Entropy of Lyapunov-optimizing measures of some matrix cocycles

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Abstract. This is an extended version of my talk at the Fractal Geometry and Stochastic V conference in Tabarz. It is based on my joint paper [BR] with Jairo Bochi (PUC Santiago). Compared with the paper, I'll skip some details of some proofs, but I'll try to explain the main idea of our approach.

1. Setting

The object we study is seemingly very simple. We are given a finite family of 2×2 matrices $A_1, \ldots, A_k \in GL(2, \mathbb{R})$. For any sequence $\omega \in \{1, \ldots, k\}^{\mathbb{N}}$ we write $A_n(\omega) = A_{\omega_{n-1}} \cdot \ldots \cdot A_{\omega_0}$ and consider the *Lyapunov exponent*

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(\omega)|, \tag{1.1}$$

whenever it exists. The maximum and minimum values λ^+, λ^- attained by the Lyapunov exponent are called the *joint spectral radius* and *joint spectral subradius*, respectively; those notions play a significant role in the control theory, see for example [J] and references therein.

The same object appears naturally in dynamical systems as well; let us explain the relation. Let us start from the main object studied in the area of multifractal formalism: the Birkhoff average. Let $T : X \to X$ be a topological dynamical system (a continuous map of a compact space into itself) and let $\Phi : X \to \mathbb{R}^+$ be a continuous function. We consider the *cocycle* $\widehat{T} : (X \times \mathbb{R}^+) \to (X \times \mathbb{R}^+)$ given by the formula

$$\widetilde{T}(x, r) = (T(x), r \cdot \Phi(x)).$$

The value

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$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \left(\Phi(x) \cdot \Phi(Tx) \cdot \ldots \cdot \Phi(T^{n-1}x)\right)$$

(whenever it is defined) is called the *exponential rate of growth* in the fiber $\{x\}$, or the *Birkhoff average* of the potential $\log \Phi$ at the point x.

Let us now consider a natural generalization of this object: *noncommutative* Birkhoff averages. That is, we replace $\mathbb{R}^+$ by some noncommutative group, and we calculate the fiber rate of growth of corresponding cocycle using some appropriate norm. In our case, the base dynamics is the full shift on k symbols, the fiber action is given by the group $GL(2, \mathbb{R})$ and the norm is the usual matrix norm:

$$\widetilde{T}(\omega, M) = (\sigma\omega, A(\omega) \cdot M)$$

(where A is a 2×2 matrix-valued potential), so

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log |A(\sigma^{n-1}\omega) \cdot \ldots \cdot A(\sigma\omega) \cdot A(\omega)|. \tag{1.2}$$

This system is quite complicated, so let us consider the special case: the *one-step* cocycle, that is, let $A(\omega)$ depend only on ω_0. This takes us exactly to the situation we considered in the beginning: denoting by A_ℓ the value of A on $\{\omega; \omega_0 = \ell\}$, (1.2) reduces to (1.1).

2. Domination

It turns out to be difficult to describe the product of matrices, in particular, the norm of such a product can strongly depend on the order in which we multiply the matrices. For this reason the usual dynamical approach is to forget about the geometry of matrix product, and use only the subadditivity property of the (logarithm of) norm. The theory of *subadditive thermodynamical formalism* has recently developed strongly, let us just mention the book [B] and the papers [FH, FS].

We will apply an alternative approach, coming from the paper [BM]. That is, instead of considering a product of matrices and asking how fast the norm grows, we will multiply this product by a given vector, and will ask how fast the length of the vector grows:

$$\lambda(\omega, v) = \lim_{n\to\infty} \frac{1}{n} \log |A_n(\omega) \cdot v| .$$

The main difference is that we can write

$$\log |A_n(\omega) \cdot v| = \sum_{\ell=0}^{n-1} \log \left| A_{\omega_\ell} \frac{A_\ell(\omega)v}{|A_\ell(\omega)v|} \right| .$$

That is, we replace a noncommutative cocycle over a simple dynamical system (full shift) by a commutative cocycle but over a considerably more complicated system (action of matrices $\{A_i\}$ on $\mathbb{P}^1$). However, we need to explain why the growth rate of the length of a vector is related to the growth rate of the matrix norm in our original problem.

It will be more convenient for us to work with cocycles over the full shift on bi-infinite sequences (Σ, σ), where $\Sigma = \{1, \ldots, k\}^{\mathbb{Z}}$ and σ is the usual left shift. Naturally, the Lyapunov exponent $\lambda(\omega)$ can be defined on this space as well; it will only depend on the positive coordinates $\omega_+ = \{\omega_i, i \geqslant 0\}$. We will distinguish between the action of a matrix on $\mathbb{R}^2_* = \mathbb{R}^2 \backslash \{0,0\}$ and the action on $\mathbb{P}^1$ by the following notation: when we have $A : \mathbb{R}^2_* \to \mathbb{R}^2_*$, we write $A' : \mathbb{P}^1 \to \mathbb{P}^1$. Similarly, if M is a union of a family of lines in $\mathbb{R}^2_*$ passing through the origin, we denote by M' the corresponding subset of $\mathbb{P}^1$.

We say that the 2×2 matrix cocycle is *dominated* (or *exponentially separated*) if for each $\omega \in \Sigma$ we are given a splitting of $\mathbb{R}^2$ as the sum of two one-dimensional subspaces $e_1(\omega)$, $e_2(\omega)$ such that the following properties hold:

- equivariance:

$$A(\omega)(e_i(\omega)) = e_i(\sigma\omega) \quad \text{for all } \omega \in \Sigma \text{ and } i \in \{1, 2\}; \tag{2.1}$$

- dominance: there are constants $c > 0$ and $\delta > 0$ such that

$$\frac{|A^{(n)}(\omega) e_1(\omega)|}{|A^{(n)}(\omega) e_2(\omega)|} \geqslant c e^{\delta n} \quad \text{for all } \omega \in \Sigma \text{ and } n \geqslant 1. \tag{2.2}$$

This definition works for general cocycles, in our case there exists another, equivalent, definition. We define the *standard positive cone* in $\mathbb{R}^2_* := \mathbb{R}^2 \backslash \{0\}$ as

$$C_+ = \{(x, y) \in \mathbb{R}^2_*;\ xy \geqslant 0\}.$$

A *cone* in $\mathbb{R}^2_*$ is an image of C_+ by a linear isomorphism. A *multicone* in $\mathbb{R}^2_*$ is a disjoint union of finitely many cones. It was proved in [ABY, BG] that the one-step cocycle generated by $\{A_1, \ldots, A_k\}$ is dominated if and only if it has a *forward-invariant multicone*, that is, when there exists a multicone M such that its image $\bigcup_i A_i(M)$ is contained in the interior of M.

We can choose on M' a generalization of the Hilbert metric, that is a bounded metric d (depending on $\{A_1, \ldots, A_k\}$) in which all the maps A'_i are uniformly contracting:

Lemma 2.1. *Let $\{A_1, \ldots, A_k\}$ be a dominated cocycle with forward-invariant multicone M. Then there exists a metric d on M' and constants $c_0 > 1, 0 < \tau < 1$ such that for $v', w' \in M'$ we have*

$$d\left(A'_i v', A'_i w'\right) \leqslant \tau d\left(v', w'\right) \quad \textit{for all } i \in \{1, \ldots, k\}, \tag{2.3}$$

$$c_0^{-1} \angle(v, w) \leqslant d\left(v', w'\right) \leqslant c_0 \angle(v, w). \tag{2.4}$$

If M is forward-invariant for $\{A_1, \ldots, A_k\}$ then $M_c = \overline{(\mathbb{R}^2_* \backslash M)}$ is forward-invariant for $\{A_1^{-1}, \ldots, A_k^{-1}\}$. Moreover,

$$e'_1(\omega) = \bigcap_{n=1}^{\infty} A'_{\omega_{-1}} \cdot \ldots \cdot A'_{\omega_{-n}}(M')$$

and

$$e_2'(\omega) = \bigcap_{n=1}^{\infty} (A'_{\omega_{n-1}} \cdot \ldots \cdot A'_{\omega_0})^{-1}(M_c').$$

Let$\omega = (\omega_-, \omega_+)$, where $\omega_- = \{\omega_i, i \leqslant -1\}$. Then $e_1(\omega) = e_1(\omega_-)$, $e_2(\omega) = e_2(\omega_+)$. We have $e_1(\Sigma) \subset M$, $e_2(\Sigma) \subset M_c$.

For 2×2 matrices $\lambda(\omega, v) = \lambda(\omega)$ for all $v \notin e_2(\omega)$. As $e_2(\omega) \subset M_c$ for all $\omega \in \Sigma$, we get

$$\lambda(\omega, v) = \lambda(\omega)$$

for all $\omega \in \Sigma$ and $v \in M$.

Given $\omega \in \Sigma$, consider the pair $(e_1(\omega), e_2(\omega))$. This behaves very nicely under action of the shift:

$$(e_1'(\sigma\omega), e_2'(\sigma\omega)) = (A'_{\omega_0} e_1'(\omega), A'_{\omega_0} e_2'(\omega)). \tag{2.5}$$

We say that the cocycle $\{A_1, \ldots, A_k\}$ satisfies the *forward non-overlapping condition* if we can choose a forward-invariant multicone M in such a way that $A_i(M) \cap A_j(M) = \varnothing$ for $i \neq j$. It satisfies the *backward non-overlapping condition* if we can choose a forward-invariant multicone M such that $A_i^{-1}(M_c) \cap A_j^{-1}(M_c) = \varnothing$ for $i \neq j$. If the cocycle satisfies both forward and backward non-overlapping condition (not necessarily for the same multicone), we say it satisfies the *non-overlapping condition* (NOC). The NOC is not only a geometric condition, it has a dynamical meaning as well: it is a necessary and sufficient condition for the map $\omega \to (e_1(\omega), e_2(\omega))$ to be a bijection.

3. Statement of results

The paradigm of *ergodic optimization* (see [Je]) says that for typical potentials the optimizing orbits (sets $\{\omega; \lambda(\omega) = \lambda^{\pm}\}$) should have low dynamical complexity. This is true in the commutative case, see [C], [M]. In the noncommutative situation it is probably not true in general, at least for the joint spectral subradius. However, in the open set of cocycles dominating and satisfying NOC, it is satisfied for all (not just typical) cocycles.

We will define *upper* and *lower Mather sets* K^+, K^- for a dominated cocycle $\{A_1, \ldots, A_k\}$ as follows: K^+ (resp. K^-) is the union of supports of all σ-invariant measures μ on Σ such that $\lambda(\mu) = \lambda^+$ (resp. λ^-).

Theorem 3.1. *For a dominated cocycle, the Mather sets K^+, K^- are compact, nonempty, and invariant under σ. Moreover, every measure μ supported on K^+ (resp. K^-) satisfies $\lambda(\mu) = \lambda^+$ (resp. λ^-).*

Our main result is the following:

Theorem 3.2. *For a dominated cocycle satisfying NOC, the Mather sets K^+, K^- have zero topological entropy under σ.*

Both assumptions of Theorem 3.2 are necessary. For example, the cocycle

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 3 & 0 \\ 1 & 1/3 \end{pmatrix}$$

satisfies domination and the forward NOC, but still it does not satisfy the assertion of Theorem 3.2 (in this case, $K^+ = K^- = \Sigma$). For cocycles not satisfying domination the situation for joint spectral subradius is even worse: if we restrict our attention to cocycles $\{A_1, \dots, A_k\} \in SL(2, \mathbb{R})^k$, among cocycles not satisfying domination one can find an open and dense subset for which there exists an invariant positive topological entropy subset of Σ for which the norms $|A_n(\omega)|$ are uniformly bounded for all n (this corresponds to $\lambda^- = 0$). That is, the ergodic optimization fails.

The behaviour of the joint spectral radius is unknown in this case, but by a long-standing conjecture the ergodic optimization holds.

4. Barabanov functions and proof of Theorem 3.1

If the cocycle $\{A_1, \dots, A_k\}$ is *irreducible* (has no nontrivial invariant subspace) then one can construct a *Barabanov norm*, that is a norm $|\cdot|_B$ on $\mathbb{R}^2$ such that for any $v \in \mathbb{R}^2$

$$\max_i |A_i v|_B = e^{\lambda^+} \cdot |v|_B. \tag{4.1}$$

In fact, such a norm can be defined in a much more general situation (i.e. for any irreducible compact subset of $GL(n, \mathbb{R})$), see [Ba, W].

Unfortunately, in general there cannot exist a norm satisfying the analogue of (4.1) for the joint spectral subradius. However, we are able to construct a replacement (based on a similar idea in [BM]). Given a dominated cocycle $\{A_1, \dots, A_k\}$ with a forward-invariant multicone M, a pair of functions $p^+, p^- : M \to \mathbb{R}$ will be called *Barabanov functions* if they have the following properties:

- extremality: for all $v \in M$,

$$\max_{i \in \{1,\dots,k\}} p^+(A_i v) = p^+(v) + \lambda^+ , \tag{4.2}$$

$$\min_{i \in \{1,\dots,k\}} p^-(A_i v) = p^-(v) + \lambda^- ; \tag{4.3}$$

- log-homogeneity: for all $v \in M$, and $t \in \mathbb{R}_*$,

$$p^\pm(tv) = p^\pm(v) + \log|t| ; \tag{4.4}$$

- regularity: there exists $c_1 > 0$ such that for all v_1, $v_2 \in M$,

$$p^\pm(v_1) - p^\pm(v_2) \leqslant c_1 \angle(v_1, v_2) + \log|v_1| - \log|v_2| . \tag{4.5}$$

Theorem 4.1. *For any dominated cocycle $\{A_1, \dots, A_k\}$ there exist Barabanov functions p^+, p^-.*

Proof. For each i let
$$h_i(v) = \log \frac{|A_i v|}{|v|}.$$
This function does not change under multiplying v by a scalar, hence it can be defined on $\mathbb{P}^1$. Let c_2 be the common Lipschitz constant of all h_is:
$$\left|h_i(v') - h_i(w')\right| \leqslant c_2 \angle(v, w) \quad \text{for all } i \in \{1, \ldots, k\}, \text{ for all } v,\, w \in \mathbb{R}^2_*.$$

Let $c_3 = c_0 c_2/(1-\tau)$. Let $\mathbb{K}$ be the space of c_3-Lipschitz functions (in d) from M' to $\mathbb{R}$ endowed with sup metric. For $f \in \mathbb{K}$ let
$$(T^+ f)(v') = \max_{i \in \{1,\ldots,k\}} \left[f\left(A_i' v'\right) + h_i(v') \right],$$
$$(T^- f)(v') = \min_{i \in \{1,\ldots,k\}} \left[f\left(A_i' v'\right) + h_i(v') \right].$$

One can check that $T^+, T^- : \mathbb{K} \to \mathbb{K}$. We also have
$$T^{\pm}(f + c) = c + T^{\pm} f,$$
hence we can define T^+, T^- on the quotient $\hat{\mathbb{K}}$ of $\mathbb{K}$ by the subspace of constant functions. $\hat{\mathbb{K}}$ is convex and (by Arzela-Ascoli) compact, hence T^+ and T^- have fixed points in $\hat{\mathbb{K}}$ that we will denote by f_0^+, f_0^-. That is, there exist constants β^+, β^- such that
$$T^{\pm} f_0^{\pm} = f_0^{\pm} + \beta^{\pm}.$$
This immediately implies that the functions
$$p^{\pm}(v) = f_0^{\pm}(v') + \log|v|$$
satisfy all the required properties of Barabanov functions, with $\beta^{\pm}$ in place of $\lambda^{\pm}$. The only thing left is to check that $\beta^{\pm}$ cannot be different from $\lambda^{\pm}$.

Let us present this argument for β^+. For any vector $v \in M$ there exists a (not necessarily unique) $\omega_1^+ \in \{1, \ldots, k\}$ such that $p^+(A_{\omega_1^+} v) = p^+(v) + \beta^+$. We can then find ω_2^+ such that $p^+(A_{\omega_2^+} A_{\omega_1^+} v) = p^+(v) + 2\beta^+$, and so on. Thus, β^+ is the maximal growth rate of p^+ for any vector $v \in M$. At the same time, by log-homogeneity of Barabanov functions, $p^+(v)$ can differ from $\log|v|$ by at most a constant. Hence, the growth rate of p^+ must be the same as the growth rate of $\log|\cdot|$, and we are done. □

The statement of Theorem 3.1 follows easily (once again, we will only construct K^+). Above we constructed for any vector $v \in M$ a set of infinite sequences $\Omega^+(v) \subset \{1, \ldots, k\}^{\mathbb{N}}$ such that for every $\omega \in \Omega^+(v)$
$$p^+\left(A_n(\omega) v\right) = p^+(v) + n\lambda^+.$$
Consider the set $K_0^+ \subset \Sigma$ of the following form: $\omega = (\omega_-, \omega_+)$ belongs to K_0^+ if and only if $\omega_+ \in \Omega^+(e_1(\omega_-))$. Clearly, $\sigma K_0^+ \subset K_0^+$. We define
$$K^+ = \bigcap_{j=0}^{\infty} \sigma^j K_0^+.$$

This set is nonempty and compact, and has the following property: let $\omega \in K^+$ and $j \in \mathbb{Z}$. Let $\sigma^j\omega = (\omega_-^{(j)}, \omega_+^{(j)})$. Then

$$\omega_+^{(j)} \in \Omega^+(e_1(\omega_-^{(j)})).$$

It follows that every measure supported on K^+ has the maximal growth of p^+. Vice versa, every measure giving maximal growth of p^+ must for almost every past ω_- give full probability to futures from $\Omega^+(e_1(\omega_-))$, hence it must be supported on K^+. As the growth of p^+ must be the same as the growth of the length of any vector from M, this proves that the constructed set K^+ is the Mather set.

5. Proof of Theorem 3.2

The strategy of the proof is quite simple. We consider the space $\{(e_1(\omega), e_2(\omega)); \omega \in K^\pm\}$ with the dynamics given by (2.5). We will use Barabanov functions and geometric arguments to prove that this dynamical system has zero entropy (this result does not use NOC, only domination). We will then use NOC to transport the entropy result back to the full shift (Σ, σ).

Let us start with a simple lemma.

Lemma 5.1. *Let $\omega = (\omega_-, \omega_+) \in K^\pm$. Choose any $x \in e_1(\omega_-)$. If $y \in M$ is such that $x - y \in e_2(\omega_+)$ then*

$$p^+(x) \leqslant p^+(y) \quad \text{if } \omega \in K^+,$$
$$p^-(x) \geqslant p^-(y) \quad \text{if } \omega \in K^-.$$

Proof. Consider the case $\omega \in K^+$, the other is analogous. As $y - x \in e_2(\omega_+)$,

$$p^+(A_n(\omega)x) - p^+(A_n(\omega)y) \to 0.$$

At the same time,

$$p^+(A_n(\omega)x) - p^+(x) = n\lambda^+ \geqslant p^+(A_n(\omega)y) - p^+(y).$$

□

Given vectors x_1, y_1, x_2, $y_2 \in \mathbb{R}^2_*$, no three of them collinear, we define their *cross-ratio*

$$[x_1, y_1; x_2, y_2] := \frac{x_1 \times x_2}{x_1 \times y_2} \cdot \frac{y_1 \times y_2}{y_1 \times x_2} \in \mathbb{R} \cup \{\infty\},$$

where $\times$ denotes the cross-product in $\mathbb{R}^2$, i.e. the determinant. The cross-ratio depends only on the directions of the four vectors, hence we can define it on $(\mathbb{P}^1)^4$. See [BK, Section 6].

Applying Lemma 5.1 twice, we get

Lemma 5.2. *Let $\omega, \tau \in K^\pm$. Then*

$$|[e_1(\omega_-), e_1(\tau_-); e_2(\omega_+), e_2(\tau_+)]| \geqslant 1 \quad \text{if } \omega, \tau \in K^+,$$
$$|[e_1(\omega_-), e_1(\tau_-); e_2(\omega_+), e_2(\tau_+)]| \leqslant 1 \quad \text{if } \omega, \tau \in K^-.$$

Proof. We will consider the case $\omega, \tau \in K^+$, the other one is analogous. We choose $x_1 \in e_1(\omega_-), x_2 \in e_2(\omega_+), y_1 \in e_1(\tau_-), y_2 \in e_2(\tau_+)$. We can write

$$x_1 = \alpha x_2 + \beta y_1 \quad \text{and} \quad y_1 = \gamma y_2 + \delta x_1.$$

Applying Lemma 5.1 twice, we get

$$p^+(x_1) \leqslant p^+(\beta y_1) \leqslant p^+(\beta\delta x_1) = p^+(x_1) + \log|\beta\delta|.$$

Hence, $|\beta\delta| \geqslant 1$. Substituting

$$\beta = \frac{x_1 \times x_2}{y_1 \times x_2} \quad \text{and} \quad \delta = \frac{y_1 \times y_2}{x_1 \times y_2}$$

we obtain the assertion. □

We now use the hyperbolic geometry representation of $\mathbb{P}^1$. We identify $(\cos\theta, \sin\theta) \in \mathbb{P}^1$ with the point $e^{2\theta i}$ on the boundary $\partial\mathbb{D}$ of the unit disk $\mathbb{D}$. We endow $\mathbb{D}$ with the Poincaré hyperbolic metric. Given two points $x, y \in \partial\mathbb{D}$, we consider their connecting geodesic $\vec{xy} \in \mathbb{D}$.

Let $(x_1, y_1; x_2, y_2)$ be a 4-tuple of distinct points in $\mathbb{P}^1$. Then one and only one of the following possibilities holds:

- *antiparallel configuration*: $x_1 < y_2 < y_1 < x_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Figure 1);
- *coparallel configuration*: $x_1 < y_1 < y_2 < x_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Figure 2);
- *crossing configuration*: $x_1 < y_1 < x_2 < y_2 < x_1$ for some cyclic order $<$ on $\mathbb{P}^1$ (see Figure 3).

We say that two geodesics $\overrightarrow{x_2x_1}$ and $\overrightarrow{y_2y_1}$ with distinct endpoints are *antiparallel*, *coparallel*, or *crossing* according to the configuration of the 4-tuple $(x_1, y_1; x_2, y_2)$.

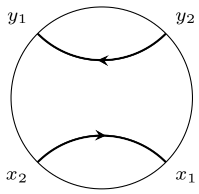

Figure 1. Antiparallel configuration

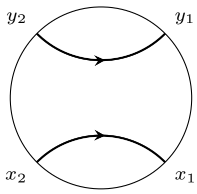

Figure 2. Coparallel configuration

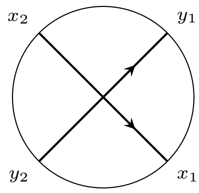

Figure 3. Crossing configuration

The configuration can be expressed in terms of the cross-ratio as follows:

Lemma 5.3. *Consider a 4-tuple $(x_1, y_1; x_2, y_2)$ of distinct points in $\mathbb{P}^1$. Then:*

- *the configuration is* antiparallel *iff* $[x_1, y_1; x_2, y_2] < 0$,
- *the configuration is* coparallel *iff* $0 < [x_1, y_1; x_2, y_2] < 1$,
- *the configuration is* crossing *iff* $[x_1, y_1; x_2, y_2] > 1$.

Hence, Lemma 5.2 implies that for two sequences $\omega, \tau \in K^{\pm}$ the corresponding geodesics $\overrightarrow{e_1(\omega_-)e_2(\omega_+)}, \overrightarrow{e_1(\tau_-)e_2(\tau_+)}$ cannot be in coparallel (if $\omega, \tau \in K^+$) or crossing (if $\omega, \tau \in K^-$) configuration.

We will not formulate the last part of the proof for the dynamical system acting on pairs $(e_1(\omega_-), e_2(\omega_+))$ but directly for $(K^{\pm}, \sigma)$. We recall that NOC guarantees that the two systems are conjugated. For K^+, K^- let us consider the sets of pasts with more than one future and sets of futures with more than one past. Formally, consider

$$N_1^+ = \{\omega_-;\ \text{there exists more than one } \omega_+ \text{ such that } (\omega_-, \omega_+) \in K^+\}, \tag{5.1}$$

$$N_2^+ = \{\omega_+;\ \text{there exists more than one } \omega_- \text{ such that } (\omega_-, \omega_+) \in K^+\}. \tag{5.2}$$

We define N_1^-, N_2^- analogously.

Lemma 5.4. *The sets $N_1^+, N_2^+, N_1^-, N_2^-$ are countable.*

Proof. Consider N_1^+ first (the case of N_2^+ is analogous). Let $\omega_- \in N_1^+$. Denote by $I^+(\omega_-)$ the convex hull (taken in $\mathbb{P}^1 \backslash \{e_1(\omega_-)\}$) of the points $e_2(\omega_+)$ for ω_+ such that $(\omega_-, \omega_+) \in K^+$. Then for different $\omega_-, \tau_- \in N_1^+$ the intervals $I^+(\omega), I^+(\tau_-)$ have disjoint interiors. Indeed, otherwise some pairs of geodesics $\overrightarrow{e_1(\omega_-)e_2(\omega_+)}, \overrightarrow{e_1(\tau_-)e_2(\tau_+)}$ would have to be in coparallel configuration, see Figure 4

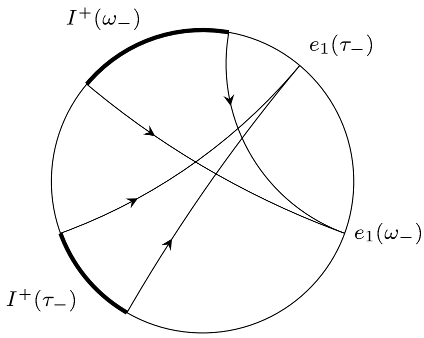

Figure 4. Disjoint arcs

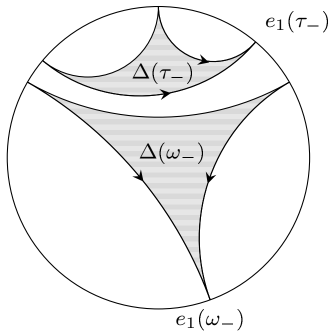

Figure 5. Disjoint geodesic triangles

Consider now the case N_1^- (or N_2^-). For any $\omega_- \in N_1^-$ we construct $I^-(\omega_-)$ analogously to $I^+(\omega_-)$ above, and then we construct the geodesic triangle $\Delta(\omega_-)$ with vertices $e_1(\omega_-)$ and the two endpoints of $I^-(\omega_-)$. Then for any two sequences $\omega_-, \tau_- \in N_1^-$ the triangles $\Delta(\omega_-), \Delta(\tau_-)$ have disjoint interiors (otherwise some pair of geodesics $\overrightarrow{e_1(\omega_-)e_2(\omega_+)}, \overrightarrow{e_1(\tau_-)e_2(\tau_+)}$ would have to be in crossing configuration), see Figure 5.

The assertion follows by the separability of $\partial\mathbb{D}$ and $\mathbb{D}$. ▫

Thus, in either K^+ or K^- every past (except countably many) has a unique future and every future (except countably many) has a unique past. Such sets have zero topological entropy:

Lemma 5.5. *Let K be a compact σ-invariant subset of a two-sided shift. Define N_1, N_2 as in* (5.1), (5.2)*. If N_1 and N_2 are countable then K has zero topological entropy.*

Proof. It is enough to prove that every ergodic invariant measure has zero metric entropy. The atomic measures have entropy zero. The nonatomic measures do not see N_1, N_2, hence the past uniquely determines the future (and vice versa). This means that the conditional entropy of the generating partition with respect to the past/future is zero. ▫

6. Open questions

There are many open questions. In particular:

- What happens for more general potentials (i.e. not piecewise constant)?
- What happens for more general base systems (for example, for subshifts of finite type)?
- What happens for matrices of size greater than 2×2?
- What happens in the generic situation?

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