Hyper-Stonean envelopes of locally compact spaces

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References


Banach space preliminaries

Take Banach spaces $E$ and $F$.

Then $E$ and $F$ are \textbf{isomorphic} if they are linearly homeomorphic. Write $E \sim F$.

Also $E$ and $F$ are \textbf{isometrically isomorphic} if there is an isometric linear bijection $T : E \to F$. Write $E \cong F$.

The \textbf{dual} of a Banach space $E$ is denoted by $E'$ and the \textbf{bidual} is $E'' = (E')'$; we regard $E$ as a subspace of $E''$. A \textbf{predual} of $E$ is a Banach space $F$ with $F' \cong E$.

\textbf{Example}: Let $K$ be a locally compact space. Then $C_0(K)$ and $C^b(K)$ are the Banach algebras of all continuous functions on $K$ that vanish at infinity and all bounded continuous functions on $K$, taken with the uniform norm

$$|f|_K = \sup\{|f(x)| : x \in K\}.$$ 

The dual of $C_0(K)$ is $M(K)$, the complex-valued, regular Borel measures on $K$. \hfill \square
Banach $A$-bimodules

Let $A$ be a Banach algebra. Then the bidual $A''$ is a Banach $A$-bimodule for the maps

$$(a, M) \mapsto a \cdot M, \quad (a, M) \mapsto M \cdot a.$$ 

There are two products on $A''$ extending the module maps. For $a, b \in A$, and $\lambda \in A'$, define:

$$\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle, \quad \langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle.$$ 

Then, for $a \in A$, $\lambda \in A'$, and $M \in A''$, define:

$$\langle a, \lambda \cdot M \rangle = \langle M, a \cdot \lambda \rangle, \quad \langle a, M \cdot \lambda \rangle = \langle M, \lambda \cdot a \rangle.$$ 

Let $M, N \in A''$. For each $\lambda \in A'$, define

$$\langle M \Box N, \lambda \rangle = \langle M, N \cdot \lambda \rangle, \quad \langle M \Diamond N, \lambda \rangle = \langle N, \lambda \cdot M \rangle.$$ 

**Easier to understand:** take $M = \lim_{\alpha} a_\alpha$ and $N = \lim_{\beta} b_\beta$ in $A''$ (in the weak-* topology) for nets $(a_\alpha)$ and $(b_\beta)$ in $A$. Then

$$M \Box N = \lim_{\alpha} \lim_{\beta} a_\alpha b_\beta, \quad M \Diamond N = \lim_{\beta} \lim_{\alpha} a_\alpha b_\beta.$$  


Arens’ theorem

**Theorem** (Arens 1951) Let $A$ be a Banach algebra. Then $(A'', \square)$ and $(A'', \diamond)$ are two Banach algebras, each containing $A$ as a closed subalgebra.

Arens regularity

The algebra $A$ is **Arens regular** if

$$M \square N = M \diamond N \quad (M, N \in A''),$$

and **strongly Arens irregular = SAI** if the opposite extreme holds: if $M \square N = M \diamond N$ for all $N \in A''$, then necessarily $M \in A$ - and similarly on the other side.

Closed subalgebras and quotients of Arens regular algebras are Arens regular.
Examples

Arens regularity/SAI gives a very sharp contrast between two classic classes of Banach algebras.

(I) Every $C^*$-algebra, including $C_0(K)$, is Arens regular - and its bidual is a $C^*$-algebra, called the **enveloping von Neumann algebra**.

(II) Let $G$ be a locally compact group. Every group algebra $L^1(G)$ is SAI (Lau-Losert); the measure algebra $M(G)$ is SAI (Losert, Neufang, Pachl, and Steprāns).

The ‘topological centre’ of $L^1(G)$ is determined by just two elements of $L^1(G)''$. How many such points are needed for $M(G)$?

There are examples that are neither Arens regular nor SAI.
Topological preliminaries

A topological space is extremely disconnected if the closure of every open set is itself open. A Stonean space is a compact topological space that is extremely disconnected.

Example: $\beta\mathbb{N}$ is Stonean.

Let $U$ be a dense subset of a Stonean space $K$. Then $\beta U = K$. Each infinite Stonean space $K$ contains a copy of $\beta\mathbb{N}$, and so $|K| \geq 2^\mathfrak{c}$.

The Souslin number $c(K)$ of $K$ is the minimum cardinality $\kappa$ such that each family of non-empty, pairwise-disjoint, open subsets has cardinality at most $\kappa$; $K$ satisfies CCC, the countable chain condition, iff $c(K) \leq \aleph_0$ is countable.
Injective spaces

A Banach space $E$ is **injective** if, for every Banach space $F$, every closed subspace $G$ of $F$, and every $T \in \mathcal{B}(G, E)$, there is an extension $\tilde{T} \in \mathcal{B}(F, E)$ of $T$; the space $E$ is **$\lambda$-injective** if, further, we can always find such a $\tilde{T}$ with $\|\tilde{T}\| \leq \lambda \|T\|$.

It is standard that an injective Banach space is $\lambda$-injective for some $\lambda \geq 1$.

Dual spaces

A Banach space $E$ is **isomorphically dual** if there is a Banach space $F$ such that $E \sim F'$.  

A Banach space $E$ is **isometrically dual** if there is a Banach space $F$ such that $E \cong F'$.

A Banach space can be isomorphically dual, but not isometrically dual; see later.
Boolean algebras

Let $B$ be a Boolean algebra (e.g., the Borel sets $\mathcal{B}_K$ for a locally compact space $K$). An ultrafilter on $B$ is a subset $p$ that is maximal with respect to the property that

$$b_1, \ldots, b_n \in p \Rightarrow b_1 \land \cdots \land b_n \neq 0.$$  

The family of ultrafilters on $B$ is the **Stone space** of $B$, denoted by $St(B)$. A topology on $St(B)$ is defined by taking the sets

$$\{p \in St(B) : b \in p\}$$

for $b \in B$ as a basis of the open sets of $St(B)$. In this way, $St(B)$ is a totally disconnected compact space; it is extremely disconnected if and only if $B$ is complete as a Boolean algebra.

**Example** $B = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. Then $St(B)$ is just $\beta \mathbb{N}$, and it is the character space of $\ell^\infty = C(\beta \mathbb{N})$. $\square$
Theorem (Banach–Stone) Let $K$ and $L$ be two non-empty, compact spaces. Then the following are equivalent:

(a) $K$ and $L$ are homeomorphic;

(b) $C(L) \cong C(K)$;

(c) $C(L)$ and $C(K)$ are $C^*$-isomorphic;

(d) there is an algebra isomorphism from $C(L)$ onto $C(K)$;

(e) there is a Banach-lattice isometry from $C(L)$ onto $C(K)$;

(f) there is an isometry from $C_R(L)$ onto $C_R(K)$.
\[ C(K) \sim C(L) \]

**Theorem (Milutin)** Suppose that \( K \) and \( L \) are uncountable, metrizable, compact. Then \( C(K) \sim C(L) \). \[ \square \]

**Theorem (Cengiz)** Suppose that \( C(K) \sim C(L) \). Then \( K \) metrizable iff \( L \) is; \( w(K) = w(L) \); \( |K| = |L| \). \[ \square \]

**Fact** \( K \) metrizable implies \( C(K) \sim C(L) \) for some \( L \) with \( L \) totally disconnected (take \( L \) to be the Cantor set). \[ \square \]

But:

**Example (Koszmider)** There is a connected, compact \( K \) such that \( C(K) \not\sim C(L) \) for any totally disconnected \( L \). \[ \square \]

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**Gleason’s theorem**

Here is the key theorem, mainly due to Gleason, but some others.

*Theorem* Let $K$ be a compact space. Then the following are equivalent:

(a) the lattice $C^*_R(K)$ is **Dedekind complete**;

(b) $K$ is Stonean;

(c) $C(K)$ is injective in the category of commutative $C^*$-algebras and continuous $*$-homomorphisms;

(d) $K$ is projective in the category of compact spaces;

(e) $C(K)$ is 1-injective as a Banach space;

and about 4 other standard properties. \[\square\]

Each compact $K$ has a **Gleason cover** $G_K$. 

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Injective spaces - questions

**Theorem** Let $E$ be a 1-injective Banach space. Then $E \sim C(K)$ for a Stonean space $K$. \qed

**Open Question** What if $E$ is just $\lambda$-injective. Is the same true? (Yes if $\lambda < 2$.)

**Open Question** Suppose that $C(K)$ is injective. Is $C(K) \sim C(L)$ for some Stonean space $L$? Is $K$ totally disconnected?

By **Amir’s theorem**, $C(K)$ injective implies that $K$ contains a dense, open, extremely disconnected subset, and so $K$ is not connected.

**Proposition** $C(K)$ injective implies $K$ totally disconnected when $c(K) < \mathfrak{c}$. \qed
Injectivity and dual spaces

**Theorem** Suppose that $C(K)$ is isomorphically dual. Then $C(K)$ is an injective space. \(\square\)

The converse is not true (Rosenthal). So ‘isomorphically dual’ is stronger than ‘injective’.

**Open Question** Suppose that $C(K)$ is isomorphically dual. Is $C(K) \sim C(L)$ for a Stonean space $L$? Is $K$ totally disconnected?
Normal measures on $K$

Let $K$ be a locally compact space.

**Definition** A (positive) measure $\mu$ on $K$ is normal if it is order-continuous, i.e., $\langle f_\alpha, \mu \rangle \to 0$ for each net $(f_\alpha)$ in $C(K)^+ \cap C(K)$ such that $f_\alpha \to 0$ in order.

**Proposition** A measure $\mu$ on $K$ is normal iff $|\mu|(L) = 0$ for every compact subset $L$ of $K$ with $\text{int} \, L = \emptyset$.

The normal measures form a closed subspace $N(K)$ of $M(K)$.

A point mass is normal iff the point is isolated.
Examples of $N(K)$

**Example 1** All measures on $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ are $\sigma$-normal, but $N(\mathbb{N}^*) = \{0\}$. □

**Example 2** $N(K) = \{0\}$ for each separable $K$ without isolated points. E.g., $N(G_I) = \{0\}$. □

**Example 3** $N(G) = \{0\}$ for each non-discrete, locally compact group. □

**Example 4** $N(K) = \{0\}$ for each locally connected space without isolated points. □

**Example 5** $N(K) = \{0\}$ for each connected $F$-space. □

However we have the following example of Grzegorz Plebanek.

**Example 6** There is a connected, compact space $K$ satisfying CCC with $N(K) \neq \{0\}$. □
Theorem - Dixmier, 1952, and Grothendieck, 1955 Let $K$ be a compact space. Then the following are equivalent, and define a hyper-Stonean space:

(a) $K$ is Stonean and the normal measures separate the elements of $C(K)$;

(b) $C(K)$ is lattice-isomorphic to the dual of a Banach lattice;

(c) $C(K)$ is isometrically dual as a Banach space (so that $C(K)$ is a von Neumann algebra);

(d) there is a locally compact space $\Gamma$ and a positive measure $\mu$ with $C(K) = L^\infty(\Gamma, \mu)$.

[and several other equivalences - but no purely topological one].

Certainly: $K$ hyper-Stonean $\Rightarrow$ $K$ Stonean.
Examples

In the above case, the isometric predual of \( C(K) \) is unique - it is \( N(K) \), so that \( C(K)_* = N(K) \).

Thus, when \( N(K) = \{0\} \), \( C(K) \) is not isometrically a dual space.

**Example** \( \beta \mathbb{N} \) is hyper-Stonean - because \( C(\beta \mathbb{N}) = \ell^\infty = (\ell^1)' \).

Can have \( C_0(K) \) isomorphically, but not isometrically, dual with

(i) \( K \) non-compact;

(ii) \( K \) compact and not Stonean;

(iii) \( K \) Stonean, but not hyper-Stonean.

For (iii), take \( K = G_\mathbb{I} \), the Gleason cover of \( \mathbb{I} \). Here \( C(K) \) is isomorphic to \( \ell^\infty \) (and so is isomorphic to a bidual space). However \( N(K) = \{0\} \), and so \( K \) is not hyper-Stonean.
A fixed measure on $K$

Fix a positive measure $\mu$ on $K$. Then $L^1(K, \mu)$ is the closed subspace of $M(K)$ consisting of the measures which are absolutely continuous with respect to $\mu$. The dual space is

$$L^1(K, \mu)' = L^\infty(K, \mu),$$

and this is a commutative $C^*$-algebra, with character space $\Phi_\mu$, say. The Gel’fand transform is

$$\mathcal{G}_\mu : L^\infty(K, \mu) \to C(\Phi_\mu).$$

The space $\Phi_\mu$ is hyper-Stonean.

For example, if $\mu$ is counting measure on $\mathbb{N}$, then $\Phi_\mu$ is $\beta\mathbb{N}$. 
A Boolean algebra

Let $K$ be a compact space, and fix $\mu \in M(K)^+$. Then $\mathcal{B}_\mu$ is $\mathcal{B}_K$ modulo the $\mu$-null sets. The Stone space $St(\mathcal{B}_\mu)$ of $\mathcal{B}_\mu$ is exactly $\Phi_\mu$, described above. So $\varphi \in \Phi_\mu$ is an ultrafilter, and we can consider ‘limits along the ultrafilter’, say $\lim_{B \to \varphi}$.

This approach gives us some useful formulae. Indeed,

$$\lim_{B \to \varphi} \frac{1}{\mu(B)} \int_B \lambda \, d\mu = G_\mu(\lambda)(\varphi)$$

for each $\lambda \in L^\infty(K, \mu)$. 
A characterization of $\Phi_\mu$

**Theorem** Let $K$ and $\mu$ be as above, and suppose that $|\mathcal{B}_\mu| = \kappa$, an infinite cardinal. Then $\Phi_\mu$ is a hyper-Stonean space satisfying CCC. Further,

$$w(\Phi_\mu) = \kappa \quad \text{and} \quad |\Phi_\mu| \leq 2^\kappa.$$ 

**Theorem** A hyper-Stonean space $X$ has the form $\Phi_\mu$ for some $\mu$ if and only if $X$ satisfies CCC.

**Classical theorem** Suppose that $\mu$ is a continuous measure and that $\mathcal{B}_\mu$ is separable. Then $(\mathcal{B}_\mu, \mu)$ is the same as the special case in which $K = \mathbb{I}$ and $\mu$ is Lebesgue measure.

In this special case, $\Phi_\mu$ is $\mathbb{H}$, called the hyper–Stonean space of the unit interval.
The bidual of $C(K)$

Let $(K, \tau)$ be a compact space. Then $C(K)$ is Arens regular, and $C(K)''$ is a commutative $C^*$-algebra. So, by Gel'fand,

$$C(K)'' = C(\tilde{K})$$

for a compact, hyper-Stonean space $(\tilde{K}, \sigma)$, called the **hyper-Stonean envelope** of $K$.

There is a continuous embedding $\iota : K \to \tilde{K}$ and a continuous projection $\pi : \tilde{K} \to K$ such that $\pi \circ \iota$ is the identity on $K$.

The map $\iota$ is not usually a homeomorphism. Indeed, $K$ consists exactly of the isolated points of $(\tilde{K}, \sigma)$, and so $K$ is open in $(\tilde{K}, \sigma)$. The closure of $K$ in $(\tilde{K}, \sigma)$ is identified with $\beta K_d$.

For $x \in K$, set $K\{x\} = \pi^{-1}(\{x\})$, the **fibre** of $x$. Then $C(K)$ is identified with the algebra of functions in $C(\tilde{K})$ that are constant on fibres.
Singular measures

A family $\mathcal{F}$ of positive measures on a compact space $K$ is singular if any two distinct measures in $\mathcal{F}$ are mutually singular. Let $U_\mathcal{F}$ be the space that is the disjoint union of the sets $\Phi_\mu$ with the topology in which each $\Phi_\mu$ is compact and open.

There is a maximal such family; we may suppose that it contains all point masses.

Fact Let $K$ be an uncountable, compact, metrizable space (e.g., $\mathbb{I} = [0, 1]$). Then there is such a family $\mathcal{F}$ consisting of just $\mathfrak{c}$ point masses and $\mathfrak{c}$ continuous measures (and each such family has these cardinalities). \qed
'Constructions' of $\widehat{K}$

Let $K$ be a locally compact space. We have obtained $\widehat{K}$ abstractly.

Take $P(K)$ to be the probability measures on $K$.

**Theorem** Take $\mathcal{F}$ to be a maximal singular family in $P(K)$. Then the map

$$\Lambda \mapsto (\Lambda \mid L^1(K, \mu) : \mu \in \mathcal{F})$$

from $C_0(K)''$ onto $\mathfrak{A} = \bigoplus_\infty \{C(\Phi_\mu) : \mu \in \mathcal{F}\}$ is a $C^*$-isomorphism. It follows from this that $C_0(K)$ is Arens regular and that we can identify $\widehat{K}$ with the hyper-Stonean space $\Phi_{\mathfrak{A}} = \beta U_{\mathcal{F}}$.

Set $U_K = \bigcup \{\Phi_\mu : \mu \in P(K)\}$. Then $\widehat{K} = \beta U_K$. $\square$
A second construction

Take $K$ to be locally compact. For $\mu, \nu \in M(K)^+$, set $\mu \sim \nu$ if $\mu \ll \nu$ and $\nu \ll \mu$. The equivalence class of $\mu$ is $[\mu]$. Set $[\mu] \leq [\nu]$ if $\mu \ll \nu$. Then

$$(M(K)^+ / \sim, \leq)$$

is a distributive lattice with a minimum element; it is a Dedekind complete Boolean ring. Its Stone space, called $S_K$, is an extremely disconnected, locally compact space. For each $\mu \in M(K)^+$, the Stone space $St(\mathcal{B}_\mu)$ is compact and open in $S_K$.

**Theorem** Take $\mathcal{F}$ to be a maximal singular family in $P(K)$. Then $U_\mathcal{F}$ is a dense open subspace of $S_K$, $C^b(U_\mathcal{F}) \cong C(\widetilde{K})$, and $\widetilde{K}$ is homeomorphic to $\beta S_K$. \qed
A third construction

Let $L$ be a convex subset of a real-linear space. The family $\mathcal{F}(L)$ of all faces of $L$ is a lattice (not generally distributive).

The set $L$ is a simplex when the ambient space is a Riesz space, $L$ is a subset of the positive cone, and every element of this cone is a positive multiple of an element of $L$.

Take $K$ compact. The space $P(K)$ is a (Choquet) simplex in the ambient space $M_\mathbb{R}(K)$, and the family $\text{Comp}_{P(K)}$ of complemented faces of $P(K)$ is a complete Boolean algebra.

**Theorem** For $K$ compact, the space $\widetilde{K}$ is homeomorphic to the Stone space $St(\text{Comp}_{P(K)})$ (and more information). \( \square \)
A fourth construction

Let $E$ be a Banach space with closed subspaces $F$ and $G$ such that $E = F \oplus_1 G$. Then we have an **L-decomposition**; the corresponding projections are **L-projections**.

The collection of L-projections on $E$ is denoted by $\text{Proj}_E$. It is a Boolean algebra for the operations

$$P \land Q = PQ, \quad P \lor Q = P + Q - PQ, \quad P' = I_E - P.$$  

The closed linear span of $\text{Proj}_E$ is a subalgebra of $\mathcal{B}(E)$ that is a commutative $C^*$-algebra, called the **Cunningham algebra** of $E$.

**Theorem** Let $K$ be compact. Then $\tilde{K}$ is homeomorphic to the Stone space $St(\text{Proj}_{M(K)})$ (and more information). \qed
Bounded Borel functions

Let $B^b(K)$ denote the $C^*$-algebra of bounded Borel functions on $K$.

For $\lambda \in B^b(K)$, define $\kappa_E(\lambda)$ on $C(K)' = M(K)$ by

$$\langle \kappa_E(\lambda), \mu \rangle = \int_K \lambda \, d\mu \quad (\mu \in M(K)).$$

This extends the canonical embedding of $C(K)$.

Thus we identify $B^b(K)$ as a closed subalgebra of $C(\tilde{K})$. By Stone–Weierstrass, it does not separate the points of $\tilde{K}$. Set

$\varphi \sim \psi$ if $\kappa_E(\lambda)(\varphi) = \kappa_E(\lambda)(\psi) \quad (\lambda \in B^b(K)).$

This is an equivalence relation; the equivalence class that contains $\varphi$ is denoted by $[\varphi]$.

The character space of $B^b(K)$ is $\tilde{K}/\sim$, a quotient of $\tilde{K}$. This space is totally disconnected, but usually not extremely disconnected.

The space $B^b(K)$ is not injective, and hence it is not isomorphic to a dual space.
Submodules of $M(K)$

The $\| \cdot \|$-closed, $C(K)$-submodules of $M(K)$ correspond to the clopen subspaces of $\tilde{K}$.

Thus $M(K) = \ell^1(K) \oplus M_c(K)$ gives a partition
\[ \{ \beta K_d, \tilde{K}_c \} \]
of $\tilde{K}$ into clopen subspaces.

Fix a $\mu \in M(K)^+$. Then
\[
\begin{array}{c}
B^b(K) \xrightarrow{\kappa_E} C(\tilde{K}) \\
\downarrow q_{\mu} \quad \downarrow \rho_{\mu} \\
L^{\infty}(K, \mu) \xrightarrow{G_{\mu}} C(\Phi_{\mu})
\end{array}
\]
is commutative, and $\kappa_E(B^b(K)) \mid \Phi_{\mu} = C(\Phi_{\mu})$.

Further,
\[
M(K) = \ell^1(K) \oplus L^1(K, \mu) \oplus M_s(K, \mu)
\]
gives a partition
\[ \{ \beta K_d, \Phi_{\mu}, \Phi_{\mu,s} \} \]
of $\tilde{K}$ into clopen subspaces.
A characterization

Let \((K, \tau)\) be an uncountable, compact, metrizable space. Then the hyper-Stonean envelope \(X = (\tilde{K}, \sigma)\) has the following properties:

(i) \(X\) is a hyper-Stonean space;

(ii) the set \(S\) of isolated points of \(X\) has cardinality \(\mathfrak{c}\), the closure \(Y\) of \(S\) in \(X\) is a clopen subspace of \(X\), and \(Y\) is homeomorphic to \(\beta S_d\);

(iii) \(X \setminus Y\) contains a family of \(\mathfrak{c}\) pairwise disjoint, clopen subspaces, each homeomorphic to \(\mathbb{H}\);

(iv) the union \(U_\mathcal{F}\) of the above sets is dense in \(X \setminus Y\) and is such that \(\beta U_\mathcal{F} = X \setminus Y\).

Further, any two spaces \(X_1\) and \(X_2\) satisfying the above properties are mutually homeomorphic.

So \(X = \tilde{I}\) is a very special compact set related to \(\beta \mathbb{N}\), but bigger.
A key preliminary result

Let $S$ be a non-empty set, and let $\kappa$ be an infinite cardinal. Then a $\kappa$-uniform ultrafilter on $S$ is an ultrafilter $\mathcal{U}$ on $S$ such that each set in $\mathcal{U}$ has cardinality at least $\kappa$.

Let $\mathcal{A}$ be a non-empty family of subsets of $S$. Then $\mathcal{A}$ has the $\kappa$-uniform finite intersection property if each non-empty, finite subfamily of $\mathcal{A}$ has an intersection of cardinality at least $\kappa$.

Let $S$ be an infinite set of cardinality $\kappa$, and let $\mathcal{A}$ be a non-empty family of at most $\kappa$ subsets of $S$ such that $\mathcal{A}$ has the $\kappa$-uniform finite intersection property. Then there are at least $2^{2^\kappa}$ $\kappa$-uniform ultrafilters on $S$ that contain $\mathcal{A}$. □
Some cardinalities

Here $K$ be an infinite, compact, metrizable space and $X = \sim K$.

**Fact** Then $|B^b(K)| = c$ and $|\Phi_{B^b(K)}| = 2^c$.

**Theorem** (i) $|C(X)| = 2^c$ and $|X| = 2^{2^c}$;
(ii) $|U_K| = 2^c$ and $w(U_K) = c$;
(iii) $|\sim K \setminus U_K| = 2^{2^c}$.

For $\mu \in M(K)^+$, set
$$[\Phi_{\mu}] := \bigcup\{[\varphi] : \varphi \in \Phi_{\mu}\}.$$ Easy that $[\Phi_{\mu}]$ is a closed subset of $\sim K$. It seemed possible that it would be the case that $\bigcup\{[\Phi_{\mu}] : \mu \in M(K)^+\}$ would be equal to the whole of $\sim K$; its complement consists of dark matter. However:

**Fact** $|\beta K_d \setminus [U_K]| = |K_c \setminus [U_K]| = 2^{2^c}$.  

Detour on \( C(X) \) as a bidual space

What if \( C(X) \) is isometrically isomorphic to the bidual of a Banach space?

**Theorem** Take \( X \) infinite and compact, and suppose that \( C(X) \cong E'' \), where \( E \) is a **separable** Banach space. Then there are exactly two possibilities:

(a) the set of isolated points of \( X \) is countably infinite, and then \( C(X) \cong C(\beta\mathbb{N}) = \ell^\infty = c_0'' \), and so \( X \) is homeomorphic to \( \beta\mathbb{N} \);

(b) the set of isolated points of \( X \) has cardinality \( \mathfrak{c} \), and then \( C(X) \) is isometrically isomorphic to \( C(\mathbb{I})'' \), and so \( X \) is homeomorphic to \( \tilde{\mathbb{I}} \). \( \square \)

**Open problem**: What happens if \( C(K) \cong E'' \) with \( E \) not necessarily separable? Does there exists a locally compact \( K \) with \( C(X) = C_0(K)'' \)? At least there is a compact space \( K \) and a clopen subspace \( V \) of \( \tilde{K} \) with \( X \) homeomorphic to \( V \).
A theorem

Take $K$ to be the semi-group $(\mathbb{N},+)$. Then $\beta \mathbb{N}$ is also a semigroup for an operation $\Box$ (or $+$) such that $\delta_u \Box \delta_v$ (a product in $\ell^1(\mathbb{N})'' = M(\beta \mathbb{N})$) is $\delta_u \Box v$. Analogue for $K$ the compact group $\mathbb{T}$. 

Some quite heavy calculations involving singular measures sitting on Cantor sets and limits along ultrafilters show that $\delta_u \Box \delta_v$ can have a variety of properties for $u,v \in \tilde{\mathbb{T}}$. For example:

**Theorem** There are a positive, singular measure $\mu$ in $M(\mathbb{T})$, elements $v \in \mathbb{T}^*_d$, and a closed subset $L$ of $\tilde{\mathbb{T}}$ such that $(\delta_u \Box \delta_v)(L) = 1/2$ for each $u \in \Phi_\mu$. In particular, $\delta_u \Box \delta_v$ is not a point mass. One can arrange that $\delta_u \Box \delta_v$ is neither a discrete nor a continuous measure on $\tilde{\mathbb{T}}$. \qed