On some non-conformal fractals

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Abstract

This paper presents a simple method of calculating the Hausdorff dimension for a class of non-conformal fractals.

1 Introduction

An iterated function scheme acting on a complete metric space X is a finite family of contracting maps $\mathcal{F} = \{f_k\}_{k=1}^n; f_k : X \to X$. As noted by Hutchinson [Hu], the related multimap

$$F(\cdot) = \bigcup_{k=1}^{n} f_k(\cdot)$$

(acting on the space $B(X)$ of nonempty compact subsets of X, considered with the Hausdorff metric) is also a contraction. Hutchinson proved that if X is complete, so is $B(X)$. Hence, by the Banach fixed point theorem, there exists a unique nonempty compact set Λ satisfying

$$\Lambda = F(\Lambda) = \lim_{n\to\infty} F^n(A).$$

The limit does not depend on the choice of $A \in B(X)$. Λ is called the limit set of the iterated function scheme $\mathcal{F}$.

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By similar reasoning, if we have a finite number of iterated function schemes $\{\mathcal{F}_i\}_{i=1}^m$ acting on X and apply them in any order, the pointwise limit

$$\Lambda_\omega = \lim_{n\to\infty} F_{\omega_1} \circ \ldots \circ F_{\omega_n}(A)$$

exists for all $\omega \in \Omega = \{1,\ldots,m\}^{\mathbb{N}}$ and does not depend on $A \in B(X)$.

The question we want to answer (motivated by [Lu], see also [N], [GL], [GL2] and the incoming paper [Re]) is: when the iterated function schemes $\mathcal{F}_i$ are of some special class (for which we can calculate the Hausdorff dimension of the limit set of any deterministic iterated function scheme from this class) and the sequence ω is chosen, what will be the value of the Hausdorff dimension of Λ_ω?

We will present a simple method of dealing with this question, working for Lalley-Gatzouras maps [LG], Barański maps [B] and higher dimensional affine-invariant sets of Kenyon and Peres [KP]. The only assumption about ω we need is that each symbol i has a limit frequency of appearance. For simplicity, we will only present the proof for an example: a class of iterated function schemes considered by Lalley and Gatzouras.

We refer the reader interested in other non-conformal random iterated constructions to [F], [GL2] and references therein.

2 Lalley-Gatzouras schemes

The Lalley-Gatzouras scheme $\mathcal{F}$ is a self-affine IFS given by a family of maps $f_{i,j} : \mathbb{R}^2 \to \mathbb{R}^2$, $(i,j) \in A$:

$$f_{i,j}(x,y) = (a_{ij}x + c_{ij}, b_i y + d_i),$$

where the alphabet A of allowed symbols is $1 \le i \le m_1, 1 \le j \le m_2(i)$. We will assume that for all $(i,j) \in A$, $b_i \ge a_{ij}$ (that is, the contraction in the horizontal direction is not weaker than the contraction in the vertical direction for all maps).

We will also assume that for all $(i,j) \in A$, $0 < a_{ij} < 1$ and $0 \le c_{i1} < \ldots < c_{im_2(i)} \le 1 - a_{im_2(i)}, c_{ij+1} \ge a_{ij} + c_{ij}$ and that $0 < b_i < 1$ and $0 \le d_1 < \ldots < d_{m_1} \le 1 - b_{m_1}, d_{i+1} \ge b_i + d_i$. We will say that the *separation condition* holds if we actually have $c_{ij+1} > a_{ij} + c_{ij}$ and $d_{i+1} > b_i + d_i$.

The main result of [LG] is the formula for the Hausdorff dimension of the limit set Λ:

$$\dim_H(\Lambda) = \max\left\{\frac{\sum_i \sum_j p_{ij}\log p_{ij}}{\sum_i \sum_j p_{ij}\log a_{ij}} + \sum_i q_i \log q_i \left(\frac{1}{\sum_i q_i \log b_i} - \frac{1}{\sum_i \sum_j p_{ij}\log a_{ij}}\right)\right\},$$

where $\{p_{ij}\}$ is a probability distribution on A, $q_i = \sum_j p_{ij}$ and the maximum is over all possible $\{p_{ij}\}$.

Consider now a family of Lalley-Gatzouras schemes $\{\mathcal{F}_k\}_{k=1}^m$ with alphabets A_k and maps $f_{i,j}^{(k)}$. As mentioned above, we can apply them in any order $F_{\omega_1} \circ F_{\omega_2} \circ \ldots$, $\omega = \omega_1\omega_2\ldots \in \Omega = \{1,\ldots,m\}^{\mathbb{N}}$ and obtain some limit set Λ_ω. We will assume that the limits

$$P_k = \lim_{n\to\infty} \frac{1}{n}\sharp\{1 \le l \le n; \omega_l = k\} \tag{2.1}$$

exist and are positive. We will ask what is the value of $\dim_H(\Lambda_\omega)$.

Before formulating the answer, let us note that any finite product $F_{\omega_1} \circ \ldots \circ F_{\omega_n}$ is again a Lalley-Gatzouras scheme. It follows that we can calculate the Hausdorff dimension of Λ_ω for any periodic sequence ω. Given a *rational* probabilistic vector $Q = (Q_1, \ldots, Q_m)$, we can choose a periodic sequence $\omega(Q)$ in which the frequency of symbol k is Q_k. Let us write

$$Ł(Q) = \dim_H \Lambda_{\omega(Q)}.$$

Our main result is as follows.

Theorem 2.1. *The function $Ł(Q)$ is well defined, does not depend on the choice of $\omega(Q)$. We can extend it by continuity to the whole simplex of probabilistic vectors (we will keep the notation $Ł(Q)$ for the extended function). We have*

$$\dim_H(\Lambda_\omega) = Ł(P).$$

3 Proof of Theorem 2.1

Let us start by presenting a more detailed description of Λ_ω (compare [Hu]). Let $A_\omega = A_{\omega_1} \times A_{\omega_2} \times \ldots$. We define a projection $\pi_\omega : A_\omega \to \Lambda_\omega$ by the formula

$$\pi_\omega((i_1, j_1), (i_2, j_2), \ldots) = \lim_{n\to\infty} f_{i_1,j_1}^{(\omega_1)} \circ \ldots \circ f_{i_n,j_n}^{(\omega_n)}(0,0),$$

$(i_k, j_k) \in A_k$. We get

$$\Lambda_\omega = \pi_\omega(A_\omega).$$

Because of the nonconformality of the system, the most natural class of subsets of A_ω to study are not cylinders but *rectangles* (in particular, *approximate squares*). The rectangle is defined as follows: given a sequence $(i,j) \in A_\omega$ and two natural numbers $n_1 \le n_2$ we define

$$R_{n_1,n_2}(i,j) = \{(i',j') \in A_\omega; i'_k = i_k \forall k \le n_2, j'_k = j_k \forall k \le n_1\}.$$

We will call

$$d_1(R_{n_1,n_2}(i,j)) = \prod_{k=1}^{n_1} a_{i_k j_k}^{(\omega_k)}$$

the *width* and

$$d_2(R_{n_1,n_2}(i,j)) = \prod_{k=1}^{n_2} b_{i_k}^{(\omega_k)}$$

the *height* of the rectangle $R_{n_1,n_2}(i,j)$. Indeed, the projection of a rectangle under π_ω is the intersection of Λ_ω with a geometric rectangle of the same width and of the same height. The rectangle of approximately (up to a constant) equal width and height is called an approximate square.

Our main step is the following proposition.

Proposition 3.1. *For Q a rational probabilistic vector sufficiently close to P and for any choice of $\omega(Q)$, there exists $K > 0$ and for every $d > 0$ there exists $\varepsilon > 0$ with $\varepsilon(d) \to 0$ as $d \to 0$ such that we can construct a bijection $\tau : A_{\omega(Q)} \to A_\omega$ with the following properties. Let $R = R_{n_1,n_2}^{(\omega(Q))}(i,j)$ be an approximate square in $A_{\omega(Q)}$ of width d. Then $\tau(R)$ contains an approximate square in A_ω of width at least $d^{1+K\delta+\varepsilon}$ and is contained in an approximate square in A_ω of width at most $d^{1-K\delta-\varepsilon}$, where $\delta = \max |P_k - Q_k|$.*

Proof. We will need the following simple statement (a reformulation of (2.1)):

Lemma 3.2. *For every n there exists $\varepsilon(n)$ such that for each k the n-th appearance of symbol k in the sequence ω takes place between positions $n/P_k(1-\varepsilon(n/P_k))$ and $n/P_k(1+\varepsilon(n/P_k))$. Moreover, $\varepsilon(n)$ goes monotonically to 0 as n goes to ∞.*

Consider now the pair of sequences: ω, the sequence we work with, and $\omega(Q)$, a periodic sequence with frequencies Q. We will assume that Q is δ-close to P and that both probabilistic vectors are positive. Obviously, in the sequence $\omega(Q)$ the n-th appearance of symbol k is at position n/Q_k, give or take a constant.

We will define $\chi_{\omega,\omega(Q)}$ as a permutation of $\mathbb{N}$ in the following way: if l_1 is the place of n-th appearance of symbol k in the sequence ω and l_2 is the place of n-th appearance of symbol k in the sequence $\omega(Q)$, we set $\chi_{\omega,\omega(Q)}(l_1) = l_2$. We can then construct a bijection $\tau : A_{\omega(Q)} \to A_\omega$ as

$$\tau((i_1, j_1), (i_2, j_2), \ldots) = (i_{\chi_{\omega,\omega(Q)}(1)}, j_{\chi_{\omega,\omega(Q)}(1)}) \ldots$$

Denote

$$D_1 = \chi_{\omega,\omega(Q)}(\{1, \ldots, n_1\})$$

and

$$D_2 = \chi_{\omega,\omega(Q)}(\{n_1 + 1, \ldots, n_2\})$$

We remind that the rectangle R is defined as the set of sequences $(i', j') \in A_{\omega(Q)}$ for which we fix the first n_1 (i'_k, j'_k) and the following $n_2 - n_1$ i'_k. Hence, the set $\tau(R)$ is the set of sequences $(i', j') \in A_\omega$ for which we fix (i'_k, j'_k) for $k \in D_1$ and we fix i'_k for $k \in D_2$.

Denote

$$r_1 = \inf(\mathbb{N} \setminus D_1) - 1,$$
$$r_2 = \inf(\mathbb{N} \setminus (D_1 \cup D_2)) - 1,$$
$$s_1 = \sup(D_1),$$
$$s_2 = \sup(D_1 \cup D_2).$$

We have

$$R^{(\omega)}_{s_1,s_2}(\tau(i, j)) \subset \tau(R) \subset R^{(\omega)}_{r_1,r_2}(\tau(i, j)).$$

Assume δ is much smaller than any P_k. By Lemma 3.2,

$$r_1 \geq n_1(1 - K_0\varepsilon(n_1) - K_0\delta),$$
$$r_2 \geq n_2(1 - K_0\varepsilon(n_2) - K_0\delta),$$
$$s_1 \leq n_1(1 + K_0\varepsilon(n_1) + K_0\delta),$$
$$s_2 \leq n_2(1 + K_0\varepsilon(n_2) + K_0\delta)$$

for some $K_0 > 0$ depending only on the iterated schemes.

Consider the width of $R^{(\omega)}_{r_1,r_2}(\tau(i, j))$ versus the width of R. The latter is a product of n_1 numbers $a^{(k)}_{ij}$, the former it the subproduct of r_1 of those numbers. As all $a^{(k)}_{ij}$ are uniformly bounded away from 0 and 1,

$$d_1(R^{(\omega)}_{r_1,r_2}(\tau(i,j))) \leq d^{1-K\varepsilon(n_1)-K\delta}$$

for some uniformly chosen K, depending only on the iterated schemes. Similar reasoning proves

$$d_2(R^{(\omega)}_{r_1,r_2}(\tau(i,j))) \leq d^{1-K\varepsilon(n_2)-K\delta}.$$

Consider now the width of $R^{(\omega)}_{s_1,s_2}(\tau(i,j))$ versus the width of R. The former a product of s_1 numbers $a^{(k)}_{ij}$, the latter is the subproduct of n_1 of those numbers, the same reasoning as before gives us

$$d_1(R^{(\omega)}_{s_1,s_2}(\tau(i,j))) \geq d^{1+K\varepsilon(n_1)+K\delta},$$

$$d_2(R^{(\omega)}_{s_1,s_2}(\tau(i,j))) \geq d^{1+K\varepsilon(n_2)+K\delta}.$$

The rectangles $R^{(\omega)}_{r_1,r_2}(\tau(i,j))$ and $R^{(\omega)}_{s_1,s_2}(\tau(i,j))$ are not necessarily approximate squares, but we can easily replace the former by some slightly larger rectangle which is an approximate square and we can replace the latter by some slightly smaller rectangle which is an approximate square. We are done. □

Remark. We can introduce a metric on A_ω, defining the distance between two points as the sum of width and height of the smallest rectangle containing them both. This metric is natural because if the maps satisfy the separation condition, π_ω is bi-Lipschitz (without separation condition it will only be a Lipschitz projection). In this metric, the maps τ, τ^{-1} are Hölder continuous with every exponent smaller than 1 (if $\delta = 0$) or with exponent $1 - K\delta$ (if δ is positive but small).

This proposition basically ends the proof of Theorem 2.1. By Proposition 3.3 and Lemma 5.2 in [LG], for any Lalley-Gatzouras scheme there exists a probabilistic measure μ supported on $A^{\mathbb{N}}$ such that

i) for a μ-typical point (i,j) and the decreasing sequence of all approximate squares $R_k = R_{n_1(k),n_2(k)}(i,j)$,

$$\frac{\log \mu(R_k)}{\log d_1(R_k)} \to \dim(\Lambda),$$

ii) for every point $x \in A^{\mathbb{N}}$ there exists a decreasing sequence of approximate squares $R_k = R_{n_1(k),n_2(k)}(i,j)$ for which

$$\frac{\log \mu(R_k)}{\log d_1(R_k)} \to \dim(\Lambda).$$

We can define such measure μ_Q supported on $A_{\omega(Q)}$ for any rational Q (because this is again a Lalley-Gatzouras scheme). We can then transport this measure to A_ω by the map τ. We obtain a measure ν_Q such that

i) for a ν_Q-typical point (i, j) and the decreasing sequence of all approximate squares $R_k = R_{n_1(k),n_2(k)}(i, j)$,

$$\liminf \frac{\log \nu_Q(R_k)}{\log d_1(R_k)} \geq Ł(Q)(1 - K\delta),$$

ii) for every point $x \in A_\omega$ there exists a decreasing sequence of approximate squares $R_k = R_{n_1(k),n_2(k)}(i, j)$ for which

$$\limsup \frac{\log \nu_Q(R_k)}{\log d_1(R_k)} \leq Ł(Q)(1 + K\delta).$$

It implies that

$$Ł(Q)(1 - K\delta) \leq \dim_H \Lambda_\omega \leq Ł(Q)(1 + K\delta),$$

the proof is as in [LG]. □

This result has immediate applications for random systems, obtained by choosing ω randomly with respect to some Bernoulli measure on Ω.

On the other hand, this method is not going to work for stochastically-selfsimilar systems considered in [F] or [GaL]. For such systems we would not have a single sequence ω but instead ω would depend on the point in the fractal. While we would still be able to define τ almost everywhere, the sequences $\omega(x)$ at different points $x \in \Lambda$ would not all satisfy Lemma 3.2, and hence τ would not everywhere have nice Hölder properties.

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