On weak and pointwise topologies in function spaces

Mikołaj Krupski and Witold Marciszewski
University of Warsaw

Transfinite methods in Banach spaces and algebras of operators
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For a compact space $K$, $C_w(K)$ ($C_p(K)$) is the space of continuous real-valued functions on $K$ endowed with the weak (pointwise) topology.

If $K$ is infinite then the pointwise topology is strictly weaker than the weak one.

Example Let $\sigma = \{ x \in \ell^2 : x_n = 0 \text{ for all but finitely many } n \}$. $\sigma$ equipped with the norm topology is homeomorphic to $\sigma$ equipped with the pointwise topology (inherited from $\mathbb{R}^\omega$).

Problem 1 Can $C_p(K)$ and $C_w(K)$ be homeomorphic for an infinite compact space $K$?

Problem 2 Can $C_p(K)$ and $C_w(L)$ be homeomorphic for infinite compact spaces $K$ and $L$?
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Can $C_p(K)$ and $C_w(L)$ be homeomorphic for infinite compact spaces $K$ and $L$?
For a compact space $K$, by $M(K)$ denote the space of all Radon measures on $K$, which can be identified with the dual space $C(K)^*$. $B_{M(K)}$ stands for the unit ball of $M(K)$, equipped with the weak* topology inherited from $C(K)^*$. 
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**Fact**

For a compact space $K$, $C_w(K)$ is linearly homeomorphic to a closed linear subspace of $C_p(B_{M(K)})$. 
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$$i(f)(\mu) = \mu(f) \quad \text{for } f \in C(K), \mu \in B_{M(K)}$$
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**Fact**

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For a topological space $X$ and $x \in X$ by $\chi(x, X)$ we denote the minimal cardinality of a base of neighborhoods of $x$. We put $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ - the *character* of $X$. 
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**Fact**

*If \( K \) is infinite compact such that \( |K| < |M(K)| \), then \( \chi(C_p(K)) = |K| \) and \( \chi(C_w(K)) = |M(K)| \).*
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\(|K| < |M(K)|\) if the cofinality of \(|K|\) is countable.
For a topological space $X$ and $x \in X$ by $\psi(x, X)$ we denote the minimal cardinality of a family $\mathcal{U}$ of open sets such that $\bigcap \mathcal{U} = \{x\}$. We put $\psi(X) = \sup \{\psi(x, X) : x \in X\}$ - the pseudocharacter of $X$. 

Fact

Let $K$ be a non-separable compact space such that there is a family $\{\mu_n \in M(K) : n \in \omega\}$ of functionals separating elements of $C(K)$ (equivalently, there is a linear continuous injection $T : C(K) \to \ell^\infty$).

Then $\psi(C_p(K)) = d(K) > \omega$ and $\psi(C_w(K)) = \omega$. 

A concrete example of a compact space $K$ with above properties is the Stone space of the measure algebra associated with the Lebesgue measure $\lambda$ on $[0,1]$. $C(K)$ is a function space representation of the algebra $L^\infty[0,1]$. 
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Fact
Let $K$ be an infinite scattered compact space. Then $C_p(K)$ is Fréchet-Urysohn and $C_w(K)$ is not.

Proposition
For any infinite compact spaces $K$ and $L$ the spaces $C_w(K)$ and $C_p(L)$ are not uniformly homeomorphic.
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Main Theorem

Let $K$ be a compact strongly countable-dimensional space and $L$ be a compact space such that $C_w(L)$ is homeomorphic to $C_w(M) \times E$ for some uncountable metrizable compact space $M$ and some topological space $E$. Then $C_p(K)$ and $C_w(L)$ are not homeomorphic.
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**Corollary**

If $K$ is a compact strongly countable-dimensional space and $L$ is a compact space containing a closed uncountable metrizable subspace, then $C_p(K)$ and $C_w(L)$ are not homeomorphic.
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For a set $\Gamma$, $\Sigma(\Gamma)$ is the $\Sigma$-product of real lines indexed by $\Gamma$, i.e., the subspace of $\mathbb{R}^\Gamma$ consisting of functions with countable supports.
**Corollary**

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For a set $\Gamma$, $\Sigma(\Gamma)$ is the $\Sigma$-product of real lines indexed by $\Gamma$, i.e., the subspace of $\mathbb{R}^\Gamma$ consisting of functions with countable supports.

A compact space $K$ is a *Valdivia* compact space if, for some set $\Gamma$, there exists an embedding $i : K \rightarrow \mathbb{R}^\Gamma$ such that the intersection $i(K) \cap \Sigma(\Gamma)$ is dense in $i(K)$.
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If $K$ is a compact finite-dimensional space and $L$ is a compact space containing a closed uncountable metrizable subspace, then $C_p(K)$ and $C_w(L)$ are not homeomorphic.

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**Proposition**

Every Valdivia compact space is either scattered or contains a closed uncountable metrizable subspace.
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If $K$ is an infinite finite-dimensional Valdivia compact space, then $C_p(K)$ and $C_w(K)$ are not homeomorphic.
The **double arrow space** \( \mathbb{K} \) is the set \( \mathbb{K} = ((0, 1] \times \{0\}) \cup ([0, 1) \times \{1\}) \) equipped with the order topology given by the lexicographical order (i.e., \((s, i) \succ (t, j)\) if either \(s < t\), or \(s = t\) and \(i < j\)).
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**Proposition (M.)**

*For each nonempty compact metrizable space $M$, the Banach spaces $C(\mathcal{K})$ and $C(\mathcal{K}) \times C(M)$ are isomorphic.*
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**Theorem (M. Krupski)**

If $K$ is infinite metrizable compact C-space, then $C_p(K)$ and $C_w(K)$ are not homeomorphic.
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Corollary
If $K$ is infinite compact metrizable C-space and $L$ is an arbitrary compact space, then $C_p(K)$ and $C_w(L)$ are not homeomorphic.
Question

Is it true that $C_p([0, 1]^{\omega})$ and $C_w([0, 1]^{\omega})$ are not homeomorphic?
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