ldots,d\}$. Hence, the restricted system (product of all $E^{(j)}$ except $E^{(i)}$) satisfies assumption of Proposition 20 and we can apply the induction assumption. This means that there are at most $K_{d-1}dc_7n^{(1+\varepsilon)(d-3)}$ such $(y_1,\ldots,y_d)$.
We can now apply Lemma 21 to the dependency graph to divide all the events $h_n(x_1,\ldots,x_d)(t)$ into $K_{d-1}dc_7n^{(1+\varepsilon)(d-3)}+1$ subfamilies of independent events. This ends the proof of the second claim.
From the two claims, the assertion follows easily. Let

$$g_n = \sup_t g_n(t).$$

Then, as soon as n is big enough for $B(n,t)$ and $A(n,t)$ (and hence, $C(n,t)$ as well) always to happen, we will have

$$g_n \leq (1+\varepsilon)pM^{d-1}g_{n-1} + c_6 n^{(1+\varepsilon)(d-2)} + c_5, \tag{3.13}$$

where the first term comes from large $H_n^i(t)$, the second term comes from small $H_n^i(t)$, and the third term from lipschitz approximation ((3.12) only gives us $g_n(t)$ for $t \in T_n$). As $pM^{d-1} < 1$, the inductive formula (3.13) implies the assertion. □

3.3. **Proof of Theorem 16.** For $d = 2$ Theorem 16 was already proven in [5] but only for the case when the angle is 45° . Our proof, however, is not similar to [5], rather it has the same flavour as the proof of the main result of [21]. We will begin the proof by strengthening the assumptions.

We will study the $d-1$-dimensional volume of $H_t \cap E_n$, we will denote this random variable by $M^{-n(d-1)} \cdot g_n(t)$. We will consider $g_{n+1}(t)$ as a random variable depending on $g_n(t)$. We have equation (3.8). We define $h_n(x_1, \ldots, x_d)(t)$ by (3.9). We are going to estimate the volume of $H_t \cap E_n$ from below, using Azuma-Hoeffding Theorem. The main difficulty in the proof is the dependence problem, which we will deal with like in the proof of Proposition 20.

The dependence problem is nonexisting for $d = 2$. Indeed, any nonhorizontal and nonvertical line ℓ^α will intersect only a bounded number of level n squares in any given row (or column). Hence, in this case the argument given in [21] works with slight modifications. In what follows, $d \geq 3$.

Given n, t, we divide random variables $h_n(\cdot)(t)$ into subfamilies $H_n^i(t)$ such that inside each $H_n^i(t)$ all the events $h_n(\cdot)(t)$ are independent. Like in the proof of Proposition 20, we say that the event $B(n,t)$ holds if we can have $c_8 n^{(1+\varepsilon)(d-3)}$ or less families $H_n^i(t)$. We denote $z_n^i(t)$ and $Z_n^i(t)$ as in (3.10),(3.11). We will say that $H_n^i(t)$ is *large* if

$$z_n^i(t) > M^{-n(d-1)} n^{1+\varepsilon},$$

otherwise it is *small*.

Conditioned on E being nonempty, almost surely the d-dimensional volume of E_n is $\approx cp^n$. Hence, almost surely we will be able to find infinitely many $N_j > N$ and corresponding t_j such that

$$g_{N_j}(t_j) > p^{N_j} M^{N_j(d-1)(1-\varepsilon)} > e^{\varepsilon N_j} + c_7.$$

Without weakening the assumptions, $N_j > j$. For $n > N_j$ let $T_n^{(j)}$ be a M^{-nd}-dense subset of $I_j = (t_j - M^{-N_j d}, t_j + M^{-N_j d})$ satisfying $T_{n+1}^{(j)} \supset T_n^{(j)}$. In particular,

$$g_{N_j}(t) > e^{\varepsilon N_j}$$

for all $t \in I_j$. For $t \in I_j$ for every small $H_n^i(t)$ we can write

$$Z_n^i(t) \geq 0.$$

For large $H_n^i(t)$ the Azuma-Hoeffding inequality gives

$$\mathbb{P}(Z_n^i(t) > (1-\varepsilon)pz_n^i(t)) > 1 - \gamma^{n^{1+\varepsilon}} \tag{3.14}$$

for some $\gamma < 1$. We say the event $D(n,t)$ holds if $Z_n^i(t) > (1-\varepsilon)pz_n^i(t)$ for all large $H_n^i(t)$. We define

$$E(n,t) = D(n,t) \vee \neg B(n,t).$$

As

$$\sup_j \sum_{n \geq N_j} \gamma^{n^{1+\varepsilon}} \cdot \sharp T_n^{(j)} < \infty,$$

almost surely there exist infinitely many j's for which the events $E(n,t)$ hold for all $n \geq N_j$ and $t \in T_n^{(j)}$. Because (3.6) holds, we can apply Proposition 20 to prove that, almost surely, events $B(n,t)$ hold for all sufficiently big n for all $t \in T_n^{(j)}$ for all $N_j \leq n$ (like in the proof of the second claim in the main proof of Proposition 20). Hence, we can choose j with arbitrarily big N_j such that both $B(n,t)$ and $D(n,t)$ hold for all $n \geq N_j$ for all $t \in T_n^{(j)}$.

We denote

$$g_n = \inf_{I_j} g_n(t).$$

We have

$$g_{n+1} \geq (1-\varepsilon)pM^{1-d}\left(g_n - c_8 n^{(1+\varepsilon)(d-3)} n^{1+\varepsilon}\right) - c_5,$$

where the first term comes from the growth of the part of $g_n(t)$ contained in the big $H_n^i(t)$ and the second part is the lipschitz correction (the first part we only know for $t \in T_n^{(j)}$). If N_j was big enough, we can prove inductively that

$$g_n > e^{\varepsilon n} > 0$$

for all $n \geq N_j$. In particular, the algebraic sum of sets $E^{(i)}$ will contain I_j. We are done.

Remark 24. *In the proofs of Proposition 20 and Theorem 16 we do not assume that H_t are hyperplanes. They might be any codimension 1 surface sufficiently close to a hyperplane as for the lipschitz property (Lemma 23) to hold. For example, the same argument can be used to show that the assertion of Theorem 16 holds if we replace algebraic sum with the algebraic multiplication.*

4. Distance sets for fractal percolations

In this section we are going to present a related result on distance sets. A long standing conjecture due to Falconer [17] says that for any set in $\mathbb{R}^d$ with Hausdorff dimension greater than $d/2$, the distance set has positive length. We prove that for fractal percolation it is enough to require that the Hausdorff dimension is greater than 1/2 for the distance set to contain an interval.

The proof is almost identical as the proof of Theorem 16, so we are only going to sketch it.

For a pair of sets $A, B \in \mathbb{R}^d$ we define their *distance set* as

$$D(A, B) = \{|x - y|; x \in A, y \in B\}.$$

Similarly,

$$D(A) := D(A, A).$$

Theorem 25. *Let E_1, E_2 be nonempty realizations of two fractal percolations in $\mathbb{R}^d$ with common scale M and with probabilities p_1, p_2. Assume $p_1, p_2 > M^{-d}$ and*

$$p_1 p_2 > M^{-2d+1}.$$

Then, almost surely $D(E_1, E_2)$ contains an interval.

Theorem 26. *Let E be a nonempty realization of a fractal percolation in $\mathbb{R}^d$ for probability $p > M^{-d+1/2}$. Then, almost surely $D(E)$ contains an interval.*

Proof. Both theorems are proven in basically the same way. For Theorem 25 almost surely we can find two cubes: $K_n(x_1, \ldots, x_d)$ with nonempty intersection with E_1 and $K_n(y_1, \ldots, y_d)$ with nonempty intersection with E_2. For Theorem 26 we find two distinct cubes with nonempty intersection with E. By going to subcubes, we can freely assume that $x_i \neq y_i$ for all i and that the two cubes are in large distance relative to their size. We can then consider the cartesian product $(E_1 \cap K_n(x_1, \ldots, x_d)) \times (E_2 \cap K_n(y_1, \ldots, y_d)$ (or $(E \cap K_n(x_1, \ldots, x_d)) \times (E \cap K_n(y_1, \ldots, y_d))$ as product of two independent random constructions.

This product is similar to one constructed in section 3.1, but it has fewer dependencies. We can consider its intersections with surfaces

$$H_t = \{(x,y); \rho(x,y) = t\}.$$

Those surfaces are sufficiently close to hyperplanes that Lemma 23 still holds, though maybe with different constant. The proof of Theorems 25 and 26, now reduces to the proof of Theorem 16. □

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