MARKOV SEMIGROUPS AND THEIR APPLICATIONS

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Abstract. Some recent results concerning asymptotic properties of Markov operators and semigroups are presented. Applications to diffusion processes and to randomly perturbed dynamical systems are given.

1. Introduction

Markov operators were introduced to study dynamical systems and dynamical systems with stochastic perturbations. These systems describe a movement of points. If we look at such a system statistically, then we observe the evolution of a probability measure describing the distribution of points on the phase space X. In this way we obtain a transformation P defined on the space of probability measures. Assume that P is defined by a transition probability function, i.e. the transformation of Dirac measures δ_x determines P. Then P is linear. If there is some standard measure m on the space X, then we can only consider measures which are absolutely continuous with respect to m. In that case instead of the transformation of measures we consider the transformation of densities of these measures. In this way we obtain a linear transformation of the space of integrable functions which preserves the set of densities. Such a transformation is called a Markov operator.

It should be noted that also nonlinear Markov operators and semigroups appear in applications. For example Boltzmann equation [2, 65] and its simplified version Tjon-Wu equation [30, 61] generate a nonlinear Markov semigroups. Also coagulation-fragmentation processes are described by nonlinear Markov semigroups [4, 13, 25]. Though it is a little easier to study Markov operators on densities, sometimes it is more convenient to consider Markov operators on measures. Such a situation appears in constructions of fractal measures [5, 29, 31].

The main subject of our paper are Markov operators and Markov semigroups acting on the set of densities. Such operators and semigroups have been intensively studied because they play a special role in applications. The book of Lasota and Mackey [27] is an excellent survey of many results on this subject. Semigroups of Markov operators are generated by partial differential equations (transport equations). Equations of this type appear in the theory of stochastic processes (diffusion

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processes and jump processes), in the theory of dynamical systems and in population dynamics.

In this paper we present recent results in the theory of Markov operators and semigroups and illustrate them by some physical and biological applications. Presented results are based on papers [44, 45, 46, 47, 54].

The organization of the paper is as follows. Section 2 contains the definitions of a Markov operator and a Markov semigroup. Then we give examples of Markov operators connected with dynamical systems and dynamical systems with stochastic perturbations and of Markov semigroups generated by generalized Fokker-Planck equations and transport equations. In Section 3 we study asymptotic properties of Markov operators and semigroups: asymptotic stability and sweeping. Theorems concerning asymptotic stability and sweeping allow us to formulate the Foguel alternative. This alternative says that under suitable conditions a Markov operator (semigroup) is asymptotically stable or sweeping. Then we define a notion called a Hasminskiĭ function. This notion is very useful in proofs of asymptotic stability of Markov semigroups. In Section 4 we give some applications of general results to differential equations connected with diffusion and jump processes. In Section 5 we present some results concerning other asymptotic properties of Markov operators: completely mixing and limit distribution [8, 52, 53].

2. Markov operators and semigroups

2.1. **Definitions.** Let the triple (X,Σ,m) be a σ-finite measure space. Denote by D the subset of the space $L^1=L^1(X,\Sigma,m)$ which contains all densities

$$D=\{f\in L^1:\ f\ge 0,\ \|f\|=1\}.$$

A linear mapping $P:L^1\to L^1$ is called a *Markov operator* if $P(D)\subset D$.

One can define a Markov operator by means of a *transition probability function.* We recall that $\mathcal{P}(x,A)$ is a transition probability function on (X,Σ) if $\mathcal{P}(x,\cdot)$ is a probabilistic measure on (X,Σ) and $\mathcal{P}(\cdot,A)$ is a measurable function. Assume that $\mathcal{P}$ has the following property

$$m(A)=0\Longrightarrow \mathcal{P}(x,A)=0 \text{ for } m\text{-a.e. } x. \tag{1}$$

Then for every $f\in D$ the measure

$$\mu(A)=\int f(x)\mathcal{P}(x,A)\,m(dx)$$

is absolutely continuous with respect to the measure m and the formula $Pf=d\mu/dm$ defines a Markov operator $P:L^1\to L^1$. Moreover, if $P^*:L^\infty\to L^\infty$ is the adjoint operator of P then $P^*g(x)=\int g(y)\,\mathcal{P}(x,dy)$. There are Markov operators which are not given by transition probability functions [17]. But if X is a Polish space (i.e. a complete separable metric space), $\Sigma=\mathcal{B}(X)$ is the σ-algebra of Borel subsets of X, and m is a probability Borel measure on X then every Markov operator on $L^1(X,\Sigma,m)$ is given by a transition probability function [23].

A family $\{P(t)\}_{t\ge0}$ of Markov operators which satisfies conditions:

(a) $P(0)=\mathrm{Id}$,
(b) $P(t+s)=P(t)P(s)$ for $s,\,t\ge 0$,
(c) for each $f\in L^1$ the function $t\mapsto P(t)f$ is continuous

is called a *Markov semigroup.*

Now we give some examples of Markov operators and Markov semigroups.

2.2. **Frobenius–Perron operator.** This operator describes statistical properties of simple point to point transformations [27]. Let (X, Σ, m) be a σ-finite measure space and let S be a measurable transformation of X. If a measure μ describes the distribution of points in the phase space X, then the measure ν given by the formula $\nu(A) = \mu(S^{-1}(A))$ describes the distribution of points after the action of the transformation S. Assume that the transformation S is non-singular, that is if $m(A) = 0$ then $m(S^{-1}(A)) = 0$. If the measure μ is absolutely continuous with respect to the measure m, then the measure ν is also absolutely continuous. If f is the density of μ and if g is the density of ν then we define the operator P_S by $P_S f = g$. This operator can be extended to a linear operator $P_S : L^1 \to L^1$. In this way we obtain a Markov operator which is called the *Frobenius–Perron operator* for the transformation S.

Remark 1. Frobenius–Perron operators can be successfully used to study ergodic properties of transformations [27]. The general rule is: the better ergodic properties a transformation has the stronger convergence of the iterates of Frobenius–Perron operator is. Namely, if the measure m is probabilistic and invariant with respect to S then S is ergodic, mixing or exact if for each density f the sequence $P^n f$ is, respectively, Cesàro, weakly or strongly convergent to $\mathbf{1}_X$.

2.3. **Iterated Function System.** Let $S_1, \dots, S_n$ be non-singular transformations of the space X. Let $P_1, \dots, P_n$ be the Frobenius–Perron operators corresponding to the transformations $S_1, \dots, S_n$. Let $p_1(x), \dots, p_n(x)$ be non-negative measurable functions defined on X such that $p_1(x) + \cdots + p_n(x) = 1$ for all $x \in X$. We consider the following process. Take a point x. We choose a transformation S_i with probability $p_i(x)$ and $S_i(x)$ describes the position of x after the action of the system. The evolution of densities of the distribution is described by the Markov operator

$$Pf = \sum_{i=1}^{n} P_i(p_i f).$$

2.4. **Integral operator.** If $k : X \times X \to [0, \infty)$ is a measurable function such that

$$\int_X k(x, y)\, m(dx) = 1$$

for almost all $y \in X$, then

$$Pf(x) = \int_X k(x, y) f(y)\, m(dy) \tag{2}$$

is a Markov operator. The function k is called a *kernel* of the operator P.

Many biological and physical processes can be modelled by means of randomly perturbed dynamical systems whose stochastic behaviour is described by integral Markov operators. Such systems are generally of the form

$$X_{n+1} = S(X_n, \xi_{n+1}), \tag{3}$$

where $(\xi_n)_{n=1}^{\infty}$ is a sequence of independent random variables (or elements) with the same distribution and the initial value of the system X_0 is independent of the sequence $(\xi_n)_{n=1}^{\infty}$. Studying systems of the form (3) we are often interested in the behaviour of the sequence of the measures (μ_n) defined by

$$\mu_n(A) = \mathrm{Prob}(X_n \in A).$$

The evolution of these measures can be described by a Markov operator P given by $\mu_{n+1} = P\mu_n$. The operator P is defined on the space of probability measures. Assume that for almost all y the distribution μ_y of the random variable $S(y, \xi_n)$ is absolutely continuous with respect to m. Let $k(x, y)$ be the density of μ_y and the operator P be given by (2). Then P describes the evolution of the system (3).

Integral Markov operators appear in a two phase model of cell cycle proposed by J. Tyrcha [63] which generalizes the model of Lasota–Mackey [28] and the tandem model of Tyson–Hannsgen [64].

2.5. **Fokker-Planck equation.** Consider the Stratonovitch stochastic differential equation

$$dX_t = \sigma(X_t) \circ dW_t + \sigma_0(X_t)\, dt, \tag{4}$$

where W_t is a m-dimensional Brownian motion, $\sigma(x) = [\sigma_j^i(x)]$ is a $d \times m$ matrix and $\sigma_0(x)$ is a vector in $\mathbb{R}^d$ with components $\sigma_0^i(x)$ for every $x \in \mathbb{R}^d$. We assume that for all $i = 1, ..., d$, $j = 0, ..., m$ the functions σ_j^i are sufficiently smooth and have bounded derivatives of all orders, and the coefficients of the matrix σ are also bounded. Recall that the Itô equivalent equation is of the form

$$dX_t = \sigma(X_t)\, dW_t + b(X_t)\, dt, \tag{5}$$

where $b_i = \sigma_0^i + \frac{1}{2}\sum_{k=1}^m \sum_{j=1}^d \sigma_k^j \frac{\partial \sigma_k^i}{\partial x_j}$. Assume that X_t is a solution of (4) or (5) such that the distribution of X_0 is absolutely continuous and has the density $v(x)$. Then X_t has also the density $u(x, t)$ and u satisfies the Fokker-Planck equation:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^d \frac{\partial(\sigma_0^i(x)u)}{\partial x_i}, \tag{6}$$

where $a_{ij}(x) = \frac{1}{2}\sum_{k=1}^m \sigma_k^i(x)\sigma_k^j(x)$. Equation (6) can be written in another equivalent form

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^d \frac{\partial^2(a_{ij}(x)u)}{\partial x_i \partial x_j} - \sum_{i=1}^d \frac{\partial(b_i(x)u)}{\partial x_i}. \tag{7}$$

Note that the $d \times d$-matrix $a = [a_{ij}]$ is symmetric and nonnegative definite, i.e. $a_{ij} = a_{ji}$ and

$$\sum_{i,j=1}^d a_{ij}(x)\lambda_i\lambda_j \geq 0 \tag{8}$$

for every $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$, so we only assume weak ellipticity of the operator on the right hand side of equation (6). Let consider the operator

$$Af = \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}(x)\frac{\partial f}{\partial x_j}\right) - \sum_{i=1}^d \frac{\partial(\sigma_0^i(x)f)}{\partial x_i}$$

on the set $E = \{f \in L^1(\mathbb{R}^d) \cap C_b^2(\mathbb{R}^d) : Af \in L^1(\mathbb{R}^d)\}$, where $C_b^2(\mathbb{R}^d)$ denotes the set of all twice differentiable bounded functions whose derivatives of order ≤ 2 are continuous and bounded. If $v \in C_b^2(\mathbb{R}^d)$ then equation (6) has in any time interval $[0, T]$ a unique classical solution u which satisfies the initial condition $u(x, 0) = v(x)$ and this solution and its spatial derivatives up to order 2 are uniformly bounded on $[0, T] \times \mathbb{R}^d$ (see [59], [21]). But if the initial function has a compact support, i.e. $v \in C_c^2(\mathbb{R}^d)$, then the solution $u(x,t)$ of (6) and its spatial derivatives converge exponentially to 0 as $\|x\| \to \infty$. From the Gauss-Ostrogradski theorem it follows that the integral $\int u(x,t)\,dx$ is constant. Let $P(t)v(x) = u(x,t)$ for $v \in C_c^2(\mathbb{R}^d)$ and $t \ge 0$. Since the operator $P(t)$ is a contraction on $C_c^2(\mathbb{R}^d)$ it can be extended to a contraction on $L^1(\mathbb{R}^d)$. Thus the operators $\{P(t)\}_{t\ge 0}$ form a Markov semigroup. We have $P(t)(C_c^2(\mathbb{R}^d)) \subset C_b^2(\mathbb{R}^d)$ for $t \ge 0$. According to Proposition 1.3.3 of [18] the closure of the operator A generates the semigroup $\{P(t)\}_{t\ge 0}$. The adjoint operators $\{P^*(t)\}_{t\ge 0}$ form a semigroup on $L^\infty(\mathbb{R}^d)$ given by the formula

$$P^*(t)g(x) = \int_{\mathbb{R}^d} g(y)\mathcal{P}(t,x,dy) \quad \text{for } g \in L^\infty(\mathbb{R}^d),$$

where $\mathcal{P}(t,x,A)$ is the transition probability function for the diffusion process X_t, i.e. $\mathcal{P}(t,x,A) = \operatorname{Prob}(X_t \in A)$ and X_t is a solution of equation (4) with the initial condition $X_0 = x$.

2.6. **Liouville equation.** If we assume that $a_{ij} \equiv 0$ in equation (7), then we obtain the *Liouville equation*

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i(x)u\big). \tag{9}$$

As in the previous example, equation (9) generates a Markov semigroup given by $P(t)v(x) = u(x,t)$, where $v(x) = u(x,0)$. The semigroup $\{P(t)\}_{t\ge 0}$ can be given explicitly. Namely, for each $\bar{x} \in X$ denote by $\pi_t\bar{x}$ the solution $x(t)$ of the equation

$$x'(t) = b(x(t)) \tag{10}$$

with the initial condition $x(0) = \bar{x}$. Then

$$P(t)f(x) = f(\pi_{-t}x)\det\big[\frac{d}{dx}\pi_{-t}x\big] \quad \text{for } f \in L^1$$

is the Frobenius-Perron operator corresponding to the map $x \mapsto \pi_t x$. Equation (9) has the following interpretation. In the space $\mathbb{R}^d$ we consider the movement of points given by equation (10). We look at this movement statistically, that is, we consider the evolution of densities of the distribution of points. Then this evolution is described by (9).

2.7. **Transport equations.** If the equation $\frac{\partial u}{\partial t} = Au$ generates a Markov semigroup $\{S(t)\}_{t\ge 0}$, K is a Markov operator, and $\lambda > 0$, then the equation

$$\frac{\partial u}{\partial t} = Au - \lambda u + \lambda K u \tag{11}$$

also generates a Markov semigroup. From the Phillips perturbation theorem [15], equation (11) generates a Markov semigroup $\{P(t)\}_{t\ge 0}$ on L^1 given by

$$P(t)f = e^{-\lambda t}\sum_{n=0}^{\infty} \lambda^n S_n(t)f, \tag{12}$$

where $S_0(t) = S(t)$ and

$$S_{n+1}(t)f = \int_0^t S(t-s)KS_n(s)f\,ds, \quad n \ge 0.$$

Equations of this type appear in such diverse areas as population dynamics [36, 40], in the theory of jump processes [49, 62], and in astrophysics – where describes the fluctuations in the brightness of the Milky-Way [12].

In many applications A is the operator from equation (9) and the Markov operator K corresponds to some transition probability function $\mathcal{P}(x,E)$. In this case equation (11) has an interesting probabilistic interpretation. Consider a collection of particles moving under the action of the equation $x' = b(x)$. This motion is modified in the following way. In every time interval $[t, t+\Delta t]$ a particle with the probability $\mathcal{P}(x,E)\Delta t + o(\Delta t)$ changes its position from x to a point from the set E. Then any solution of (11) is the probability density function of the position of the particle at time t.

Time and size dependent models of populations can be described by a transport equation of the form (11), namely

$$\frac{\partial u}{\partial t} + \frac{\partial (V(x)u)}{\partial x} = -u(x,t) + Ku(x,t). \tag{13}$$

Here the function $V(x)$ is the velocity of the growth of the size of a cell and K is a Markov operator describing the process of replication. If we assume that the size of a daughter cell is exactly a half of the size of the mother cell, then $Kf(x) = 2f(2x)$. If we consider unequal division then K is some integral operator.

It is interesting that more advanced models of population dynamics lead to equations similar to (13), but instead of the operator $K - I$ on the right-hand side of (13) appears a non-bounded linear operator (e.g. [16]). Also these equations often generate Markov semigroups [57].

Equation (11) also describes the distribution of the solutions of a Poisson driven stochastic differential equation ([62]):

$$dX_t = b(X_t)\,dt + f(X_t)\,dN_t,$$

where N_t is the Poisson process. Here $Au = -\sum_{i=1}^d \frac{\partial}{\partial x_i}\big(b_i(x)u\big)$ and K is the Frobenius-Perron operator corresponding to the transformation $T(x) = x + f(x)$.

2.8. **Randomly flashing diffusion.** Consider the stochastic equation

$$dX_t = (Y_t\sigma(X_t))\,dW_t + b(X_t)\,dt, \tag{14}$$

where Y_t is a homogeneous Markov process with values 0 and 1 independent of W_t and X_0. Equation (14) describes the process which randomly jumps between stochastic and deterministic states. Such processes appear in transport phenomena in sponge–type structures [3, 10, 35]. This process also generates a Markov semigroup but on the space $L^1(\mathbb{R}\times\{0,1\})$. The densities of the distribution of this process satisfies the following system of equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -pu_1 + qu_0 + \dfrac{\partial^2}{\partial x^2}\big(a(x)u_1\big) - \dfrac{\partial}{\partial x}\big(b(x)u_1\big) \\ \dfrac{\partial u_0}{\partial t} = pu_1 - qu_0 - \dfrac{\partial}{\partial x}\big(b(x)u_0\big). \end{cases} \tag{15}$$

In a similar way we can introduce a notion of a multistate diffusion process on $\mathbb{R}^d$ and check that it generates a Markov semigroup [54]. Let Y_t be a continuous time Markov chain on the phase space $\Gamma = \{1,\dots,k\}$, $k \ge 2$, such that the transition probability from the state j to the state $i \ne j$ in time interval Δt equals $p_{ij}\Delta t + o(\Delta t)$. We assume that $p_{ij} > 0$ for all $i \ne j$. Let b be a $d-$dimensional vector function defined on $\mathbb{R}^d \times \Gamma$. Let X_0 be a $d-$dimensional random variable independent of Y_t. Consider the stochastic differential equation

$$dX_t = \sigma(X_t, Y_t)\, dW_t + b(X_t, Y_t)\, dt.$$

The pair (X_t, Y_t) constitutes a Markov process on $\mathbb{R}^d \times \Gamma$. We assume that the random variable X_0 has an absolutely continuous distribution. Then the random variable X_t has an absolutely continuous distribution for each $t > 0$. Define the function u by the formula

$$\operatorname{Prob}((X_t, Y_t) \in E \times \{i\}) = \int_E u(x, i, t)\, dx.$$

Denote by A_l the differential operators

$$A_l f = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij}(x, l) f)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i(x, l) f)}{\partial x_i}.$$

Let $p_{ii} = -\sum_{j \neq i} p_{ji}$ and denote by M the matrix $[p_{ij}]$. We use the notation $u_i(x, t) = u(x, i, t)$ and $u = (u_1, \ldots, u_k)$ is a vertical vector. Then the vector u satisfies the following equation

$$\frac{\partial u}{\partial t} = Mu + Au, \tag{16}$$

where $Au = (A_1 u_1, \ldots, A_k u_k)$ is also a vertical vector. The operator A_l generates a semigroup $\{S(t)(l)\}_{t \geq 0}$ of Markov operators on the space $L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$, where μ is the Lebesgue measure.

Let $\mathcal{B}(\mathbb{R}^d \times \Gamma)$ be the σ–algebra of Borel subsets of $\mathbb{R}^d \times \Gamma$ and let m be the product measure on $\mathcal{B}(\mathbb{R}^d \times \Gamma)$ given by $m(B \times \{i\}) = \mu(B)$ for each $B \in \mathcal{B}(\mathbb{R}^d)$ and $1 \leq i \leq k$. The operator A generates a Markov semigroup $\{S(t)\}_{t \geq 0}$ on the space $L^1(\mathbb{R}^d \times \Gamma, \mathcal{B}(\mathbb{R}^d \times \Gamma), m)$ given by the formula

$$S(t)f = (S(t)(1)f_1, \ldots, S(t)(k)f_k),$$

where $f_i(x) = f(x, i)$ for $x \in \mathbb{R}^d$, $1 \leq i \leq k$. Now, let λ be a constant such that $\lambda = \max\{-p_{11}, \ldots, -p_{kk}\}$ and $K = \lambda^{-1} M + I$. Then (16) can be written in the form

$$\frac{\partial u}{\partial t} = Au - \lambda u + \lambda K u \tag{17}$$

and the matrix K is a Markov operator on $L^1(\mathbb{R}^d \times \Gamma, \mathcal{B}(\mathbb{R}^d \times \Gamma), m)$. Equation (17) has the form (11) and generates a Markov semigroup $\{P(t)\}_{t \geq 0}$ given by (12).

If $\sigma \equiv 0$ then the process X_t describes the movement of points under the action of k dynamical systems $\pi_t^i(x)$ corresponding to the equations $x' = b(x, i)$, $i = 1, \ldots, k$. The Markov chain Y_t decides which dynamical system acts at time t. We will call such a stochastic process a *randomly controlled dynamical system* and we will study it in subsection 4.6. Let E be a Borel subset of $\mathbb{R}^n$. If for each $1 \leq i \leq k$ and for all $t \geq 0$ we have $\pi_t^i(E) \subset E$, then the operator A generates a semigroup $\{S(t)\}_{t \geq 0}$ of Markov operators on the space $L^1(E \times \Gamma, \mathcal{B}(E \times \Gamma), m)$.

3. Asymptotic properties of Markov operators and semigroups

Now we introduce some notions which characterize the behaviour of Markov semigroups $\{P(t)\}_{t\ge 0}$ when $t \to \infty$ and powers of Markov operators P^n when $n \to \infty$. Since the powers of Markov operators also form a (discrete time) semigroup we will consider only Markov semigroups.

3.1. **Asymptotic stability.** Consider a Markov semigroup $\{P(t)\}_{t\ge 0}$. A density f_* is called *invariant* if $P(t)f_* = f_*$ for each $t > 0$. The Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for} \quad f \in D.$$

If the semigroup $\{P(t)\}_{t\ge 0}$ is generated by some differential equation then the asymptotic stability means that all solutions of the equation starting from a density converge to the invariant density.

In order to formulate the main result of this section we need an auxiliary definition. A Markov semigroup $\{P(t)\}_{t\ge 0}$ is called *partially integral* if there exist $t_0 > 0$ and a measurable non-negative function $q(x, y)$ such that

$$\int_X \int_X q(x, y)\, m(dx)\, m(dy) > 0 \tag{18}$$

and

$$P(t_0)f(x) \ge \int_X q(x, y) f(y)\, m(dy) \quad \text{for every } f \in D. \tag{19}$$

The main result of this part is the following

Theorem 1 ([54]). *Let $\{P(t)\}_{t\ge 0}$ be a partially integral Markov semigroup. Assume that the semigroup $\{P(t)\}_{t\ge 0}$ has an invariant density f_* and has no other periodic points in the set of densities. If $f_* > 0$ a.e. then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

The proof of Theorem 1 is based on the theory of Harris operators given in [19, 24]. Now we formulate corollaries which are often used in applications. Let f be a measurable function. The *support* of f is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x \in X : f(x) \ne 0\}.$$

We say that a Markov semigroup $\{P(t)\}_{t\ge 0}$ *spreads supports* if for every set $A \in \Sigma$ and for every $f \in D$ we have

$$\lim_{t\to\infty} m(\operatorname{supp} P(t)f \cap A) = m(A)$$

and *overlaps supports* if for every $f, g \in D$ there exists $t > 0$ such that

$$m(\operatorname{supp} P(t)f \cap \operatorname{supp} P(t)g) > 0.$$

Corollary 1 ([54]). *A partially integral Markov semigroup which spreads supports and has an invariant density is asymptotically stable.*

Corollary 2 ([54]). *A partially integral Markov semigroup which overlaps supports and has an invariant density $f_* > 0$ a.e. is asymptotically stable.*

These corollaries generalize some earlier results [6, 37, 50, 53] for integral Markov semigroups. Another proof of Corollary 2 is given in [7].

Corollary 1 remains true also for the Frobenius-Perron operators. Precisely, let S be a double-measurable transformation of a probabilistic measure space (X, Σ, m). If S preserves the measure m and the Frobenius-Perron operator P_S spreads supports, then the powers of P_S are asymptotically stable [54]. It is interesting that if we assume only that a Markov operator (or semigroup) P has an invariant density f_* and spreads supports, then P is weakly asymptotically stable (*mixing*). It means that for every $f \in D$ the sequence $P^n f$ converges weakly to f_*. One can expect that we can omit in Corollary 1 the assumption that the semigroup is partially integral. But it is not longer true. Indeed, in [56] we construct a Markov operator $P : L^1[0,1] \to L^1[0,1]$ which spreads supports and $P\mathbf{1} = \mathbf{1}$ but it is not asymptotically stable.

If $\{P(t)\}_{t\geq 0}$ is a continuous time Markov semigroup then we can strengthen considerably Theorem 1.

Theorem 2 ([47]). *Let $\{P(t)\}_{t\geq 0}$ be a continuous time partially integral Markov semigroup. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has the only one invariant density f_*. If $f_* > 0$ a.e. then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.*

Remark 2. In applications we often replace the assumption that the invariant density is unique by the following one. We assume that there does not exist a set $E \in \Sigma$ such that $m(E) > 0$, $m(X \setminus E) > 0$ and $P(t)E = E$ for all $t > 0$. Here $P(t)$ is the operator acting on the σ-algebra Σ given by: if $f \geq 0$, $\operatorname{supp} f = A$ and $\operatorname{supp} Pf = B$ then $PA = B$.

3.2. **Sweeping.** A Markov semigroup $\{P(t)\}_{t\geq 0}$ is called *sweeping* with respect to a set $A \in \Sigma$ if for every $f \in D$

$$\lim_{t\to\infty} \int_A P(t)f(x)\, m(dx) = 0. \tag{20}$$

The notion of sweeping was introduced by Komorowski and Tyrcha [26]. The crucial role in theorems concerning sweeping plays the following condition.

(KT): There exists a measurable function f_* such that: $0 < f_* < \infty$ a.e., $P(t)f_* \leq f_*$ for $t \geq 0$, $f_* \notin L^1$ and $\int_A f_*\, dm < \infty$.

Theorem 3 ([26]). *Let $\{P(t)\}_{t\geq 0}$ be an integral Markov semigroup which has no invariant density. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ and a set $A \in \Sigma$ satisfy condition (KT). Then the semigroup $\{P(t)\}_{t\geq 0}$ is sweeping with respect to A.*

In paper [54] it was shown that Theorem 3 holds for a wider class of operators than integral ones. In particular, the following result was proved (see [54] Corollary 4 and Remark 6).

Theorem 4. *Let $\{P(t)\}_{t\geq 0}$ be a Markov semigroup which overlaps supports. Assume that the semigroup $\{P(t)\}_{t\geq 0}$ and a set $A \in \Sigma$ satisfy condition (KT). Then the semigroup $\{P(t)\}_{t\geq 0}$ is sweeping with respect to A.*

The main difficulty in applying Theorems 3 and 4 is to prove that a Markov semigroup satisfies condition (KT). Now we formulate a criterion for sweeping which will be useful in applications.

Theorem 5 ([54]). *Let X be a metric space and Σ be the σ–algebra of Borel sets. We assume that a Markov semigroup $\{P(t)\}_{t\geq 0}$ has the following properties:*

(a) *for every* $f \in D$ *we have* $\int_0^\infty P(t)f\,dt > 0$ *a.e. or* $\sum_{n=0}^\infty P^n f > 0$ *a.e. if* $\{P(t)\}_{t\ge 0}$ *is a discrete time semigroup,*
(b) *for every* $y_0 \in X$ *there exist* $\varepsilon > 0$ *and a measurable function* $\eta \ge 0$ *such that* $\int \eta\,dm > 0$ *and*

$$q(x,y) \ge \eta(x)\mathbf{1}_{B(y_0,\varepsilon)}(y),$$

where q *is a function satisfying (18) and (19). If the semigroup* $\{P(t)\}_{t\ge 0}$ *has no invariant density then it is sweeping with respect to compact sets.*

3.3. **Foguel alternative.** We say that a Markov semigroup $\{P(t)\}_{t\ge 0}$ satisfies the *Foguel alternative* if it is asymptotically stable or sweeping from a sufficiently large family of sets. For example this family can be all compact sets.

From Corollary 1 and Theorem 5 it follows immediately

Theorem 6. *Let* X *be a metric space and* Σ *be the* σ*–algebra of Borel sets. Let* $\{P(t)\}_{t\ge 0}$ *be a Markov semigroup. We assume that there exist* $t > 0$ *and a continuous function* $q : X \times X \to (0,\infty)$ *such that*

$$P(t)f(x) \ge \int_X q(x,y)f(y)\,m(dy) \quad \text{for } f \in D. \tag{21}$$

Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.

Using Theorem 6 one can check that the Foguel alternative holds for multistate diffusion processes [35, 45, 54], diffusion with jumps [46] and transport equations (11) [44].

More general results concerning Foguel alternative can be found in [54]. These results were applied to the Markov operator P considered in the cell cycle model [63].

3.4. **Hasminskiĭ function.** Now we consider only continuous time Markov semigroups. Sometimes we know that a given semigroup satisfies the Foguel alternative. We want to prove that this semigroup is asymptotically stable. In order to exclude sweeping we introduce a notion called a Hasminskiĭ function.

Consider a Markov semigroup $\{P(t)\}_{t\ge 0}$ and let A be the infinitesimal generator of $\{P(t)\}_{t\ge 0}$. Let $\mathcal{R} = (I-A)^{-1}$ be the resolvent operator at point 1. A measurable function $V : X \to [0,\infty)$ is called a *Hasminskiĭ function* for the Markov semigroup $\{P(t)\}_{t\ge 0}$ and a set $Z \in \Sigma$ if there exist $M > 0$ and $\varepsilon > 0$ such that

$$\int_X V(x)\mathcal{R}f(x)\,dm(x) \le \int_X (V(x)-\varepsilon)f(x)\,dm(x) + \int_Z M\mathcal{R}f(x)\,dm(x). \tag{22}$$

Theorem 7. *Let* $\{P(t)\}$ *be a Markov semigroup generated by the equation*

$$\frac{\partial u}{\partial t} = Au.$$

Assume that there exists a Hasminskiĭ function for the semigroup $\{P(t)\}_{t\ge 0}$ *and a set* Z*. Then the semigroup* $\{P(t)\}$ *is not sweeping with respect to the set* Z*.*

In application we take V such that the function A^*V is "well defined" and it satisfies the following condition $A^*V(x) \le -c < 0$ for $x \notin Z$. Then we check that V satisfies inequality (22). This method was applied to multistate diffusion processes [45] and diffusion with jumps [46], where inequality (22) was proved by using some generalization of the maximum principle. This method was also applied to transport equations (11) in [44] but the proof of inequality (22) is different and based on an approximation of V by a sequence of elements from the domain of the operator A^*.

The function V was called a Hasminskiĭ function because he showed [22] that the semigroup generated by the Fokker-Planck equation (7) has an invariant density if there exists a positive function V such that $A^*V(x) \le -c < 0$ if $\|x\| \ge r$.

4. Applications

4.1. **The Fokker-Planck equation.** If we assume that the functions a_{ij} satisfy the uniform elliptic condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \ge \alpha|\lambda|^2 \tag{23}$$

for some $\alpha > 0$ and every $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ then the Markov semigroup generated by the Fokker-Planck equation (6) is an integral semigroup. That is

$$P(t)f(x) = \int_{\mathbb{R}^d} q(t,x,y)f(y)\,dy, \quad t > 0$$

and the kernel q is continuous and positive. From the Foguel alternative follows

Corollary 3. *Let $\{P(t)\}_{t\ge 0}$ be a Markov semigroup generated by the Fokker-Planck equation. Then this semigroup is asymptotically stable or is sweeping with respect to compact sets.*

It is easy to check that if this semigroup is not asymptotically stable, then it is sweeping with respect to the family of sets with finite Lebesgue measures.

The operator A^* is given by the formula

$$A^*V = \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i}.$$

If there exist a non-negative C^2-function V, $\varepsilon > 0$ and $r \ge 0$ such that

$$A^*V(x) \le -\varepsilon \qquad \text{for} \qquad \|x\| \ge r$$

then the Markov semigroup generated by the Fokker-Planck equation is asymptotically stable. This theorem generalizes earlier results [14, 58].

Now we give an example of application of Theorem 4 to study sweeping property. Consider the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial(b(x)u)}{\partial x}. \tag{24}$$

Let $\{P(t)\}_{t\ge 0}$ be a Markov semigroup generated by equation (24) and let $f_*(x) = \exp\{\int_0^x b(s)\,ds\}$. Observe that if $\int_{-\infty}^{\infty} f_*(x)\,dx < \infty$ then the semigroup $\{P(t)\}_{t\ge 0}$ has an invariant density $f_*/\|f_*\|$ and consequently it is asymptotically stable. If $\int_{-\infty}^{\infty} f_*(x)\,dx = \infty$ then the semigroup $\{P(t)\}_{t\ge 0}$ is sweeping from bounded sets. But if additionally $\int_0^{\infty} f_*(x)\,dx < \infty$ then the semigroup $\{P(t)\}_{t\ge 0}$ is also sweeping from intervals $[c,\infty)$, $c \in \mathbb{R}$. Indeed, since $f_* > 0$, $Af_* \le 0$ and $\int_c^{\infty} f_*(x)\,dx < \infty$ the semigroup $\{P(t)\}_{t\ge 0}$ and the set $[c,\infty)$ satisfy condition (KT). Thus Theorem 4 implies that the semigroup $\{P(t)\}_{t\ge 0}$ is sweeping from $[c,\infty)$. Theorems 3 and 4 can be applied to study the sweeping property in the cell cycle model ([34],[54]).

Now we consider degenerate diffusion processes. Here instead of (23) we assume (8). The fundamental theorem on the existence of smooth densities of the transition probability function for degenerate diffusion processes is due to Hörmander. In a series of papers [38, 39] Malliavin has developed techniques, called Malliavin calculus, to give probabilistic proof of this fact. Now we recall some results from this theory. Let $a(x)$ and $b(x)$ be two vector fields on $\mathbb{R}^d$. The Lie bracket $[a, b]$ is a vector field given by

$$[a,b]_j(x) = \sum_{k=1}^{d} \left(a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x) \right).$$

Consider the Stratonovitch stochastic differential equation (4), i.e. equation

$$dX_t = \sigma(X_t) \circ dW_t + \sigma_0(X_t)\, dt.$$

Let $\sigma_j(x)$ be a vector in $\mathbb{R}^d$ with components $\sigma_j^i(x)$ for every $x \in \mathbb{R}^d$. We assume Hörmander's condition as in [42]

(H): For every $x \in \mathbb{R}^d$ vectors

$$\sigma_1(x), \dots, \sigma_m(x),\ [\sigma_i, \sigma_j](x)_{0\le i,j\le m},\ [\sigma_i, [\sigma_j, \sigma_k]](x)_{0\le i,j,k\le m}, \dots$$

span the space $\mathbb{R}^d$.

Note that the vector σ_0 appears only through brackets. The reason why σ_0 does not appear in condition (H) can be seen by considering $(X^1(t), X^2(t)) = (W(t), t)$, which certainly does not have a density in $\mathbb{R}^2$.

Theorem 8 (Hörmander)**.** *Under hypothesis (H) the transition probability function $\mathcal{P}(t, x, A)$ has a density $k(t, y, x)$ and $k \in C^\infty((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$.*

Remark 3. Note that in the uniformly elliptic case the vectors $\sigma_1(x)$, $\sigma_2(x)$,..., $\sigma_m(x)$ span $\mathbb{R}^d$, so that the hypothesis (H) is satisfied and a smooth transition density exists.

To formulate the Foguel alternative for the semigroup $\{P(t)\}_{t\ge0}$ generated by equation (4) we need the following condition

(I): For every open set $U \subset \mathbb{R}^d$ and every measurable set A with a positive Lebesgue measure there exists $t > 0$ such that

$$\int_U \int_A k(t, x, y)\, dx\, dy > 0. \tag{25}$$

Theorem 9. *Assume that conditions (H) and (I) hold. Then the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable or is sweeping with respect to compact sets. Moreover, if there exist a nonnegative C^2-function V and $r > 0$ such that*

$$\sup_{||x||>r} A^*V(x) < 0, \tag{26}$$

then the semigroup $\{P(t)\}_{t\ge0}$ is asymptotically stable.

Proof. From (H) it follows that the semigroup $\{P(t)\}_{t\ge0}$ is integral and given by

$$P(t)f(x) = \int_{\mathbb{R}^d} k(t, x, y) f(y)\, dy$$

for $f \in L^1(\mathbb{R}^d)$. Let f be a density. Then for $t > 0$ the function $P(t)f$ is continuous and condition (I) implies that

$$\int_0^\infty P(t)f\,dt > 0 \text{ a.e.} \tag{27}$$

If the semigroup $\{P(t)\}_{t\ge 0}$ has an invariant density f_* then from (I) it follows that f_* is a unique invariant density and $f_* > 0$ a.e. According to Theorem 2 the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable. If the semigroup $\{P(t)\}_{t\ge 0}$ has no invariant density then according to Theorem 5 this semigroup is sweeping with respect to compact sets. Using similar arguments to that of [45] one can check that V is a Hasminskiĭ function for the semigroup and the closed ball $\{x : \|x\| \le r\}$, which completes the proof. □

In order to verify condition (I) we describe a method based on support theorems ([1, 9, 60]) for checking positivity of k. Let $U(x_0,T)$ be the set of all points y for which we can find a $\phi \in L^2([0,T];\mathbb{R}^m)$ such that there exists a solution of the equation

$$x_\phi(t) = x_0 + \int_0^t \big(\sigma(x_\phi(s))\phi(s) + \sigma_0(x_\phi(s))\big)\,ds \tag{28}$$

satisfying the condition $x_\phi(T) = y$. From the support theorem for diffusion processes it follows that the topological support of the measure $\mathcal{P}(T,x_0,\cdot)$ coincides with closure in $\mathbb{R}^d$ of the set $U(x_0,T)$. Let $D_{x_0,\phi}$ be the Frechét derivative of the function $h \mapsto x_{\phi+h}(T)$ from $L^2([0,T];\mathbb{R}^m)$ to $\mathbb{R}^d$. By $\widetilde{U}(x_0,T)$ we denote all points y such that $x_\phi(T) = y$ and the derivative $D_{x_0,\phi}$ has rank d. Then

$$\widetilde{U}(x_0,T) = \{y : k(T,y,x_0) > 0\} \quad \text{and} \quad \operatorname{cl}\widetilde{U}(x_0,T) = \operatorname{cl} U(x_0,T),$$

where cl = closure. The derivative $D_{x_0,\phi}$ can be found by means of the perturbation method for ordinary differential equations. Let

$$\Lambda(t) = \frac{d\sigma_0}{dx}(x_\phi(t)) + \sum_{i=1}^m \frac{d\sigma_i}{dx}(x_\phi(t))\phi_i(t) \tag{29}$$

and let $Q(t,t_0)$, for $T \ge t \ge t_0 \ge 0$, be a matrix function such that $Q(t_0,t_0) = I$ and $\dfrac{\partial Q(t,t_0)}{\partial t} = \Lambda(t)Q(t,t_0)$. Then

$$D_{x_0,\phi}h = \int_0^T Q(T,s)\sigma(x_\phi(s))h(s)\,ds. \tag{30}$$

Example. Consider the Newton equation with stochastic perturbation

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \psi(x) = \frac{dW_t}{dt}, \tag{31}$$

where $\beta > 0$. Equation (31) describes the dynamics of mechanical systems perturbed by white noise [27]. Let $X_t = x(t)$ and $Y_t = x'(t)$. Then equation (31) is equivalent to the system

$$dX_t = Y_t\,dt, \quad dY_t = dW_t - (\beta Y_t + \psi(X_t))\,dt. \tag{32}$$

Then $\sigma_1 \equiv [0,1]$, $\sigma_0(x,y) = [y, -\beta y - \psi(x)]$, $[\sigma_0,\sigma_1] \equiv [1,-\beta]$ and condition (H) holds. System (28) corresponding to (32) can be written in the following way

$$x'_\phi = y_\phi, \quad y'_\phi = \phi - \beta y_\phi - \psi(x_\phi). \tag{33}$$

For given $x_0, x_1, y_0, y_1 \in \mathbb{R}$ and $T > 0$ there exist functions ϕ and x_ϕ of the form $x_\phi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ such that x_ϕ and $y_\phi = x'_\phi$ satisfy system (33) and boundary conditions $x_\phi(0) = x_0$, $x_\phi(T) = x_1$, $y_\phi(0) = y_0$, $y_\phi(T) = y_1$. In our case

$$\Lambda(t) = \begin{bmatrix} 0 & 1 \\ -\psi'(x_\phi(t)) & -\beta \end{bmatrix}, \quad \sigma = \sigma_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $\varepsilon \in (0, T)$ and $h = \mathbf{1}_{[T-\varepsilon,T]}$. Since $Q(T, s) = I - \Lambda(T)(T - s) + o(T - s)$, from (29) we obtain

$$D_{x_0,y_0;\phi} h = \varepsilon\sigma_1 - \tfrac{1}{2}\varepsilon^2 \Lambda(T)\sigma_1 + o(\varepsilon^2). \tag{34}$$

Since $\sigma_1 \equiv [0, 1]$ and $\Lambda(T)\sigma_1 \equiv [1, -\beta]$, these vectors are linearly independent and the derivative $D_{x_0,y_0;\phi}$ has rank 2. Thus the system (32) generates an integral Markov semigroup $\{P(t)\}_{t\ge 0}$ with a continuous and strictly positive kernel k. Consider the Fokker-Planck equation corresponding to (32):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}[(\beta y + \psi(x))u]. \tag{35}$$

Let $\Psi(x) = \int_0^x \psi(s)\, ds$. Then the function $u_*(x, y) = \exp\{-\beta y^2 - 2\beta\Psi(x)\}$ is a stationary solution of (35). If $\int_{-\infty}^{\infty} e^{-2\beta\Psi(x)}\, dx < \infty$ then, according to Corollary 1, the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable. If $\int_{-\infty}^{\infty} e^{-2\beta\Psi(x)}\, dx = \infty$ then the function u_* satisfies condition (KT) for every set of the form $A(L) = \{(x, y) : x \in [-L, L],\ y \in \mathbb{R}\}$ because $P(t)u_* = u_*$ and $\iint_{A(L)} u_*(x, y)\, dx\, dy < \infty$. According to Theorem 4 this semigroup is sweeping from the sets $A(L)$.

Remark 4. Since a lot of transport equations generates a partially integral semigroup which spreads supports we can obtain similar results for these equations. Consider, for example, a stochastic equation $dX_t = -\lambda X_t\, dt + dC_t$, where C_t is a Cauchy process [20]. The semigroup generated by this equation is an integral semigroup with a continuous and positive kernel. From the Foguel alternative this semigroup is asymptotically stable or sweeping from compact sets. If $\lambda > 0$ then $f_*(x) = \lambda/\pi(\lambda^2 x^2 + 1)$ is an invariant density for semigroup $\{P(t)\}_{t\ge 0}$ and consequently it is asymptotically stable.

4.2. **Diffusion with jumps.** Consider the following equation

$$\frac{\partial u}{\partial t} = Au - \lambda u + \lambda P u, \tag{36}$$

where $\lambda > 0$,

$$Au = \sum_{i,j=1}^{d} \frac{\partial^2 (a_{ij} u)}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i} \tag{37}$$

and P is a Markov operator corresponding to the iterated function system

$$(S_1(x), \dots, S_N(x), p_1(x), \dots, p_N(x)).$$

The probabilistic interpretation of equation (36) is similar to that of equation (11). We assume that for each j we have

$$\lim_{\|x\|\to\infty} \|S_j(x)\| = \infty.$$

Assume that

$$\lim_{\|x\|\to\infty} 2\langle x, b(x)\rangle + \lambda \sum_{j=1}^{n} p_j(x)\left(\|S_j(x)\|^2 - \|x\|^2\right) = -\infty,$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^d$. Then a Markov semigroup $\{P(t)\}_{t\geq 0}$ generated by equation (36) is asymptotically stable [46].

4.3. **Randomly interrupted diffusion.** This process was described by the following system of equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = -pu_1 + qu_0 + A_1 u_1 \\ \dfrac{\partial u_0}{\partial t} = pu_1 - qu_0 + A_0 u_0. \end{cases}$$

A semigroup generated by this system satisfies the Foguel alternative. In order to prove asymptotic stability it is sufficient to construct a proper Hasminskiĭ function. One can check that if there exist non-negative C^2-functions V_1 and V_2 such that

$$-p(x)V_1(x) + p(x)V_2(x) + A_1^* V_1(x) \leq -\varepsilon,$$
$$q(x)V_1(x) - q(x)V_2(x) + A_2^* V_2(x) \leq -\varepsilon$$

for $\|x\| \geq r$, then the corresponding Markov semigroup is asymptotically stable [45].

4.4. **Transport equation.** Consider a partial differential equation with an integral perturbation

$$\frac{\partial u}{\partial t} + \lambda u = -\sum_{i=1}^{d} \frac{\partial(b_i u)}{\partial x_i} + \lambda \int k(x,y)u(y,t)\,dy. \tag{38}$$

If $k(x,y)$ is a continuous and strictly positive function and there exists a C^1-function $V : X \to [0,\infty)$ such that

$$\sum_{i=1}^{d} b_i \frac{\partial V}{\partial x_i} - \lambda V(x) + \lambda \int k(y,x)V(y)\,dy \leq -c < 0$$

for $\|x\| \geq r$, $r > 0$, then a Markov semigroup $\{P(t)\}_{t\geq 0}$ generated by equation (38) is asymptotically stable [44].

Remark 5. Consider the transport equation

$$\frac{\partial u}{\partial t} + \lambda u = Au + \lambda K u, \tag{39}$$

where A is a generator of the Markov semigroup $\{S(t)\}_{t\geq 0}$ and K is a Markov operator. If the semigroup $\{S(t)\}_{t\geq 0}$ is partially integral or the operator K is partially integral then from (12) it follows that the semigroup $\{P(t)\}_{t\geq 0}$ is partially integral. From (12) and continuity of the semigroups $\{S(t)\}_{t\geq 0}$ and $\{P(t)\}_{t\geq 0}$ it follows that for a measurable set E we have $P(t)E \subset E$ for all $t \geq 0$ if and only if $KE \subset E$ and $S(t)E \subset E$ for all $t \geq 0$. Let $\mathcal{P}(x,E)$ be the transition probability function corresponding to K. Then $KE \subset E$ if and only if $\mathcal{P}(x,E) = 1$ for a.e. $x \in E$. In the next subsections we consider two examples of random movement of this type. In these examples both the semigroup $\{S(t)\}_{t\geq 0}$ and the operator K are singular (have no integral parts) but the semigroup $\{P(t)\}_{t\geq 0}$ is partially integral. Moreover we give sufficient conditions for asymptotic stability of these semigroups which are based on Theorem 2.

4.5. **Jump process.** We consider equation (36) but instead of the operator (37) we consider the Liouville operator

$$Au = -\sum_{i=1}^{d} \frac{\partial(b_i u)}{\partial x_i}.$$

The probabilistic interpretation of this equation was given in Subsection 2.7.

Theorem 10 ([47])**.** *Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has a non-zero invariant function and has no non-trivial invariant sets. Let $(i_1,\dots,i_d)$ be a given sequence of integers from the set $\{1,\dots,k\}$. Let $x_0\in X$ be a given point and let $x_j = S_{i_j}(x_{j-1})$ for $j=1,\dots,d$. Set*

$$v_j = S'_{i_d}(x_{d-1})\dots S'_{i_j}(x_{j-1})b(x_{j-1}) - b(x_d)$$

for $j=1,\dots,d$. Assume that $p_{i_j}(x_{j-1})>0$ for all $j=1,\dots,d$ and suppose that the vectors $v_1,\dots,v_d$ are linearly independent. Then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

4.6. **Randomly controlled dynamical system.** Now we consider a stochastic process introduced in the end of subsection 2.8. We recall that we have k dynamical systems $\pi_t^i(x)$ corresponding to the equations $x' = b(x,i)$, $i=1,\dots,k$ and we exchange their randomly. Denote by $\{P(t)\}_{t\geq 0}$ the semigroup corresponding to this system. Let $(i_1,\dots,i_{d+1})$ be a sequence of integers from the set $\Gamma=\{1,\dots,k\}$. For $x\in X$ and $t>0$ we define the function $\psi_{x,t}$ on the set $\Delta_t=\{\tau=(\tau_1,\dots,\tau_d):\tau_i>0, \tau_1+\dots+\tau_d\leq t\}$ by

$$\psi_{x,t}(\tau_1,\dots,\tau_d) = \pi^{i_{d+1}}_{t-\tau_1-\tau_2-\dots-\tau_d}\circ\pi^{i_d}_{\tau_d}\circ\dots\circ\pi^{i_2}_{\tau_2}\circ\pi^{i_1}_{\tau_1}(x).$$

Theorem 11 ([47])**.** *Assume that the semigroup $\{P(t)\}_{t\geq 0}$ has a non-zero invariant function and has no non-trivial invariant sets. Suppose that for some $x_0\in X$, $t_0>0$ and $\tau^0\in\Delta_{t_0}$ we have*

$$\det\left[\frac{d\psi_{x_0,t_0}(\tau^0)}{d\tau}\right]\neq 0. \tag{40}$$

Then the semigroup $\{P(t)\}_{t\geq 0}$ is asymptotically stable.

Remark 6. A measurable set $E\subset X\times\Gamma$ is invariant with respect to the semigroup $\{P(t)\}_{t\geq 0}$ if and only if E is of the form $E=E_0\times\Gamma$ and

$$\pi_t^i(E_0)=E_0 \qquad \text{for } t\geq 0 \text{ and } i=1,\dots,k.$$

Remark 7. Condition (40) can be formulated using Lie brackets. Let $b_i(x)=b(x,i)$. If vectors

$$b_2(x_0)-b_1(x_0),\dots,b_k(x_0)-b_1(x_0),\ [b_i,b_j](x_0)_{1\leq i,j\leq k},\ [b_i,[b_j,b_l]](x_0)_{1\leq i,j,l\leq k},\dots$$

span the space $\mathbb{R}^d$ then (40) holds.

4.7. **Population dynamics equation.** Some models of size-structured cell populations lead to transport equations similar to (11), but these equations do not generate Markov semigroups. Also in these cases we can often apply results presented in Section 3. We consider here a model derived in [57], which generalized some earlier models of cell populations (e.g. [16]).

We assume that a cell is fully characterized by its size x which ranges from $x=a$ to $x=1$. The cell size grows according to equation $x'=g(x)$. Cells can die or divide with rates $\mu(x)$ and $b(x)$. We assume that the cells cannot divide before they have reached a minimal maturation $a_0 \in (a,1)$. Since the cells have to divide before they reach the maximal size $x=1$, we assume that $\lim_{x\to 1}\int_a^x b(\xi)\,d\xi=\infty$. If $x \ge a_0$ is the size of a mother cell at the point of cytokinesis, then a new born daughter cell has the size which is randomly distributed in the interval $(a, x-h]$, where h is a positive constant. We denote by $\mathcal{P}(x,[x_1,x_2])$ the probability for a daughter cell born from a mother cell of size x to have a size between x_1 and x_2.

The function $N(x,t)$ describing the distribution of the size satisfies the following equation

$$\frac{\partial N}{\partial t} = -\frac{\partial (gN)}{\partial x} - (\mu + b)N + 2P(bN), \tag{41}$$

where $P : L^1(a,1) \to L^1(a,1)$ is a Markov operator such that $P^*\mathbf{1}_B(x) = \mathcal{P}(x,B)$. The main result concerning equation (41) is the following

Theorem 12. *There exist $\lambda \in \mathbb{R}$ and continuous and positive functions f_* and w defined on the interval $(a,1)$ such that $e^{-\lambda t}N(\cdot,t)$ converges to $f_*\Phi(N)$ in $L^1(a,1)$, where $\Phi(N) = \int_a^1 N(x,0)w(x)\,dx$.*

The proof of Theorem 12 goes as follows. Equation (41) can be written as an evolution equation $N'(t) = AN$. First we show that A is an infinitesimal generator of a continuous semigroup $\{T(t)\}_{t\ge 0}$ of linear operators on $L^1(a,1)$. Then we prove that there exist $\lambda \in \mathbb{R}$ and continuous and positive functions v and w such that $Av = \lambda v$ and $A^*w = \lambda w$. From this it follows that the semigroup $\{P(t)\}_{t\ge 0}$ given by $P(t) = e^{-\lambda t}T(t)$ is a Markov semigroup on the space $L^1(X,\Sigma,m)$, where m is a Borel measure on the interval $[a,1]$ given by $m(B) = \int_B w(x)\,dx$. Moreover, for some $c > 0$ the function $f_* = cv$ is an invariant density with respect to $\{P(t)\}_{t\ge 0}$. Finally, from Theorem 1 we conclude that this semigroup is asymptotically stable. Since the Lebesgue measure and the measure m are equivalent we obtain that $e^{-\lambda t}N(\cdot,t)$ converges to $f_*\Phi(N)$ in $L^1(a,1)$.

5. Other asymptotic properties

In this section we give some results concerning other asymptotic properties of Markov operators: completely mixing and limit distribution.

5.1. Completely mixing.

Semigroup $\{P(t)\}_{t\ge 0}$ is called completely mixing if for any two densities f and g

$$\lim_{t\to\infty} \|P(t)f - P(t)g\| = 0. \tag{42}$$

This notion has the following probabilistic interpretation. Let $\{P(t)\}_{t\ge 0}$ be the Markov semigroup corresponding to a diffusion process. Assume that this process describes a movement of particles. Then condition (42) means that particles are mixed in such a way that after a long time their distribution does not depend on the initial distribution. If there exists an invariant density f_* then completely mixing is equivalent to asymptotic stability. However, the semigroup $\{P(t)\}_{t\ge 0}$ can be completely mixing, but it can have no invariant density. For example, the heat equation $\dfrac{\partial u}{\partial t} = \Delta u$ generates the semigroup which is completely mixing and has no invariant density.

Completely mixing property of the semigroup $\{P(t)\}_{t\geq 0}$ is strictly connected with the notion of the relative entropy. The relative entropy can be written down in the following way

$$H(t) = \int P(t)f(x)\ln\frac{P(t)f(x)}{P(t)g(x)}\,dx, \qquad f,g\in D.$$

It is not difficult to check that if $\lim_{t\to\infty} H(t) = 0$ then the semigroup $\{P(t)\}_{t\geq 0}$ is completely mixing (see [33] for a more general result). It is also easy to check that completely mixing implies that all fixed points of the semigroup $\{P^*(t)\}_{t\geq 0}$ are constant functions.

Completely mixing property for the Fokker-Planck equation (7) was studied in the papers [11, 51]. The most general result in this direction was received in [8]. They proved that if the coefficient in the Fokker-Planck equation are bounded with their first and second partial derivatives, the diffusion term satisfies uniform elliptic condition (23) and all fixed points of the semigroup $\{P^*(t)\}_{t\geq 0}$ are constant functions then the semigroup $\{P(t)\}_{t\geq 0}$ is completely mixing. In other words the semigroup $\{P(t)\}_{t\geq 0}$ is completely mixing if and only if all bounded solutions of the elliptic equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} = 0$$

are constant. It is worth pointing out that even in one-dimensional case with constant diffusion the assumption that the drift coefficient is bounded cannot be replaced with the assumption that it grows linearly [51].

Remark 8. Let P_S be the Frobenius–Perron operator for a measurable transformation S of a σ-finite measure space (X,Σ,m). Then P_S is completely mixing if and only if $\bigcap_{n=1}^{\infty} S^{-n}\Sigma = \{\emptyset, X\}$ ([32]). If additionally the measure m is invariant then the transformation S is exact. In the paper [48] we give an example of a piecewise linear and expanding transformation of the interval $[0,1]$ which is completely mixing but for every density f the iterations $P_S^n f$ converge weakly to the standard Cantor measure. This transformation has similar properties to the Smale horseshoe.

5.2. **Limit distribution.** Let $S = \{x\in\mathbb{R}^d : \|x\| = 1\}$ and A be a measurable subset of S. Denote by $K(A)$ the cone spanned by A:

$$K(A) = \{x\in\mathbb{R}^d : x = \lambda y,\ y\in A,\ \lambda > 0\}.$$

Consider a Markov semigroup $\{P(t)\}_{t\geq 0}$ corresponding to a diffusion process. Then the function

$$p_A(t) = \int_{K(A)} P(t)f(x)\,dx, \qquad f\in D,$$

describes the mass of particles which are in the cone $K(A)$. If the semigroup $\{P(t)\}_{t\geq 0}$ is completely mixing then the asymptotic behaviour of $p_A(t)$ does not depend on f. It is interesting when there exists the limit $p_A = \lim_{t\to\infty} p_A(t)$. If $\{P(t)\}_{t\geq 0}$ is sweeping then nearly all particles are in a neighbourhood of ∞ for large t and p_A measures the *sectorial limit distribution* of particles.

The problem of finding the limit distribution for arbitrary diffusion process in d–dimensional space is difficult. Some partial results can be obtained under additional assumption that all functions a_{ij} and b_i are periodic with the same periods (we recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is periodic if there exist independent vectors $v_1, \dots, v_d$ such that $f(x + v_i) = f(x)$ for each $x \in \mathbb{R}^d$ and $i = 1, \dots, d$).

In one-dimensional space we can consider the function $p_+(t) = \int_c^\infty u(x,t)\,dx$ which describes the mass of particles in the interval (c, ∞). The paper [52] provides a criterion for the existence of the limit $\lim_{t\to\infty} p_+(t)$. In the same paper we construct an equation such that the following condition holds

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t p_+(s)\,ds = 1 \qquad \text{and} \qquad \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_+(s)\,ds = 0. \tag{43}$$

In this example $a(x) = 1$ and $b(x) \to 0$ as $|x| \to \infty$. Condition (43) means that particles synchronously oscillate between $+\infty$ and $-\infty$.

Remark 9. Many abstract results concerning completely mixing property can be found in books [41, 43]. Completely mixing property of an integral Markov operator appearing in a model of cell cycle was studied in [55]. If a Markov semigroup has no invariant density one can investigate a property of convergence after rescaling. We say that a Markov semigroup $\{P(t)\}_{t\geq 0}$ is *convergent after rescaling* if there exist a density g and functions $\alpha(t)$, $\beta(t)$ such that

$$\lim_{t\to\infty} \int_X |\alpha(t)P(t)f\big(\alpha(t)x + \beta(t)\big) - g(x)|\,dx = 0 \qquad \text{for every } f \in D. \tag{44}$$

Condition (44) implies completely mixing property. One of the weak versions of this condition is the central limit theorem. In papers [49, 50] it is shown that semigroups connected with processes with jumps satisfy condition (44), precisely, these processes are asymptotically log-normal.

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