Character of points in the corona of a metric space

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This topic lies in the intersection of three disciplines:

- Asymptotic Topology,
- General Topology,
- Set Theory.

Objects: Metric spaces,
Morphisms: Coarse maps.

A function $f : X \to Y$ between metric spaces is called *coarse* if
$$\forall\delta \in \mathbb{R}_+\ \exists\varepsilon \in \mathbb{R}_+\ \forall x, x' \in X\ \ d_X(x, x') < \delta\ \Rightarrow\ d_Y(f(x), f(x')) < \varepsilon.$$

Coarse maps are antipods of uniformly continuous maps.

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Coarse isomorphisms and coarse equivalences

Def: A coarse map $f : X \to Y$ between metric spaces is called a

- a *coarse isomorphism* if f is bijective and f^{-1} is coarse;
- a *coarse equivalence* if there exists a coarse map $g : Y \to X$ such that $\max\{d_X(g \circ f, \mathrm{id}_X), d_Y(f \circ g, \mathrm{id}_Y)\} < \infty$.

Example:

The identity embedding $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence but not a coarse isomorphism.

Asymptotic Topology studies properties of metric spaces preserved by coarse equivalences.

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Asymptotic neighborhoods

A function $f : X \to Y$ between metric spaces is *bounded-to-bounded* if a subset $B \subset Y$ is bounded iff $f^{-1}(B)$ is bounded in X.

Let $\omega^{\uparrow X}$ be the set of all bounded-to-bounded functions $\varepsilon : X \to \omega$.

For a function $\varepsilon \in \omega^{\uparrow X}$ and a subset $A \subset X$ let $B(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon(a))$.

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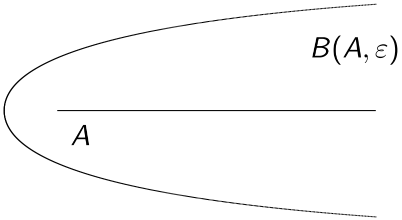

The corona of a metric space.

For a metric space X let X_d be X endowed with the discrete topology and βX_d be the Stone-Čech compactification of X_d.

Let $X^\#$ be the closed subset of βX_d consisting of all unbounded ultrafilters.

An ultrafilter $\mathcal{F}$ on X_d is *unbounded* if it contains no bounded subset of X.

Def: The *corona* of a metric space X is the quotient space $\check{X} = X^\#/_\sim$ of $X^\#$ by the equivalence relation identifying any ultrafilters $p, q \in X^\#$ such that $B(P, \varepsilon) \cap B(Q, \varepsilon) \neq \emptyset$ for any $P \in p$, $Q \in q$ and $\varepsilon \in \omega^{\uparrow X}$.

Elements of $\check{X}$ are equivalence classes $\check{p}$ of ultrafilters $p \in X^\#$.

Topology of $\check{X}$: For any ultrafilter $p \in X^\#$ the sets

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General Problems

The corona is a kind of a topological telescope which transforms a macro-object (metric space) into a compact micro-object (its corona).

Problem

Which asymptotic properties of a metric space X are reflected in topological properties of its corona $\check{X}$?

Such properties should be preserved by coarse equivalences because of

Fact

Each coarse equivalence $f : X \to Y$ between metric spaces induces a homeomorphism $\check{f} : \check{X} \to \check{Y}$ of their coronas.

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An alternative definition of corona

A metric space X is *proper* if each closed ball in X is compact.

Def: A function $f : X \to \mathbb{R}$ is called *slowly oscillating* if $\forall \varepsilon > 0 \ \forall \delta < \infty$ there is a bounded subset $B \subset X$ such that $\forall x, x' \in X \setminus B \quad d_X(x, x') < \delta \ \Rightarrow \ |f(x) - f(x')| < \varepsilon$.

Example.

The function $f : [1, \infty) \to \mathbb{R}$, $f : x \mapsto \frac{1}{x}$, is slowly oscillating.

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Higson corona of a proper metric space

For a proper metric space X let $SO(X)$ be the algebra of real-valued bounded continuous slowly oscillating functions.

This algebra determined a compactification $\bar{h}(X)$ of X called the Higson compactification of X.

The compactification $\bar{h}(X)$ is the closure of the image $h(X)$ of X under the embedding $h : X \to \mathbb{R}^{SO(X)}$, $h : x \mapsto (f(x))_{f \in SO(X)}$.

The remainder $\nu X = \bar{h}(X) \setminus h(X)$ is called the Higson corona of X.

Theorem (Protasov)

For a proper metric space X its Higson corona νX is canonically homeomorphic to the corona $\check{X}$ of X.

The corona $\check{X}$ "sees" certain asymptotic properties of X, in particular, its asymptotic dimension $\operatorname{asdim}(X)$.

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Various dimensions of metric spaces

A metric space X has

- topological dimension $\dim(X) \le n$ if for each open cover ε of X there are an open cover δ of X and a cover $\mathcal{C} \prec \varepsilon$ of X such that the δ-star $B(x,\delta) = \bigcup\{D \in \delta : x \in D\}$ of any point $x \in X$, meets at most $n+1$ elements of the cover $\mathcal{C}$;
- uniform dimension $\mathrm{udim}(X) \le n$ if for each $\varepsilon > 0$ there are $\delta > 0$ and a cover $\mathcal{C} \prec \{B(x,\varepsilon)\}_{x\in X}$ of X such that each δ-ball $B(x,\delta)$, $x \in X$, meets at most $n+1$ elements of the cover $\mathcal{C}$;
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Dimension of corona

Theorem

Let X be a proper metric space. Then

1. $\dim(\check{X}) \le \operatorname{asdim}(X)$ (Dranishnikov-Keesling-Uspenskij, 1998);
2. $\dim(\check{X}) = \operatorname{asdim}(X)$ *if* $\operatorname{asdim}(X) < \infty$ (Dranishnikov, 2000);
3. $\dim(\check{X}) = 0$ *iff* $\operatorname{asdim}(X) = 0$ (Banakh-Chervak, 2012).

Open Problem (Dranishnikov)

Is $\dim(\check{X}) = \operatorname{asdim}(X)$ for each proper metric space X?

Fact

A metric space has asymptotic dimension zero if and only if it is coarsely isomorphic to an ultrametric space.

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The topological structure of the corona

Theorem (Protasov, 2011)

For each unbounded metric separable space X with $\operatorname{asdim}(X) = 0$

1. *$\check{X}$ is a zero-dimensional compact Hausdorff space of weight $\mathfrak{c}$;*
2. *each non-empty G_δ-subset in $\check{X}$ has non-empty interior;*
3. *any two disjoint open F_σ-subsets of $\check{X}$ have disjoint closures.*

This theorem and the CH-characterization of the Stone-Čech remainder $\omega^* = \beta(\omega) \setminus \omega$ imply:

Corollary (Protasov, 2011)

Under CH the corona $\check{X}$ of an unbounded metric separable space X of $\operatorname{asdim}(X) = 0$ is homeomorphic to ω^*.

Problem (Protasov)

Is this theorem true in ZFC? No!

The topological structure of the corona

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For each unbounded metric separable space X with $\mathrm{asdim}(X) = 0$

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Minimal character of a topological space

For a topological space X its minimal character

$$m\chi(X) = \min_{x \in X} \chi(x, X)$$

where $\chi(x, X)$, the *character* of X at a point x is the smallest cardinality of a neighborhood base at x.

The cardinal $\mathfrak{u} = m\chi(\omega^*)$ is one of well-known small uncountable cardinals.

Another well-known small uncountable cardinal is $\mathfrak{d}$, the cofinality of the partially ordered set $(\omega^\omega, \leq)$.

It is known that $\mathfrak{u} = \mathfrak{d} = \mathfrak{c}$ under MA, but the strict inequalities $\mathfrak{u} < \mathfrak{d}$ and $\mathfrak{d} < \mathfrak{u}$ are consistent with ZFC.

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We say that a metric space X *has isolated balls* if there is $\varepsilon < \infty$ such that for each $\delta < \infty$ there is a point $x \in X$ with $B(x, \delta) \subset B(x, \varepsilon)$.

Example

The space $\mathbb{A} = \{n^2\}_{n \in \omega} \subset \mathbb{Z}$ has asymptotically isolated balls.

Theorem (Banakh-Chervak-Zdomskyy, 2012)

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The Cantor macro-cube is the set

$$2^{<\mathbb{N}} = \left\{(x_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : \sum_{n=1}^{\infty} x_n < \infty\right\}$$

endowed with the metric $d((x_n),(y_n)) = \sum_{n=1}^{\infty} 2^n \cdot |x_n - y_n|$.

$2^{<\mathbb{N}}$ is an asymptotic counterpart of the Cantor cube $2^{\omega} = \{0,1\}^{\omega}$.

Fact

The Cantor macro-cube $2^{<\mathbb{N}}$ is coarsely isomorphic to the Cantor macro-set $\left\{\sum_{n=1}^{\infty} 3^n 2x_n : (x_n)_{n\in\mathbb{N}} \in 2^{<\mathbb{N}}\right\} \subset \mathbb{Z}$.

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Universality of the Cantor macro-cube

It is well-known that the Cantor cube 2^ω contains a topological copy of each zero-dimensional metrizable separable space.

A similar property has the Cantor macro-cube $2^{<\mathbb{N}}$.

Definition

A metric space X has *bounded geometry* if $\exists \varepsilon < \infty\ \forall \delta < \infty\ \exists N \in \mathbb{N}$ such that each δ-ball $B(x, \delta)$, $x \in X$, can be covered by $\leq N$ ε-balls.

Theorem (Dranishnikov-Zarichnyi, 2004)

A metric space X is coarsely equivalent to a subspace of $2^{<\mathbb{N}}$ iff $\mathrm{asdim}(X) \leq 0$ *and X has bounded geometry.*

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A coarse characterization of the Cantor macro-cube

Theorem (Brouwer, 1904)

A metric space X is (uniformly) homeomorphic to 2^ω if and only if X has topological dimension zero, is compact, and contains no isolated points.

Theorem (Banakh-Zarichnyi, 2011)

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Theorem (Banakh-Chervak-Zdomskyy, 2012)

Under $\mathfrak{u} < \mathfrak{d}$ for a metric space X of bounded geometry the following conditions are equivalent:

1. *X and $2^{<\mathbb{N}}$ are coarsely equivalent;*
2. *the coronas of X and $2^{<\mathbb{N}}$ are homeomorphic;*
3. $\dim(\check{X}) = 0$ *and* $m\chi(\check{X}) = \mathfrak{d}$.

So, under $\mathfrak{u} < \mathfrak{d}$ the corona recognizes metric spaces coarsely equivalent to the Cantor macro-cube.

Under $\omega_1 = \mathfrak{c}$ the corona is “blind” and sees no difference between asymptotically zero-dimensional separable metric spaces.

Under OCA+$\text{MA}_{\aleph_1}$ the corona is able to see in another (say, infra-red) end of the asymptotic spectrum and recognizes asymptotically discrete metric spaces.

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Asymptotically discrete spaces

A metric space X is asymptotically discrete if $\exists \varepsilon < \infty \ \forall \delta < \infty$ there is a bounded subset $B \subset X$ such that $B(x, \delta) \subset B(x, \varepsilon)$ for all $x \in X \setminus B$.

Fact

1. *Each unbounded metric space contains an unbounded asymptotically discrete subspace.*
2. *A separable metric space is asymptotically discrete iff it is coarsely equivalent to the space $\mathbb{A} = \{n^2\}_{n\in\omega} \subset \mathbb{Z}$.*

So up to a coarse equivalence, $\mathbb{A} = \{n^2\}_{n\in\omega}$, is a smallest unbounded metric space, opposite to the Cantor macro-cube $2^{<\mathbb{N}}$ which is the largest metric space of bounded geometry and asymptotic dimension zero.

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Fact

The corona $\check{\mathbb{A}}$ *of the space* $\mathbb{A} = \{n^2\}_{n\in\omega}$ *is canonically homeomorphic to* ω^*.

Theorem (Banakh-Chervak-Zdomskyy, 2012)

Under OCA+MA$_{\aleph_1}$ *a metric separable space* X *is asymptotically discrete iff its corona* $\check{X}$ *is homeomorphic to* $\check{\mathbb{A}} \approx \omega^*$. *Moreover, each homeomorphism* $\check{X} \to \check{\mathbb{A}}$ *is induced by a suitable coarse equivalence* $X \to \mathbb{A}$.

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Principal Conjecture

The proof of the preceding theorem is based on the following deep:

Theorem (Veličković, 1993)

Under OCA+MA$_{\aleph_1}$ *each homeomorphism of* ω^* *is induced by a bijection between cofinite subsets of* ω*.*

Conjecture

Under OCA+MA$_{\aleph_1}$ two separable metric spaces X, Y are coarsely equivalent iff their coronas are homeomorphic.

Moreover, each homeomorphism $\check{X} \to \check{Y}$ is induced by a suitable coarse equivalence $X \to Y$.

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