Noncommutative Valdivia compacta

Marek Cúth

Workshop on set theoretic methods
in compact spaces and Banach spaces
2013, Warsaw
1 Recent history and motivation

2 Some basic facts about skeletons
   - Definitions
   - Retractional skeletons and elementary submodels
   - Skeletons, Valdivia compacta and Plichko spaces

3 Main results
   - Renorming theorems
   - Results in $\mathcal{C}(K)$ spaces
   - The relationship between projectional and retractional skeletons
   - Few words about the proofs
**Motivation:** Separable Banach spaces have nice properties (renormings, Markushevich basis). Nonseparable Banach spaces need not have those properties. But, some of them do (Hilbert spaces).

Having certain nonseparable Banach space $X$, does it share the nice properties with separable spaces?
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Banach spaces with a projectional skeleton

W. Kubiś: Banach spaces with a projectional skeleton

\[(PG) \Rightarrow \text{p-skeleton} \Rightarrow (PRI)\]

Definition

A *projectional skeleton* in a Banach space $X$ is a family of projections $\{P_s\}_{s \in \Gamma}$, indexed by an up-directed partially ordered set $\Gamma$, such that

1. $X = \bigcup_{s \in \Gamma} P_s X$ and each $P_s X$ is separable.
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(iii) Given \(s_0 < s_1 < \cdots\) in \(\Gamma\), \(t = \sup_{n \in \omega} s_n\) exists and \(P_tX = \bigcup_{n \in \omega} P_{s_n}X\).
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We shall say that \(\{P_s\}_{s \in \Gamma}\) is an *r-projectional skeleton* if it is a projectional skeleton such that \(\|P_s\| \leq r\) for every \(s \in \Gamma\).
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- We say that \(\{P_s\}_{s \in \Gamma}\) is a *commutative projectional skeleton* if \(P_s \circ P_t = P_t \circ P_s\) for every \(s, t \in \Gamma\).
On compact spaces, W. Kubiš introduced the notion of a retractional skeleton, dual to a projectional skeleton.
Compact spaces with a retractional skeleton

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A *retractional skeleton* in a compact space \( K \) is a family of retractions \( \{r_s\}_{s \in \Gamma} \), indexed by an up-directed partially ordered set \( \Gamma \), such that

(i) For every \( x \in K \), \( x = \lim_{s \in \Gamma} r_s(x) \) and \( r_s[K] \) is metrizable for each \( s \in \Gamma \).
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We say that \( \{r_s\}_{s \in \Gamma} \) is a **commutative retractional skeleton** if \( r_s \circ r_t = r_t \circ r_s \) for every \( s, t \in \Gamma \).
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The connection with elementary submodels

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**means that**

There is $\varphi_1, \ldots, \varphi_n$ and a countable set $Y$ such that whenever $M \supset Y$ is a countable set with $\varphi_1, \ldots, \varphi_n$ absolute for $M$, then ...
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**Theorem (Kubis)**

Let $K$ be a compact space, and let $D \subset K$ be a dense subset. Then the following properties are equivalent:

(i) There exists a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $K$ such that $D \subset \bigcup_{s \in \Gamma} r_s[K]$. 

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Let $K$ be a compact space, and let $D \subset K$ be a dense subset. Then the following properties are equivalent:

(i) There exists a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $K$ such that $D \subset \bigcup_{s \in \Gamma} r_s[K]$.

(ii) For every suitable elementary submodel $M$, $\mathcal{C}(K) \cap M$ separates the points of $D \cap M$. 
Banach space $X$ has a commutative p-skeleton $\iff X$ is Plichko.
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Compact space $K$ has a commutative r-skeleton $\iff K$ is Valdivia
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Recall:
- Banach space $X$ is called Plichko (resp. 1-Plichko) if there are a linearly dense set $M \subset X$ and a norming (resp. 1-norming) set $D \subset X^*$ such that for every $x^* \in D$ the set $\{m \in M : x^*(m) \neq 0\}$ is countable.
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- Let $\Gamma$ be a set. We put $\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq \omega\}$. 
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- Compact space $K$ is a \textit{Corson compact}, if $K$ is homeomorphic to a subset of $\Sigma(\Gamma)$. 
Skelettons, Valdivia compacta and Plichko spaces

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- Let $Γ$ be a set. We put $Σ(Γ) = \{x ∈ R^Γ : |\{γ ∈ Γ : x(γ) ≠ 0\}| ≤ ω\}$.
- Compact space $K$ is a *Corson compact*, if $K$ is homeomorphic to a subset of $Σ(Γ)$.
- $A \subset K$ is a *Σ-subset* of $K$ if there is a homeomorphic embedding $h : K \rightarrow [0, 1]^κ$ such that $A = h^{-1}[Σ(κ)]$. 
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- Compact space $K$ is a *Corson compact*, if $K$ is homeomorphic to a subset of $\Sigma(\Gamma)$.

- $A \subset K$ is a $\Sigma$-*subset* of $K$ if there is a homeomorphic embedding $h : K \to [0, 1]^\kappa$ such that $A = h^{-1}[\Sigma(\kappa)]$.

- Compact space $K$ is a *Valdivia compact*, if there exists a dense $\Sigma$-subset of $K$. 
Characterization of WLD spaces

Theorem

The following conditions are equivalent for a Banach space $\langle X, \| \cdot \| \rangle$:

(i) $(B_{\langle X^*, \| \cdot \| \rangle}, w^*)$ is Corson ($= X$ is WLD).
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(i) $(B_{\langle X^*, \| \cdot \| \rangle}, w^*)$ is Corson (= $X$ is WLD).

(ii) $\langle X, \| \cdot \| \rangle$ is 1-Plichko for every equivalent norm $\| \cdot \|$.
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(iii) \( \langle X, \| \cdot \| \rangle \) has a 1-projectional skeleton for every equivalent norm \( \| \cdot \| \).

(iv) \( (B_{\langle X^*, \| \cdot \| \rangle}, w^*) \) is Valdivia for every equivalent norm

(v) \( (B_{\langle X, \| \cdot \| \rangle^*, w^*) \) has a retractional skeleton for every equivalent norm \( \| \cdot \| \).
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The following conditions are equivalent for a Banach space \((X, \| \cdot \|)\):

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Characterization of Asplund spaces

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The following conditions are equivalent for a Banach space \((X, \| \cdot \|)\):

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(iii) \((B(X, \| \cdot \|)^{**}, w^*)\) has a retractional skeleton for every equivalent norm \(\| \cdot \|\).
Subspaces of $\mathcal{C}(K)$ spaces

**Theorem**

The following conditions are equivalent for a compact $K$, which is a continuous image of a space with a commutative retractional skeleton.

(i) $K$ is a Corson compact with the property $(\mathcal{M})$.
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(i) $K$ is a Corson compact with the property $(M)$.

Definition

A compact space $K$ is said to have the property $(M)$ if every Radon probability measure on $K$ has separable support.

Remark: It is not provable in ZFC whether every Corson compact $K$ has the property $(M)$. 
Subspaces of $\mathcal{C}(K)$ spaces

**Theorem**

*The following conditions are equivalent for a compact $K$, which is a continuous image of a space with a commutative retractional skeleton.*

(i) $K$ is a Corson compact with the property $(M)$.

(ii) Every subspace of $\mathcal{C}(K)$ is 1-$\text{Plichko}$.
The following conditions are equivalent for a compact $K$, which is a continuous image of a space with a commutative retractional skeleton.

(i) $K$ is a Corson compact with the property $(M)$.

(ii) Every subspace of $\mathcal{C}(K)$ is 1-Plichko.

(iv) $(B_{Y^*}, w^*)$ is Valdivia for every subspace $Y \subset \mathcal{C}(K)$. 
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The following conditions are equivalent for a compact \( K \), which is a continuous image of a space with a retractional skeleton.

(i) \( K \) is a Corson compact with the property \((M)\).

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The following conditions are equivalent for a compact $K$, which is a continuous image of a space with a retractional skeleton.

(i) $K$ is a Corson compact with the property $(M)$.

(ii) Every subspace of $C(K)$ is 1-Plichko.

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The following conditions are equivalent for a compact $K$, which is a continuous image of a space with a retractional skeleton.

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Theorem

The following conditions are equivalent for a compact space $K$, which is a continuous image of a space with a commutative retractional skeleton.

(i) $K$ is a Corson compact.

(ii) $C(L)$ is 1-Plichko for every continuous image $L$ of $K$. 
Subspaces of $\mathcal{C}(K)$ spaces

Theorem

The following conditions are equivalent for a compact space $K$, which is a continuous image of a space with a commutative retractional skeleton.

(i) $K$ is a Corson compact.

(ii) $\mathcal{C}(L)$ is 1-Plichko for every continuous image $L$ of $K$.

(iv) $(B_{\mathcal{C}(L)^*}, w^*)$ is Valdivia for every continuous image $L$ of $K$. 
The following conditions are equivalent for a compact space $K$, which is a continuous image of a space with a retractional skeleton.

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(iii) $C(L)$ has a 1-projectional skeleton for every continuous image $L$ of $K$.
(iv) $(B_{C(L)^*}, w^*)$ is Valdivia for every continuous image $L$ of $K$.
(v) $(B_{C(L)^*}, w^*)$ has a retractional skeleton for every continuous image $L$ of $K$. 

Theorem
Proposition

Let $K$ be a compact space. Consider the following conditions

(i) $K$ has a retractional skeleton

(ii) $C(K)$ has a 1-projectional skeleton

(iii) $(B_{C(K)^*}, w^*)$ has a retractional skeleton

Then the following implications hold:

$$(i) \Rightarrow (ii) \Rightarrow (iii).$$

Moreover, if $K$ has a dense set of $G_\delta$ points, then all the conditions are equivalent.
Corollary

The following conditions are equivalent for a compact space $K$.

(i) $K$ is a Corson compact.
Corollary

The following conditions are equivalent for a compact space $K$.

(i) $K$ is a Corson compact.
(ii) Every continuous image of $K$ is Valdivia.
Corollary

The following conditions are equivalent for a compact space $K$.

(i) $K$ is a Corson compact.

(ii) Every continuous image of $K$ is Valdivia.

(iii) Every continuous image of $K$ has a retractional skeleton.
Skelettons in $\mathcal{C}(K)$ spaces

**Theorem**

Let $K$ be a compact space. Then the following conditions are equivalent:

(i) $\mathcal{C}(K)$ has a 1-projectional skeleton.

(ii) There is a retractional skeleton $\{r_s\} s \in \Gamma$ in $(\mathcal{B} \mathcal{C}(K)^*, \text{w}^*)$ such that $S_s \in \Gamma_{r_s}[K]$ is a convex set.

(iii) There is a retractional skeleton $\{r_s\} s \in \Gamma$ in $\mathcal{P}(K)$ such that $S_s \in \Gamma_{r_s}[K]$ is a convex set.
Theorem

Let $K$ be a compact space. Then the following conditions are equivalent:

(i) $\mathcal{C}(K)$ has a 1-projectional skeleton.

(ii) There is a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $(B_{\mathcal{C}(K)^*}, w^*)$ such that $\bigcup_{s \in \Gamma} r_s[K]$ is a convex set.
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Let $K$ be a compact space. Then the following conditions are equivalent:

(i) $C(K)$ has a 1-projectional skeleton.

(ii) There is a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $(B_{C(K)}^*, w^*)$ such that $\bigcup_{s \in \Gamma} r_s[K]$ is a convex set.

(iii) There is a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $P(K)$ such that $\bigcup_{s \in \Gamma} r_s[K]$ is a convex set.
Theorem

Let \( (X, \| \cdot \|) \) be a Banach space. Then the following conditions are equivalent:

(i) \( X \) has a 1-projectional skeleton.
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Let \((X, \| \cdot \|)\) be a Banach space. Then the following conditions are equivalent:

(i) \(X\) has a 1-projectional skeleton.

(ii) There is a retractional skeleton \(\{r_s\}_{s \in \Gamma}\) in \((B_X^*, w^*)\) such that \(\bigcup_{s \in \Gamma} r_s[K]\) is a convex and symmetric set.
Sets induced by an r-skeleton and their role in the proofs

**Definition**

Let $s = \{ r_s \}_{s \in \Gamma}$ be a retractional skeleton in a compact space $K$ and let $D(s) = \bigcup_{s \in \Gamma} r_s[K]$. Then we say that $D(s)$ is induced by an $r$-skeleton in $K$. 

Sets induced by an r-skeleton and dense $\Sigma$-subsets have some common topological properties (dense, countably closed, Fréchet-Urysohn).
Sets induced by an r-skeleton and their role in the proofs

**Definition**

Let $\mathcal{s} = \{r_s\}_{s \in \Gamma}$ be a retractional skeleton in a compact space $K$ and let $D(\mathcal{s}) = \bigcup_{s \in \Gamma} r_s[K]$. Then we say that $D(\mathcal{s})$ is induced by an $r$-skeleton in $K$.

$K$ has an $r$-skeleton $\iff$ there exists a set $D \subset K$ induced by an $r$-skeleton.
Sets induced by an r-skeleton and their role in the proofs

Definition

Let $s = \{r_s\}_{s \in \Gamma}$ be a retractional skeleton in a compact space $K$ and let $D(s) = \bigcup_{s \in \Gamma} r_s[K]$. Then we say that $D(s)$ is induced by an $r$-skeleton in $K$.

- $K$ has an $r$-skeleton $\iff$ there exists a set $D \subset K$ induced by an $r$-skeleton
- $K$ has a commutative $r$-skeleton $\iff$ there exists a dense $\Sigma$-subset $A \subset K$
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Sets induced by an $r$-skeleton and dense $\Sigma$-subsets have some common topological properties (dense, countably closed, Fréchet-Urysohn)
Simultaneous projectional skeletons

**Theorem**

Assume $D$ is induced by a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $K$.  

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Noncommutative Valdivia compacta
Simultaneous projectional skeletons

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Assume $D$ is induced by a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $K$. Let $\{P_s\}_{s \in \Gamma}$ be the 1-projectional skeleton in $C(K)$ induced by $\{r_s\}_{s \in \Gamma}$; i.e., $P_s(f) = f \circ r_s$, $s \in \Gamma$, $f \in C(K)$. 
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Let $Y$ be a $\tau_p(D)$-closed subset of $C(K)$. 
Simultaneous projectional skeletons

Theorem

Assume $D$ is induced by a retractional skeleton $\{r_s\}_{s \in \Gamma}$ in $K$. Let $\{P_s\}_{s \in \Gamma}$ be the 1-projectional skeleton in $\mathcal{C}(K)$ induced by $\{r_s\}_{s \in \Gamma}$; i.e., $P_s(f) = f \circ r_s$, $s \in \Gamma$, $f \in \mathcal{C}(K)$.

Let $Y$ be a $\tau_p(D)$-closed subset of $\mathcal{C}(K)$. Then there is an up-directed, $\sigma$-closed and unbounded set $\Gamma' \subset \Gamma$ such that $\{P_s\upharpoonright_Y\}_{s \in \Gamma'}$ is a 1-projectional skeleton in $Y$. 
References I


Thank you for your attention!