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KERNEL ESTIMATORS AND THE DVORETZKY–KIEFER–WOLFOWITZ INEQUALITY

Abstract. It turns out that for standard kernel estimators no inequality like that of Dvoretzky–Kiefer–Wolfowitz can be constructed, and as a result it is impossible to answer the question of how many observations are needed to guarantee a prescribed level of accuracy of the estimator. A remedy is to adapt the bandwidth to the sample at hand.

1. Dvoretzky–Kiefer–Wolfowitz inequality. Let $X_1, \ldots, X_n$ be a sample from an (unknown) distribution $F \in \mathcal{F}$ where $\mathcal{F}$ is the class of all continuous distribution functions. Let

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} 1_{(-\infty,x]}(X_j)$$

be the empirical distribution function. The Dvoretzky–Kiefer–Wolfowitz inequality in its strongest version (Massart 1990) states that

$$P\{\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \geq \varepsilon\} \leq 2e^{-2n\varepsilon^2}. \tag{1}$$

Making use of this inequality, for every $\varepsilon > 0$ and every $\eta > 0$ one can easily calculate $N(\varepsilon, \eta)$ such that if $n \geq N(\varepsilon, \eta)$ then

$$P\{\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \geq \varepsilon\} \leq \eta.$$

2. Kernel estimators. The standard kernel density estimator is of the form (e.g. *Encyclopedia of Statistical Sciences* (2006))

$$\widehat{f}_n(x) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{h_n}\, k\left(\frac{x - X_j}{h_n}\right)$$

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with appropriate h_n, $n = 1, 2, \ldots$. We shall consider kernel distribution estimators of the classical form

$$\widehat{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right)$$

where $K(x) = \int_{-\infty}^{x} k(t)\,dt$, and we shall show that no inequality like (1) with $\widehat{F}_n(x)$ instead of $F_n(x)$ can be constructed.

PROPOSITION. *Let $k(\cdot)$ be any kernel such that $0 < K(0) < 1$ and $K^{-1}(t) < 0$ for some $t \in (0, K(0))$. Let $(h_n, n = 1, 2, \ldots)$ be any sequence of positive reals. Then there exist $\varepsilon > 0$ and $\eta > 0$ such that for every n there exists $F \in \mathcal{F}$ for which*

$$P\{\sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F(x)| \ge \varepsilon\} \ge \eta.$$

Proof. Obviously it is enough to demonstrate that under the assumptions of the Proposition there exist $\varepsilon > 0$ and $\eta > 0$ such that for every n there exists $F \in \mathcal{F}$ satisfying $P\{\widehat{F}_n(0) > F(0) + \varepsilon\} \ge \eta$.

Take $\varepsilon \in (0, t)$ and $\eta \in (t - \varepsilon, 1)$. Fix n. Given ε, η, and n, choose F such that $F(0) = t - \varepsilon$ and $F(-h_n K^{-1}(t)) = P\{X_j < -h_n K^{-1}(t)\} > \eta^{1/n}$. Then

$$P\left\{K\left(-\frac{X_j}{h_n}\right) > F(0) + \varepsilon\right\} > \eta^{1/n}$$

and due to the fact that

$$\bigcap_{j=1}^{n}\left\{K\left(-\frac{X_j}{h_n}\right) > F(0) + \varepsilon\right\} \subset \left\{\frac{1}{n}\sum_{j=1}^{n} K\left(-\frac{X_j}{h_n}\right) > F(0) + \varepsilon\right\}$$

we have

$$P\left\{\frac{1}{n}\sum_{j=1}^{n} K\left(-\frac{X_j}{h_n}\right) > F(0) + \varepsilon\right\} \ge \prod_{j=1}^{n} P\left\{K\left(-\frac{X_j}{h_n}\right) > F(0) + \varepsilon\right\} > \eta,$$

which ends the proof.

REMARK. By the Proposition, $\sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F(x)|$ does not converge to zero in probability, uniformly in $F \in \mathcal{F}$.

3. Random bandwidth. Let $X_{1:n} \le \cdots \le X_{n:n}$ be order statistics from the sample $X_1, \ldots, X_n$. Define

$$H_n = \min\{X_{j:n} - X_{j-1:n} : j = 2, \ldots, n\}.$$

Define the kernel estimator

$$\widetilde{F}_n(x) = \frac{1}{n}\sum_{j=1}^{n} K\left(\frac{x - X_j}{H_n}\right)$$

where for K we assume:

$$K(t) = \begin{cases} 0 & \text{for } t \le -1/2, \\ 1 & \text{for } t \ge 1/2, \end{cases}$$

$K(0) = 1/2$, $K(t)$ is continuous and nondecreasing in $(-1/2, 1/2)$.

Now, for $k = 1, \dots, n$ we have $|\widetilde{F}_n(X_{k:n}) - F_n(X_{k:n})| = 1/2n$. The kernel estimator $\widetilde{F}_n(x)$ is continuous and nondecreasing, the empirical distribution function $F_n(x)$ is a step function, and consequently $|\widetilde{F}_n(x) - F_n(x)| \le 1/2n$ for all $x \in (-\infty, \infty)$. By the triangle inequality

$$|\widetilde{F}_n(x) - F(x)| \le |F_n(x) - F(x)| + \frac{1}{2n}$$

we obtain

$$P\{\sup_{x\in\mathbb{R}} |\widetilde{F}_n(x) - F(x)| \ge \varepsilon\} \le P\left\{\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| + \frac{1}{2n} \ge \varepsilon\right\},$$

and hence by (1) we have

$$P\{\sup_{x\in\mathbb{R}} |\widetilde{F}_n(x) - F(x)| \ge \varepsilon\} \le 2e^{-2n(\varepsilon-1/2n)^2}, \qquad n > \frac{1}{2\varepsilon}, \tag{2}$$

which enables us to calculate $N = N(\varepsilon, \eta)$ that guarantees the prescribed accuracy of the kernel estimator $\widetilde{F}_n(x)$.

Comment 1. Observe that the smallest $N = N(\varepsilon, \eta)$ that guarantees the prescribed accuracy is somewhat greater for the kernel estimator $\widetilde{F}_n$ than that for the crude empirical step function F_n. For example, $N(0.1, 0.1)$ is 150 for F_n and 160 for $\widetilde{F}_n$; $N(0.01, 0.01)$ is 26 492 for F_n and 26 592 for $\widetilde{F}_n$.

Comment 2. Another disadvantage of kernel smoothing has been discovered by Hjort and Walker (2001): "kernel density estimator with optimal bandwidth lies outside any confidence interval, around the empirical distribution function, with probability tending to 1 as the sample size increases". Perhaps a reason is that smoothing adds to observations something which is rather arbitrarily chosen and which may spoil the inference.

A generalization. Inequality (2) holds for every distribution function $\widetilde{F}_n(x)$ that satisfies $|\widetilde{F}_n(X_{k:n}) - F_n(X_{k:n})| = 1/2n$, $k = 1, \dots, n$.

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