Five cardinals in Cichoń's Diagram and the strongly proper game

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In a previous paper (with Arthur Fischer, Jakob Kellner, and Saharon Shelah) we showed that the following is consistent:

$$\aleph_1 = \mathfrak{d} < \text{non(meager)} < \text{non(null)} < \text{cf(null)} < \mathfrak{c}$$

In a new paper (in progress; with Jakob Kellner, Diego Mejía, and Saharon Shelah) we plan to build a model in which the following holds:

$$\aleph_1 = \text{non(meager)} < \aleph_2 = \mathfrak{d} < \text{non(null)} < \text{cf(null)} < \mathfrak{c}$$

The construction again uses a product of creature forcings, but now also an iteration of Miller forcing to increase $\mathfrak{d}$.

A main new tool is the *strongly proper game* (Proper and Improper Forcing, chapter IX), defined as follows: Players I and II alternately play conditions $p_0, q_0, p_1, q_1, \ldots$, where player II always plays a condition q_n which is stronger than p_n. Player I wins if there is a condition q which forces that infinitely many q_n are in the generic filter.

Here is an example of an application: if P is a forcing notion for which player I has a winning strategy, then P preserves Cohen reals over countable models (and hence the old reals are still non-meager after forcing with P).

In a variant of the game, players I and II play blocks $p_n^1, \ldots, p_n^{k_n}$ and $q_n^1, \ldots, q_n^{k_n}$, with q_n^i stronger than p_n^i, and player I wins iff for each n there is at least one i with q_n^i the generic filter.

In my talk I will discuss these games, and give a sketch of our construction.