Reflecting properties of compacta in small continuous images

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Transfinite methods in Banach spaces and algebras of operators
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joint work with

Menachem Magidor (Hebrew University of Jerusalem)
Kamil Duszenko
Died on July 23, 2014 (aged 28).
He submitted his PhD thesis *Actions of Coxeter groups on negatively curved spaces* in 2013. There was no time for the final defence.

is awarded in haematology and in mathematics (geometric group theory).
The prize was partially founded by all those who generously supported his struggle with leukaemia in 2013 and 2014.
Thank you.
Reflection problems in topology

Type 1 reflection problem

Does a topological space \( X \) have a property (P) provided all its small subspaces have property (P)?

Type 2 reflection problem, Tkachuk and Tkachenko (2012, 2015)

Does a topological space \( X \) have property (P) provided every continuous image of \( X \) of small weight has property (P)?

If \( A \) is a Boolean algebra then \( \text{Ult}(A) \) is its Stone space. If \( K \) is compact zerodimensional then \( A = c\text{lop}(K) \) is the algebra of its clopen subsets.

Type 2 problem for Boolean algebras

Suppose that every subalgebra \( B \) of a Boolean algebra \( A \), if \(|B| \leq \omega_1 \) then \( B \) has property (P'). Does \( A \) have property (P')?
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Suppose that every subalgebra $\mathcal{B}$ of a Boolean algebra $\mathcal{A}$, if $|\mathcal{B}| \leq \omega_1$ then $\mathcal{B}$ has property $(P')$. Does $\mathcal{A}$ have property $(P')$?
Eberlein compacta, Corson compacta and $\omega_1$

Let $K$ be a compact space. Suppose that every continuous image of $K$ of weight $\leq \omega_1$ is Eberlein compact. Is $K$ Eberlein compact itself? Suppose that every continuous image of $K$ of weight $\leq \omega_1$ is Corson compact. Is $K$ Corson compact itself?

One (relatively consistent) answer: no. Subject to some set-theoretic assumption, there exists a compact space $K$ which is not Corson compact but all its continuous images of weight $\leq \omega_1$ are Eberlein compacta.
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$K$ always denotes a compact Hausdorff space.
### Basics

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4. If $n \in \omega$ then every compact subset of

$$\sigma_n(2^\kappa) = \{ x \in 2^\kappa : |\{ \alpha : x_\alpha \neq 0 \}| \leq n \},$$

is uniform Eberlein compact, embeds into $l_2(\kappa)$. 
The axiom

Stationary sets $F \subseteq \gamma$ is closed if it is closed in the interval topology of $\gamma = \{ \alpha : \alpha < \gamma \}$. $F$ is unbounded in $\gamma$ if for every $\beta < \gamma$ there is $\alpha \in F$ such that $\beta < \alpha$. $S \subseteq \gamma$ is stationary if $S \cap F \neq \emptyset$ for every closed and unbounded $F \subseteq \gamma$.

Axiom (*)

There is a stationary set $S \subseteq \omega_2$ such that

1. $\text{cf}(\alpha) = \omega$ for every $\alpha \in S$;
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Remarks on (*)

1. (*) follows from Jensen's principle $\Box_{\omega_1}$ and hence it holds in the constructible universe.
2. (*) is more true than untrue, to prove the consistency of $\neg(*)$ one needs large cardinals (see Magidor 1982).
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The result

Under (*) there is a scattered compact space $K$ with $K(3) = \emptyset$ such that

1. $K$ is not Corson compact;
2. whenever $L$ is a continuous image of $K$ with $w(L) \leq \omega_1$ then $L$ is uniform Eberlein compact;
3. for every $Y \subseteq K$, if $|Y| \leq \omega_1$ then $Y$ is uniform Eberlein compact.

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For $\alpha \in S$ pick $\left( p_n(\alpha) \right)_{n<\omega}$ such that $p_n(\alpha) \nearrow \alpha$. 
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The construction: Fix a set $S \subseteq \omega_2$ granted by (*).

For $\alpha \in S$ pick $(p_n(\alpha))_{n<\omega}$ such that $p_n(\alpha) \nearrow \alpha$. Let $A_\alpha = \{p_n(\alpha) : n < \omega\}$,
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1. Prove that for any family $G \subseteq \mathcal{A}$ generating $\mathcal{A}$ there is $\xi < \omega_2$ such that $\xi \in G$ for uncountably many $G \in G$.

2. Conclude that $K$ is not Corson.

3. Prove that for every $\eta < \omega$, the sets $A_\alpha$, $\alpha < \eta$ can be made disjoint by finite modifications.

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A Banach space $X$ is WCG if $X = \text{lin}(C)$ for some weakly compact $C \subseteq X$.

$K$ is Eberlein iff $C(K)$ is WCG.

If $X$ is WCG then a subspace $Y$ of $X$ need not be WCG.

Marián Fabian (1987): If $X$ is Asplund and WCG then every subspace of $X$ is WCG.

Theorem: Under (*) there is a Banach space $X$ of density $\omega_2$ which is not WCG but all its subspaces of density $\leq \omega_1$ are WCG.
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Countable functional tightness

**Definition**

For topological space $X$, $t_0(X) = \omega$ if for every $f: X \to \mathbb{R}$, if $f|_A \in C(A)$ for any countable $A \subseteq X$ then $f \in C(X)$.

**Lemma**

If $f \in C(A)$ for every countable $A \subseteq X$ then $f|_Y \in C(Y)$ for every separable $Y \subseteq X$.

In particular, $t_0(X) = \omega$ for separable $X$.

**Theorem**

1. Uspenskii (1983): $t_0(2^\kappa) = \omega$ iff there are no measurable cardinals $\leq \kappa$.

2. Talagrand (1984): the Banach space $C(2^\kappa)$ is realcompact in its weak topology iff there are no measurable cardinals $\leq \kappa$.

3. Mazur: If there are no weakly inaccessible cardinals $\leq \kappa$ then every sequentially continuous $f: 2^\kappa \to \mathbb{R}$ is continuous.

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Countable functional tightness

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Problem

Let $K$ be a compact space such that $t_0(L) = \omega$ for every its continuous image $L$ with $w(L) \leq \omega_1$. Does this imply that $t_0(K) = \omega$?
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