SOME EXERCISES FOR THE MINICOURSE: BANACH REPRESENTATIONS
OF DYNAMICAL SYSTEMS

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CONTENTS

1. Exercises 1
2. Some Solutions 4

1. Exercises

Exercise 1.1. Show that the one point compactification $A(\lambda)$ of a discrete space with cardinality $\lambda \geq \omega$ is uniformly Eberlein.

Exercise 1.2. Let $G$ be a countable discrete group. Show that there exists a topological group embedding $G \rightarrow \text{Iso}_{\text{lin}}(l_2)$.

Exercise 1.3. Let $G$ be the Polish symmetric group $S_N$ (of all permutations of $N$) with the pointwise topology. Show that there exists a topological group embedding $S_N \rightarrow \text{Iso}_{\text{lin}}(l_2)$.

Exercise 1.4. Let $V$ be a reflexive space and $B \subset V, A \subset V^*$ are bounded subsets. Show that the function $A \times B \rightarrow \mathbb{R}, (x,f) \mapsto \langle x, f \rangle = f(x)$ has DLP.

Exercise 1.5. Show that the original norm of the Banach space $c_0$ does not satisfy DLP.

Hint: Define $u_n := e_n$ and $v_m := \sum_{i=1}^{m} e_i$.

Exercise 1.6. Show that the Banach space $L_{2k}[0,1]$ has DLP for every $k \in \mathbb{N}$.

Exercise 1.7. Give an example of a bounded countable family of continuous functions $F \subset C[0,1]$ such that $F$ does not satisfy DLP (double limit property) on $[0,1]$.

Exercise 1.8. Let a topological group $G$ admit a left-invariant metric with DLP. Show that $G$ is reflexively representable.

Exercise 1.9. The regular representation of the circle group $T$ on $V := C(T)$ is continuous but not adjoint continuous.

Hint: ("Point measures are responsible for this") Indeed, the continuity of the adjoint representation $T \rightarrow \text{Iso}_{\text{lin}}(V)$ on a Banach space $V$ adjoint continuous if the adjoint representation $h^* : G \rightarrow \text{Iso}_{\text{lin}}(V^*)$ is also continuous. It is a well known phenomenon in Functional Analysis that continuous representations on general Banach spaces need not be adjoint continuous (even for compact groups).

Exercise 1.10. Let $L$ be the left uniform structure of the topological group $\text{Iso}(l_2)$. Show that the uniform space $(\text{Iso}(l_2), L)$ is uniformly embedded into the uniform space $l_2$.

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**Exercise 1.11.** Let $\pi : S_\mathbb{N} \times l_1 \to l_1$ be the natural (linear isometric) action of the Polish symmetric group $S_\mathbb{N}$ on the Banach space $l_1$ (permutations of coordinates). Show that the dual action

$$ S_\mathbb{N} \times l_\infty \to l_\infty, \ (gf)(x) := f(g^{-1}(x)) $$

on the dual Banach space $l_1^* = l_\infty$ is not continuous. So, the natural representation of $S_\mathbb{N}$ on $l_1$ is (continuous but) not adjoint continuous.

**Exercise 1.12.**

1. For a topological space $X$ consider the semigroup $(X^X, \circ)$ of all selfmaps $f : X \to X$ with respect to pointwise (=product) topology. Show that $X^X$ is a right topological semigroup.

2. $C(X, X)$ is a semitopological subsemigroup of $X^X$.

3. Prove that the left translation $l_f : X^X \to X^X$ is continuous if and only if $f \in C(X, X)$. Derive that if $X$ is $T_1$, then the right topological semigroup $X^X$ is semitopological if $X$ is discrete.

**Exercise 1.13.** Let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of $G$.

Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not topological semigroup.

**Exercise 1.14.** Let $G$ be a Hausdorff topological group and $H \leq G$ be its topological subgroup. If $H$ is locally compact then $H$ is closed in $G$.

**Exercise 1.15.** If $S$ is a compact Hausdorff topological semigroup and if $G$ is a subgroup of $S$ then $cl(G)$ is a (compact) topological group.

Hint: $e_G$ is an idempotent of $S$ and also a neutral element of $T := cl(G)$.

**Exercise 1.16.** Let $S$ be the interval $[0, 1]$ with the multiplication

$$ st = \begin{cases} t, & \text{if } 0 \leq t < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} $$

Show that: $S$ is a compact right topological semigroup with $\Lambda(S) = \emptyset$. The subset $T := [0, \frac{1}{2})$ is a subsemigroup of $S$ and $cl(T) = [0, \frac{1}{2}]$ is not a subsemigroup of $S$.

**Example 1.17.** Let $S := \mathbb{Z} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{Z}$. Extend the usual addition by:

$$ n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \ s, t \in \{-\infty, \infty\} $$

Show: $(S, +)$ is a noncommutative compact right topological semigroup having dense topological centre $\Lambda(S) = \mathbb{Z}$. $S$ is not semitopological.

**Exercise 1.18.** Show that the right topological semigroup $S$ of the previous exercise is topologically isomorphic to the *enveloping semigroup* of the invertible cascade $(\mathbb{Z}, [0, 1])$ generated by the homeomorphism $\sigma : [0, 1] \to [0, 1], \sigma(x) = x^2$.

**Exercise 1.19.** Prove that:

1. for every metric space $(M, d)$ the semigroup $S := \Theta(M, d)$ of all non-expanding maps $^1$

   $$ f : X \to X \quad \text{(that is, } d(f(x), f(y)) \leq d(x, y)) \text{ is a topological monoid with respect to the topology of pointwise convergence;} $$

2. the group $\text{Iso}(M)$ of all onto isometries is a topological group;

3. the evaluation map $S \times M \to M$ is a continuous monoidal action.

**Exercise 1.20.** Let $S \times X \to X$ be contractive action of $S$ on $(X, d)$. Show that the following conditions are equivalent:

(i) The action is continuous.

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$^1$ in another terminology: *Lipschitz 1 maps*
Exercise 1.21. Prove that $\Theta(V)$ and $L(V, V)$ are semitopological monoids with respect to the weak operator topology for every Banach space $V$.

Exercise 1.22. For every Banach space $(V, || \cdot ||)$ show that:
1. The semigroup $\Theta(V)_s$ (with SOT) is a topological monoid.
2. The subspace $\text{Iso}(V)_s$ of all linear onto isometries is a topological group.

Exercise 1.23. Let $\text{Unif}(Y, Y)$ be the set of all uniform self-maps of a uniform space $(Y, \mu)$. Denote by $\mu_{\text{sup}}$ the uniformity of uniform convergence on $\text{Unif}(Y, Y)$. Show that
1. under the corresponding topology $\text{top}(\mu_{\text{sup}})$ on $\text{Unif}(Y, Y)$ and the usual composition we get a topological monoid;
2. If $G$ is any subgroup of the monoid $\text{Unif}(Y, Y)$ then $G$ is a topological group;
3. For every subsemigroup $S \subset \text{Unif}(Y, Y)$ the induced action $S \times Y \to Y$ is continuous;

Exercise 1.24. Let $Y$ be a compact space. Show that:
1. The semigroup $C(Y, Y)$ endowed with the compact open topology is a topological monoid;
2. The subset $H(Y)$ in $C(Y, Y)$ of all homeomorphisms $Y \to Y$ is a topological group;
3. For every subsemigroup $S \subset C(Y, Y)$ the induced action $S \times Y \to Y$ is continuous;
4. Furthermore, it satisfies the following remarkable minimality property. If $\tau$ is an arbitrary topology on $S$ such that $(S, \tau) \times Y \to Y$ is continuous then $\tau_{\text{co}} \subset \tau$.

Exercise 1.25. Let $X$ be a compact space and $F \subset C(X)$ be a bounded subset. Show that $F$ has DLP on $X$ iff $F$ has DLP on $B^*$, where $B^* = B_{C(X)}$.

Exercise 1.26. Show that if $V$ is Asphuld then $\text{Iso}(V)_w$ (in WOT) is a topological group.

Exercise 1.27. If $\nu_1 : X \to Y_1$ and $\nu_2 : X \to Y_2$ are two compactifications, then $\nu_2$ dominates $\nu_1$, that is, $\nu_1 = q \circ \nu_2$ for some (uniquely defined) continuous map $q : Y_2 \to Y_1$ iff $A_{\nu_1} \subset A_{\nu_2}$. Show that if in addition, $X$, $Y_1$ and $Y_2$ are $S_\nu$-systems (i.e., all the $s$-translations on $X$, $Y_1$ and $Y_2$ are continuous) and if $\nu_1$ and $\nu_2$ are $S$-maps, then $q$ is also an $S$-map. Furthermore, if the action on $Y_1$ is (separately) continuous then the action on $Y_2$ is (respectively, separately) continuous. If $\nu_1$ and $\nu_2$ are homomorphisms of semigroups then $q$ is also a homomorphism.

Exercise 1.28. (Greatest ambit) Let $G$ be a topological group and $\beta_G : G \to \beta G$ be the compactification induced by the algebra $\text{RUC}(G)$. Show that it is the universal semigroup $G$-compactification of $G$ with jointly continuous $G$-action. (The universality means that for every semigroup $G$-compactification $\nu : G \to P$ with continuous action $G \times P \to P$ there exists a unique continuous $G$-homomorphism $q : \beta G \to P$ such that $\beta_G \circ q = \nu$.

Exercise 1.29. (Enveloping semigroups)
1. Let $X$ be a compact $S$-space with the enveloping semigroup $E(X)$ and $L$ a subset of $C(X)$ such that $L$ separates points of $X$. Then the Ellis compactification $j : S \to E(X)$ is equivalent to the compactification of $S$ which corresponds to the subalgebra $A_L := \langle m(L, X) \rangle$, the smallest norm closed $S$-invariant unital subalgebra of $C(S)$ which contains the family
$$\{m(f, x) : S \to \mathbb{R}, \ s \mapsto f(sx)\}_{f \in L, \ x \in X}.$$
2. Let $q : X_1 \to X_2$ be a continuous onto $S$-map between compact $S$-spaces. There exists a (unique) continuous onto semigroup homomorphism $Q : E(X_1) \to E(X_2)$ with $j_{X_1} \circ Q = j_{X_2}$.
3. Let $Y$ be a closed $S$-subspace of a compact $S$-system $X$. The map $r_Y : E(X) \to E(Y)$, $p \mapsto p|_Y$ is the unique continuous onto semigroup homomorphism such that $r_X \circ j_X = j_Y$.
4. Let $\alpha : S \to P$ be a right topological compactification of a semigroup $S$. Then the enveloping semigroup $E(S, P)$ of the semitopological system $(S, P)$ is naturally isomorphic to $P$.
5. If $X$ is metrizable then $E(X)$ is separable. Moreover, $j(S) \subset E(X)$ is separable.
Proof. (1) The proof is straightforward using the Stone-Weierstrass theorem.

(2) By Remark 1.27 it suffices to show that the compactification \( j_{X_1} : S \to E(X_1) \) dominates the compactification \( j_{X_2} : S \to E(X_2) \). Equivalently we have to verify the inclusion of the corresponding algebras. Let \( g(x) = y, f_0 \in C(X_2) \) and \( f = f_0 \circ g \). Observe that \( m(f_0, y) = m(f, x) \) and use (1).

(3) Is similar to (2).

(4) Since \( E(S,P) \to P, a \to a(e) \) is a natural homomorphism, \( j_P : S \to E(S,P) \) dominates the compactification \( S \to P \). So it is enough to show that, conversely, \( \alpha : S \to P \) dominates \( j_P : S \to E(S,P) \). By (1) the family of functions

\[
\{ m(f, x) : S \to \mathbb{R} \}_{f \in C(P), x \in P}
\]

generates the Ellis compactification \( j_P : S \to E(S,P) \). Now observe that each \( m(f, x) : S \to \mathbb{R} \) can be extended naturally to the function \( P \to \mathbb{R}, p \mapsto f(px) \) which is continuous.

(5) Since \( X \) is a metrizable compactum, \( C(X,X) \) is separable and metrizable in the compact open topology. Then \( j(S) \subset C(X,X) \) is separable (and metrizable) in the same topology. Hence, the dense subset \( j(S) \subset E(X) \) is separable in the pointwise topology. This implies that \( E(X) \) is separable. \( \square \)

Exercise 1.30. Let \( K \) be a compact space which is norm-fragmented in \( C(K)^\ast \). Show that \( K \) is scattered.

Exercise 1.31. If \( X \) is (locally) fragmented by \( f : X \to Y \), where \( (X, \tau) \) is a Baire space and \( (Y, \rho) \) is a pseudometric space then \( f \) is continuous at the points of a dense \( G_\delta \) subset of \( X \).

Exercise 1.32. Let \( K \) be a RN compactum. Show that \( K \) has a dense subset \( Y \subset X \) such that \( y \) has a countable local bases in \( X \) for every \( y \in Y \).

Exercise 1.33. When \( X \) is compact and \( (Y, \rho) \) metrizable uniform space then \( f : X \to Y \) is fragmented iff \( f \) has a point of continuity property (i.e., for every closed nonempty \( A \subset X \) the restriction \( f|_A : A \to Y \) has a continuity point).

Exercise 1.34. Let \( (X, \tau) \) be a separable metrizable space and \( (Y, \rho) \) a pseudometric space. Suppose that \( f : X \to Y \) is a fragmented onto map. Then \( Y \) is separable. Hint: use the idea of the Cantor-Bendixon theorem.

Exercise 1.35. Show that \( F = \{ f_i : X \to \mathbb{R} \}_{i \in I} \) is a fragmented family iff the induced map \( X \to (\mathbb{R}^F, \xi_U) \) is fragmented, where \( \xi_U \) is the uniformity of uniform convergence on \( \mathbb{R}^F \).

Exercise 1.36. Give an example of a bounded family \( F \) of continuous functions \([0,1] \to \mathbb{R}\) such that \( F \) is eventually fragmented but not fragmented.

2. Some Solutions

Definition. Let \((Y, \tau)\) be a topological space and \( X \) be a set. Denote by \( Y^X \) the set of all maps \( f : X \to Y \) endowed with the product topology of \( Y^X \). This topology has the standard base \( \alpha \) which consists of all the sets:

\[
O(x_1, \ldots, x_n; U_1, \ldots, U_n) := \{ f \in Y^X : f(x_i) \in U_i \}
\]

where, \( F := \{ x_1, \ldots, x_n \} \) is a finite subset of \( X \) and \( U_i \) are nonempty open subsets in \( Y \). Other names of this topology are: pointwise topology, point-open topology. Sometimes we use a short notation \((x_1, \ldots, x_n; U_1, \ldots, U_n)\) instead of \(O(x_1, \ldots, x_n; U_1, \ldots, U_n)\).

Exercise 2.1.

(1) For every topological space \( X \) consider the semigroup \((X^X, \circ)\) of all selfmaps \( f : X \to X \) with respect to pointwise (=product) topology. Show that \( X^X \) is a right topological semigroup.

(2) \( C(X,X) \) is a semitopological subsemigroup of \( X^X \).

Is it true that \( C([0,1], [0,1]) \) is a topological semigroup?

(3) Prove that the left translation \( l_f : X^X \to X^X \) is continuous if and only if \( f \in C(X,X) \). Derive that if \( X \) is \( T_1 \), then the right topological semigroup \( X^X \) is semitopological iff \( X \) is discrete.
Proof. First a general

**Remark 2.** The product topology on $X^X$ can be described by nets as the pointwise topology. A net (generalized sequence) $f_i$ in $X^X$ converges to $f \in X^X$ iff the net $f_i(x)$ in $X$ converges to $f(x)$ for each $x \in X$. This explains the term: “pointwise topology”.

(1) First proof: (using the nets)

We have to show that $r_h : X^X \to X^X$ is continuous for every given $h \in X^X$. It is equivalent to show

$$\lim f_i = f \Rightarrow \lim f_i h = f u$$

for every net $f_i$ in $X^X$. $\lim f_i = f$ means (see Remark 2) that $\lim f_i(x) = f(x)$ for every $x \in X$. Then substituting $h(x)$ we have $\lim f_i(h(x)) = f(h(x))$. This exactly means that $\lim f_i h = f h$

(again, Remark 2).

Second proof: (using the nbds)

First we recall the following general topological

**Fact 2.** For the continuity of a map it is enough to show that the preimage of any basic nbdl is an nbdl. Moreover, in fact, it is enough even to check the same for a subbase. ²

Consider the following family

$$\gamma := \{(x; U) : x \in X, U \in \tau\}, \quad (x; U) := \{f \in X^X : f(x) \in U\}$$

Then $\gamma$ is a subbase of the standard base (of the pointwise topology on $X^X$)

$$(x_1, \cdots, x_n; U_1, \cdots, U_n) := \{f \in Y^X : f(x_i) \in U_i\}.$$ 

Now we can prove (1) using Fact 2. Let $h \in X^X$. For every given $(x; U)$ consider the open set $(h(x); U)$. Then for every $f \in (h(x); U)$ we have $fh \in (x; U)$.

(2) $C(X, X)$ is evidently a subsemigroup of $X^X$ so it is enough to show that for $h \in C(X, X)$ the corresponding left translation $l_h : X^X \to X^X$ is continuous (i.e., $C(X, X) \subset \Lambda(X^X)$).

First proof: Let $h \in C(X, X)$. If $\lim f_i = f$ in $X^X$ then $\lim f_i(x) = f(x)$ in $X$ for every $x \in X$. Then by the continuity of $h$ we have $\lim h(f_i(x)) = h(f(x))$. This means that $\lim h(f_i) = h(f)$.

Now use Remark 2.

Second proof: Let $h \in C(X, X)$. For every standard subbase nbdl $(a; U) \in \gamma$ consider the open set $(a; h^{-1}U))$ (the continuity of $h$ guarantees that $h^{-1}(U)$ is open in $X$). Then $f \in (a; h^{-1}(U))$ implies that $h f \in (a; U)$. By Fact 2 we obtain that $l_h : X^X \to X^X$ is continuous.

$C([0,1],\gamma)$ is *not* a topological semigroup. We have to show that the multiplication $m$ (the composition) is not continuous. In fact, we will show much more that $m$ is not continuous at any point $(h_0, f_0) \in C[0,1] \times C[0,1]$. Let $a := h_0(f_0)(1)$. Consider an open nbdl $(a;(-\frac{1}{2}, \frac{1}{2}))$ of $h_0 \circ f_0$ in the space $C([0,1],\gamma)$. Then for every basic nbds

$$h_0 \in (x_1, \cdots, x_n; U_1, \cdots, U_n) \quad f_0 \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$$

there exists a pair $f, h$ such that

$$h \in (x_1, \cdots, x_n; U_1, \cdots, U_n) \quad f \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$$

but $h \circ f \notin (a;(-\frac{1}{2}, \frac{1}{2}))$. Indeed using a freedom ³ in the building of continuous functions (and the fact that each of the nbds $U_i$ and $V_k$ are infinite sets) one may choose $f \in (y_1, \cdots, y_m; V_1, \cdots, V_m)$ s.t. $f(1) \notin \{x_1, \cdots, x_n\}$. Now we can choose $h \in (x_1, \cdots, x_n; U_1, \cdots, U_n)$ s.t. $h(f(1)) \notin (-\frac{1}{2}, \frac{1}{2})$.

(3) (First part)

First proof: Let $l_h : X^X \to X^X$ be continuous. We have to show that $h \in C(X, X)$. It is equivalent to show that $h$ preserves the convergence of nets in $X$ in the following sense:

$$\lim x_i = x \Rightarrow \lim h(x_i) = h(x)$$

²Recall that a family $\gamma$ of open subsets is said to be a subbase if the finite intersections (that is, the family $\gamma^{\cap} f_{in}$) is a topological base.

³Namely the fact that every map $F \to [0,1]$ on a finite subset $F \subset [0,1]$ can be extended to a continuous map $[0,1] \to [0,1]$
For every $y \in X$ consider the constant function $c_y : X \to X, c_y(t) = t$. Then $\lim c_{x_1} = c_x$ in $X^X$. The continuity of $l_h : X^X \to X^X$ means that it preserves the convergence in $X^X$. So, in particular, we have $\lim l_h(c_{x_1}) = l_h(c_x)$. But this means that $\lim h(x_i) = h(x)$, as desired.

Second proof: (Gal Lavi and Noam Lifshitz)

We have to show that $h : X \to X$ is continuous at every given $a \in X$. Let $U \in N(h(a))$ in $X$. Consider the open nbd $(a;U)$ in $X^X$. Consider the constant function $c_a : X \to X, x \mapsto a$. Then $(h \circ c_a)(x) = h(a)$ for every $x \in X$. In particular, $h_a \in (a;U)$. By our assumption the left transition $l_h : X^X \to X^X$ is continuous. Therefore, there exists a basic nbd

$$W := (x_1, \ldots, x_n; V_1, \ldots, V_n)$$

of $c_a$ in $X^X$ s.t. $hW \subset (a;U)$. Each $V_i$ is a nbd of $a$ (because, $c_a(x_i) = a$). Then also $V := \cap_i V_i \in N(a)$. Now observe that $c_v \in W$ for every $v \in V$. On the other hand, $hW \subset (a;U)$ leads us to $h(e) = f(v) \in U$ for every $v \in V$. Hence, $h(V) \subset U$. This proves the continuity of $h$ at $a$.

(3) (Second part)

If $X$ is discrete then of course $X^X = C(X, X)$ which is semitopological by (2).

If $X^X$ is semitopological then by the first part of (3) we know that $X^X = C(X, X)$. Let $X \in T_1$. We have to show that $X$ is discrete. Since $X \in T_1$, every singleton $\{a\}$ is closed in $X$. Choose one of them. For every nonempty $A \subset X$ consider a function $f_A : X \to X$ s.t. $f_A^1(a) = A$. Since $f$ is continuous we get that $A$ is closed. So, every subset of $X$ is closed, hence $X$ is discrete.

One may show that in general if $X^X = C(X, X)$ then either $X$ is discrete or $X$ has the trivial topology. So, the assumption $X \in T_1$ can be replaced by the assumption that the topology on $X$ is not trivial.

\[\square\]

**Definition.** Let $X$ be a topological space. A compactification of $X$ is a continuous map $f : X \to Y$ where $Y$ is a compact Hausdorff space and $f(X)$ is dense in $Y$. We say: proper compactification when, in addition, $f$ is required to be a topological embedding.

One of the standard examples of a proper compactification is the so-called 1-point compactification $\nu : X \hookrightarrow X_\infty := X \cup \{\infty\}$ defined for every locally compact non-compact Hausdorff space $(X, \tau)$. Recall the topology

$$\tau_\infty := \tau \cup \{X_\infty \setminus K : K \text{ is compact in } X\}.$$ 

**Exercise 2.2.** Let $(G, \cdot, \tau)$ be a locally compact non-compact Hausdorff topological group. Denote by $S := G \cup \{\infty\}$ the 1-point compactification of $G$.

Show that $(S, \cdot, \tau_\infty)$ is a semitopological but not topological semigroup.

**Proof.** First we show that $S$ is semitopological. Let $a \in S$. We have to show that $l_a : S \to S$ and $r_a : S \to S$ are continuous. We consider only the case of $l_a$. The second case is similar. So, we have to check that $l_a : S \to S$ is continuous at every $y \in S$. For $a = \infty$ the transition $l_a$ is a constant map, WRG assume that $a \neq \infty$, hence $a \in G$. We have two cases for $y \in S$:

(a) If $y \neq \infty$ then for every open nbd $U \in N(y), U \subset G$ just take the open nbd $V := a^{-1}U \in N(a^{-1}y)$. Then $l_a(V) = U$.

(b) Let $y = \infty$ and $U \in N(\infty)$ is an open nbd. Then by the definition of the 1-point compactification topology, $U = S \setminus K$, where $K$ is compact in $G$. Then $a^{-1}K$ is also compact in $G$. So, $V := S \setminus a^{-1}K \in N(\infty)$ and $l_a(V) = U$.

Now we show that $S$ is not topological. That is, the multiplication is not continuous. Indeed, we show that the multiplication $m : S \times S \to S$ is not continuous at the point $(\infty, \infty)$.

First proof:

Choose the open nbd $U := S \setminus \{e\} \subset \infty$. It is enough to show that for every nbd $V \in N(\infty)$ we have $e \in VU$ (this will mean that $VV$ is not a subset of $U$). Observe that every $V \in N(\infty)$ contains an open nbd $S \setminus K$, where $K$ is compact and symmetric (indeed, WRG replace $K$ by $K \cup K^{-1}$). Now observe that for every $x \in S \setminus K$ we have $x^{-1} \in S \setminus K$ but $xx^{-1} = e \notin U$.

Second proof:
Assuming the contrary let $m : S \times S \to S$ be continuous. Then 

$$A := m^{-1}(\{e\}) = \{(x, x^{-1}) \in S \times S : x \in G\}$$

is a closed subset of the product $S \times S$. Since, $S$ is compact then $A$ is compact, too. Consider the projection $\pi_1 : S \times S \to S, (a, b) \mapsto a$. Then $\pi_1(A)$ is a compact subset of $S$. But $\pi_1(A) = G$. So, we obtain that $G$ is compact, a contradiction (because $G$ is assumed to be noncompact).

\[\square\]

As we know a locally compact Hausdorff group $G$ admits an embedding into a compact Hausdorff group iff $G$ is compact. Exercise 2.2 shows that such $G$ at least admits a proper semigroup compactification $\nu : G \to S$ such that $S$ is a compact semitopological monoid.

**Exercise 2.3.** Let $G$ be a Hausdorff topological group and $H \leq G$ be its topological subgroup. If $H$ is locally compact then $H$ is closed in $G$.

**Proof.** It is equivalent to prove in the case of $\text{cl}(H) = G$. So we have to show that $H$ is closed in $\text{cl}(H)$. It suffices to show that $H$ is open in $G = \text{cl}(H)$ (because every open subgroup is closed).

Since $H$ is LC one may choose a compact nbhd $K$ of $e$ in $H$.

$$\exists U \in N_G(e) \cap \tau : U \cap H \subset K$$

$$U = U \cap G = U \cap \text{cl}(H) \subset \text{cl}(U \cap H) \subset \text{cl}(K) = K$$

(remark1: for every open $O \subset X$ and $A \subset X$ we have $O \cap \text{cl}(A) \subset \text{cl}(O \cap A)$

(remark2: every compact subset is closed in a Hausdorff space)

So, $U \subset K$. Therefore, $U \subset H$. Hence, $\text{int}_G(H) \neq \emptyset$. It follows that that the subgroup $H$ is open in $\text{cl}(H)$. Hence, also closed. So, $H = \text{cl}(H)$.

\[\square\]

It is impossible to embed a locally compact noncompact group into any Hausdorff compact group. In particular, there is no finite-dimensional topologically faithful representation by linear isometries of a locally compact noncompact groups (like $\mathbb{Z}$, $\mathbb{R}$) on finite-dimensional Euclidean spaces.

**Exercise 2.4.** If $S$ is a compact Hausdorff topological semigroup and if $G$ is a subgroup of $S$ then $\text{cl}(G)$ is a (compact) topological group.

**Hint:** $e_G$ is an idempotent of $S$ and also a neutral element of $T := \text{cl}(G)$.

**Proof.** The simplest way here is to use the technique of the nets (generalized sequences).

1. $T = \text{cl}(G)$ is a **topological subsemigroup of $S$.**

   Indeed, let $x, y \in T := \text{cl}(G)$. Then there exist nets $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ such that $\lim x_i = x, \lim y_i = y$ and $x_i, y_i \in G$. 4 Then by the continuity of the multiplication we have $\lim (x_i y_i) = (\lim x_i)(\lim y_i) = xy$. Since $x_i y_i \in G$ we obtain that $xy \in \text{cl}(G)$.

2. $e_G$ is a **neutral element in $T = \text{cl}(G)$. So, $T$ is a topological monoid.**

   Indeed, if $\lim x_i = x \in \text{cl}(G)$ and $x_i \in G$ then $\lim (x_i e) = \lim x_i = x$. On the other hand, by the continuity of the multiplication we have $\lim (x_i e_G) = (\lim x_i)e_G = xe_G$. So, $xe_G = x$. Similarly, $e_G x = x$.

3. $T$ is a **group.**

   Let $t \in T$ and $g_t$ be a net in $G$ converging to $t$. By compactness we may assume that some subnet of $g_t^{-1}$ converges to some $s \in T$. For simplicity (WRG) we assume that $g_t^{-1}$ itself converges to some $s \in T$. Since $S$ is topological we have $g_t g_t^{-1}$ converges to $ts$. By the Hausdorff axiom we necessarily have $e = ts$. Similarly, $e = st$.

4. $T$ is a **(compact) topological group.**

   Now (after 1-3) it suffices to show that the inversion $j : T \to T, t \mapsto t^{-1}$ is continuous. Let $\lim t_i = t$ in $T$. We have to show that the limit $\lim t_i^{-1}$ exists in $T$ and it equals to $t^{-1}$. Consider the net $t_i^{-1}$ in $T$. Since $T$ is compact, there exists a converging subnet $t_{i_j}$. Let $\lim t_{i_j}^{-1} = y \in \text{cl}(G)$.

4Note that for every two converging nets $\nu_1 : (\Gamma_1, \leq_1) \to X$, $\nu_2 : (\Gamma_2, \leq_2) \to X$ one may assume WRG that they have the same domain $(\Gamma := \Gamma_1 \times \Gamma_2)$ for example)
A closed subset

Clearly, \( Z \)

First of all it is straightforward to see that

\[ \text{Proof.} \]

\[ x \] for every \( \in \mathbb{T} \)

addition by:

\[ \text{subsemigroup of} \ S \]

continuity of the multiplication in \( T \)

8

\[ \text{Exercise 2.5. Let} \ S \] be the interval \([0, 1]\) with the multiplication

\[ st = \begin{cases} 
    t, & \text{if } 0 \leq t < \frac{1}{2}; \\
    1, & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases} \]

Show that: \( S \) is a compact right topological semigroup with \( \Lambda(S) = \emptyset \). The subset \( T := [0, \frac{1}{2}] \) is a subsemigroup of \( S \) and \( cl(T) = [0, \frac{1}{2}] \) is not a subsemigroup of \( S \).

\[ \text{Proof.} \]

First of all it is straightforward to see that \( S \) is a semigroup and \( T \) its subsemigroup.

\( S \) is right topological. Because \( r_t : S \to S \) is a constant function (\( t \) or \( 1 \)) for every \( r \in S \).

\[ \Lambda(S) = \emptyset. \]

Indeed, for every \( s \in S \) we have that \( L_s : [0, 1] \to [0, 1] \) has a jump discontinuity point at \( \frac{1}{2} \).

\[ cl(T) = [0, \frac{1}{2}] \] is not a subsemigroup of \( S \). Indeed, \( cl(T) = [0, \frac{1}{2}] \) and \( \frac{1}{2} \times \frac{1}{2} = 1 \notin T. \]

\[ \text{Exercise 2.6. Let} \ S := \mathbb{Z} \cup \{-\infty, \infty\} \] be the two-point compactification of \( \mathbb{Z} \). Extend the usual addition by:

\[ n + t = t + n = s + t = t \quad n \in \mathbb{Z}, \quad s, t \in \{-\infty, \infty\} \]

Show: \( (S, +) \) is a noncommutative compact right topological monoid having dense commutative topological centre \( \Lambda(S) = \mathbb{Z} \). \( S \) is not semitopological.

\[ \text{Proof.} \]

First of all it is straightforward to see that \( (S, +) \) is a monoid and \( (\mathbb{Z}, +) \) its submonoid.

\( (S, +) \) is noncommutative because \( \infty + (-\infty) = -\infty \) and \( (-\infty) + \infty = \infty \).

\( S := \mathbb{Z} \cup \{-\infty, \infty\} \) carries the topology of the natural linear order. A natural subbase for the topology of \( S \) is the following family

\[ A_n := \{ x \in S : x < n \}, \quad B_m := \{ x \in S : m < x \}, \quad n, m \in \mathbb{Z} \]

Clearly, \( \mathbb{Z} \) is dense in \( S \) and every \( x \in \mathbb{Z} \) is an isolated point in \( S \). The space \( S \) is homeomorphic to a closed subset

\[ Y := \{-1\} \cup \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{1\} \]

of \([−1, 1] \), hence compact.

The right translations \( r_t : S \to S \) are continuous. Indeed, \( r_\infty \) is the constant function \( r_\infty(x) = \infty \) for every \( x \in S \). \( r_{-\infty} \) is the constant function \( r_{-\infty}(x) = -\infty \) for every \( x \in S \). \( r_k^{-1}(A_n) = A_{n-k} \), \( r_k^{-1}(B_m) = B_{m-k} \) for every \( k \in \mathbb{Z} \).

\[ \Lambda(S) = \mathbb{Z}. \]

Indeed, every \( l_k : S \to S \) is continuous for each \( k \in \mathbb{Z} \) because \( l_k^{-1}(A_n) = A_{n-k} \), \( l_k^{-1}(B_m) = B_{m-k} \).

\[ l_\infty : S \to S \] is not continuous at the point \( s = -\infty \). Take a sequence \( \{-k\}_{k \in \mathbb{N}} \). Then

\[ \lim(-k) = -\infty \quad \text{but} \quad \lim l_\infty(-k) = \infty \neq l_\infty(-\infty) = \infty + (-\infty) = -\infty \]

Similarly, \( l_{-\infty} : S \to S \) is not continuous at the point \( s = \infty \).

\[ \text{Exercise 2.7. Show that the right topological semigroup} \ S \] of the previous exercise is topologically isomorphic to the enveloping semigroup \( E \) of the invertible cascade \( (\mathbb{Z}, [0, 1]) \) generated by the homeomorphism \( \sigma : [0, 1] \to [0, 1], \sigma(x) = x^2 \).

\[ \text{Proof.} \]

Hint: Let \( E \) be the enveloping semigroup of \( (\mathbb{Z}, [0, 1]) \) and \( j : \mathbb{Z} \to E \) be the corresponding compactification. Observe that besides the points \( j(\mathbb{Z}) = \{ \sigma^n n \in \mathbb{Z} \} \) the enveloping semigroup \( E(X) \) contains two more points: \( a, b \), where \( a = \xi_{(1)} \) the characteristic function of \( \{1\} \) and \( b = 1 - \xi_0 \), where \( \xi_{(0)} \) is the characteristic function of \( \{0\} \).

\[ \square \]
Exercise 2.8. For every metric space \((M,d)\) show that:

1. The semigroup \(\Theta(M,d)\) of all \(d\)-contractive maps \(f : X \to X\) (that is, \(d(f(x), f(y)) \leq d(x, y)\)) is a topological monoid with respect to the topology of pointwise convergence;
2. The group \(\text{Iso}(M)\) of all onto isometries is a topological group;
3. The evaluation map \(S \times M \to M\) is a jointly continuous monoidal action for every submonoid \(S \subseteq \Theta(M,d)\).

Proof. (1) **Algebraic part:** it is trivial to see that the composition is well defined, associative, and \(\Theta := \Theta(M,d)\) is a monoid.

**Continuity of the multiplication**

We use the following easy reformulation of the pointwise topology:

**Fact.** Let \(X\) be any nonempty set, \((Y,d)\) be a metric space and \(\tau_p\) be the pointwise (product) topology on \(Y^X := \{ f : X \to Y \}\). Then for every \(f_0 \in Y^X\) the following family of sets is a local base at the point \(f_0\) with respect to the topology \(\tau_p\):

\[
(f_0; x_1, \cdots, x_n; \varepsilon) := \{ f \in Y^X : d(f_0(x_k), f(x_k)) < \varepsilon \forall k = 1, \cdots, n\},
\]

where \(x_1, \cdots, x_n\) is a finite subset in \(X\) and \(\varepsilon > 0\).

Now we prove the continuity of the multiplication \(m : \Theta \times \Theta \to \Theta\) at the point \((s_0, t_0) \in \Theta \times \Theta\).

We have to show that \(st\) is close to \(s_0t_0\) when \(s\) and \(t\) are close enough to \(s_0\) and \(t_0\), respectively. In order to get a "right idea for the proof" consider the following inequalities:

\[
d(s_0t_0(x_k), st(x_k)) \leq d(s_0t_0(x_k), st_0(x_k)) + d(st_0(x_k), st(x_k))
\]

\[
\leq d(s_0t_0(x_k), st_0(x_k)) + d(t_0(x_k), t(x_k))
\]

Note that in the last inequality we need to use the Lipschitz-1 property for \(s\).

Now we can easily finish the proof choosing appropriate neighborhoods for \(t_0\) and \(s_0\) for a given nbd \(O := (s_0t_0; x_1, \cdots, x_n; \varepsilon)\) of \(s_0t_0\). Indeed, take the following neighborhoods \(U := (t_0; x_1, \cdots, x_n; \frac{\varepsilon}{2})\) and \(V := (s_0; t_0(x_1), \cdots, t_0(x_n); \frac{\varepsilon}{2})\). Then for every \(t \in U, s \in V\) we have \(st \in O\), as desired.

**Remark.** Another proof can be based on nets. Namely, to the following useful (and characterizing) property of the pointwise topology.

- a net \(s_i\) converges to \(s_0\) in \(Y^X\) (with respect to pointwise topology) if and only if the net \(s_i(x_0)\) converges to \(s(x_0)\) (in \(Y\)) for every \(x_0 \in X\).

(2) For the **continuity of the inversion** \(\text{Iso}(M) \to \text{Iso}(M)\) at the point \(s_0\).

In order to estimate how close can be \(s^{-1}\) to \(s_0^{-1}\) look at the following key equality (using, this time, that \(s : M \to M\) is an isometry)

\[
d(s^{-1}(x_k), s_0^{-1}(x_k)) = d(x_k, s_0s_0^{-1}(x_k)) = d(s_0(t_k), s(t_k))
\]

with \(x_k := s_0(t_k)\).

Now the rest is easy. For a given nbd \(O(s_0^{-1}) := (s_0^{-1}; x_1, \cdots, x_n; \varepsilon)\) of \(s_0^{-1}\) choose \(U(s_0) := (s_0; t_1, \cdots, t_n; \varepsilon)\) of \(s_0\) with \(t_k := s_0^{-1}(x_k)\). Now if \(s \in U\) then \(s^{-1} \in O\).

(3) We have to prove the continuity of the action \(S \times X \to X\) at every given point \((s_0, x_0)\). We give only a key inequality (the rest will be clear):

\[
d(s_0x_0, sx_0) \leq d(s_0x_0, sx_0) + d(sx_0, sx) \leq d(s_0x_0, sx_0) + d(x_0, x).
\]

\[\square\]

An action \(S \times X \to X\) on a metric space \((X,d)\) is **non-expanding** if every \(s\)-translation \(\tilde{s} : X \to X\) lies in \(\Theta(X,d)\). It defines a natural homomorphism \(h : S \to \Theta(X,d)\).

Exercise 2.9. Let \(S \times X \to X\) be a non-expanding action of \(S\) on \((X,d)\). Show that the following conditions are equivalent:

(i) The action is continuous.
(ii) The action is separately continuous.
(iii) The natural homomorphism $h : S \to \Theta(X,d)$ of monoids is continuous.

**Proof.** (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) We have to show that $h : S \to \Theta$ is continuous. Let $s_i \to s_0$ be a converging net in $S$. We need to verify that $h(s_i) \to h(s_0)$. By the definition of pointwise topology it is equivalent to check that $h(s_i)(x_0) \to h(s_0)(x_0)$. By the definition of $h$ the latter is equivalent to $s_i(x_0) \to s_0(x_0)$. This follows from separate continuity of $S \times X \to X$.

(iii) $\Rightarrow$ (i) We know by Exercise 2.8.3 that the action $\Theta \times M \to M$ is continuous. Then $S \times M \to M$ is also continuous (because $S \times X \to \Theta \times X$ is continuous).

**Exercise 2.10.** Prove that $\Theta(V)$ and $L(V,V)$ are semitopological monoids with respect to the weak operator topology for every Banach space $V$.

**Proof.** Algebraically $\Theta(V)$ is a submonoid of $L(V,V)$. So, it is enough to show that $L(V,V)$ is a semitopological monoid with respect to the weak operator topology. Recall the definition of weak operator topology on $L(V,V)$. A net $s_i \tau_w$-converges to $s$ in $L(V,V)$ iff $f(s_i(v))$ converges to $f(s(v))$ in $\mathbb{R}$ for every $v \in V$, $f \in V^*$.

First we show that the right translations $\rho_t : L(V,V) \to L(V,V)$, $\rho_t(s) := st$ are continuous for every $t \in L(V,V)$. Indeed, let we have a convergence of nets $s_i \to s$. We have to show that $s_it \to st$. It is equivalent to see that $f(s_it(v))$ converges to $f(st(v))$ in $\mathbb{R}$. Or, equivalently, that $f(s_i(tv))$ converges to $f(s(tv))$ in $\mathbb{R}$. This is clear because $t(v) \in V$ (in the criterion we have the condition for every $v \in V$).

The case of left translations is similar by observing that $ft \in V^*$ for every $f \in V^*$ and $t \in L(V,V)$.

**Exercise 2.11.** For every Banach space $(V,||\cdot||)$ show that:

1. The semigroup $\Theta(V)_s$ (with SOT) is a topological monoid.
2. The subspace $Iso (V)_s$ of all linear onto isometries is a topological group.

**Proof.** We can apply Exercise 2.8.

**Exercise 2.12.** Let $\text{Unif}(Y,Y)$ be the set of all uniform self-maps of a uniform space $(Y,\mu)$. Denote by $\mu_{\text{sup}}$ the uniformity of uniform convergence on $\text{Unif}(Y,Y)$. Show that

1. under the corresponding topology $\text{top}(\mu_{\text{sup}})$ on $\text{Unif}(Y,Y)$ and the usual composition we get a topological monoid;
2. If $G$ is any subgroup of the monoid $\text{Unif}(Y,Y)$ then $G$ is a topological group;
3. For every subsemigroup $S \subseteq \text{Unif}(Y,Y)$ the induced action $S \times Y \to Y$ is continuous;

**Proof.** (Sketch) (1) Continuity of the multiplication. The elements $(st(x),s_0t_0(x))$ are "close enough" (uniformly for every $x \in X$) because we can force the pairs

$$(st(x),s_0t_0(x)),$$ $(s_0t(x),s_0t_0(x))$$

be sufficiently close.

(2) Let $G$ be any subgroup of the monoid $\text{Unif}(Y,Y)$. For the continuity of the inversion in $G$ note that if $(s_0(t),s(t))$ is small then $(t,s_0^{-1}s(t))$ is small for all $t \in Y$; now substituting $t = s^{-1}(x)$ we get

$$(t,s_0^{-1}s(t)) = (s^{-1}(x),s_0^{-1}(x)))$$

is small.

(3) Continuity of $S \times Y \to Y$ at point $(s_0,y_0)$.

The elements $(s_0y_0, s_0y)$ are close enough because we can force that $(s_0y_0, s_0y)$ and $(s_0y, sy)$ are sufficiently close.

**Exercise 2.13.** Let $Y$ be a compact space. Show that:

1. The semigroup $C(Y,Y)$ endowed with the compact open topology is a topological monoid;
2. The subset $H(Y)$ in $C(Y,Y)$ of all homeomorphisms $Y \to Y$ is a topological group;
3. For every subsemigroup $S \subseteq C(Y,Y)$ the induced action $S \times Y \to Y$ is continuous;
4. Furthermore, it satisfies the following remarkable minimality property. If $\tau$ is an arbitrary topology on $S$ such that $(S,\tau) \times Y \to Y$ is continuous then $\tau_{co} \subseteq \tau$.

**Proof.** (1), (2) and (3) Follow directly from the previous Exercise 2.12 taking into account that the uniformity of uniform convergence for compact $Y$ induces the compact open topology (see, for example, book of J. Kelley, General Topology).

(4) Let $(S,\tau) \times Y \to Y$ be continuous. Then by the compactness of $Y$ it is easy to see the following

$$\forall s_0 \in S \ \forall \varepsilon \in \mu_Y \ \exists U \in N_\tau(s_0) : (s_0y,sy) \in \varepsilon \ \forall y \in Y.$$  

This proves that the topology of compactness convergence $\tau_{co}$ is weaker than $\tau$.

---

**Exercise 2.14.** Let $G$ be a countable discrete group. Show that there exists a topological group embedding $G \to \text{Iso}(l_2)$.

**Proof.** It is equivalent to show that there exists a co-embedding. Indeed, for every (topological) group $G$ the inversion map $j : G \to G, j(g) = g^{-1}$ is a co-isomorphism. So, if $h : G \to P$ is a co-embedding then $h \circ j : G \to P$ is an embedding.

Let $S_{\mathbb{N}}$ be the symmetric group. Consider the natural left action $S_{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$. It induces the natural right action

$$l_2 \times S_{\mathbb{N}} \to l_2, \ (u,\sigma) \to u\sigma$$

where $(u\sigma)(k) = u(s(k))$ (we treat (the sequence) $u \in l_2$ as a function $u : \mathbb{N} \to \mathbb{R}$). Observe that it is an action “by permutations of coordinates”.

By Cayley’s theorem we have an embedding of abstract (discrete) groups $\nu : G \hookrightarrow S_G \cong S_{\mathbb{N}}$. Now consider the induced action of $G$ on $l_2$. More precisely, if $G := \{g_1,g_2,\cdots\}$ is an enumeration of $G$ then we have the action of $G$ on $\mathbb{N}$ according to its left translations $G \to G$. Consider the induced action of $G$ on $l_2$

$$\pi : l_2 \times G \to l_2.$$ 

Then we have:

1. $\pi$ is linear.
2. $\pi$ is an action by isometries.
3. $\pi$ induces a co-homomorphism $h : S_{\mathbb{N}} \to \text{Iso}(l_2)$
4. $h$ is injective.

Let $\sigma_1 \neq \sigma_2$ in $S_{\mathbb{N}}$. There exists $k \in \mathbb{N}$ such that $i = \sigma_1(k) \neq \sigma_2(k) = j$. Consider the vector $e_k \in l_2$ having the $k$-th coordinate $= 1$ and other coordinates $= 0$. Then $e_i = v_k\sigma_1 \neq v_k\sigma_2 = e_j$.

5. $h(G)$ is discrete.

It is equivalent to show that the identity operator $id = h(e)$ is isolated in $h(G)$ with respect to the strong operator topology. By the definition of strong operator topology one of the possible neighborhoods of $id$ in $h(G)$ is the following set

$$[id;e_1;\varepsilon = 1] \cap h(G) := \{h(g) \in h(G) : ||e_1 g - e_1|| < 1\}$$

where $e_1 := (1,0,0,\cdots)$. By the definition of $\pi$ it is clear that $[id;e_1;1] \cap h(G) = \{id\}$ because any nontrivial left translation $L_g : G \to G$ moves any point of itself. So, $h(G)$ is discrete because its neutral element is isolated.

---

**Exercise 2.15.** If $X$ is (locally) fragmented by $f : X \to Y$, where $(X,\tau)$ is a Baire space and $(Y,\rho)$ is a pseudometric space then $f$ is continuous at the points of a dense $G_\delta$ subset of $X$.
Proof. For a fixed $\epsilon > 0$ consider

$$O_\epsilon := \{\text{union of all } \tau\text{-open subsets } O \text{ of } X \text{ with } \text{diam}_\rho f(O) \leq \epsilon\}.$$  

The local fragmentability implies that $O_\epsilon$ is dense in $X$. Clearly, $\bigcap \{O_\frac{\epsilon}{n} : n \in \mathbb{N}\}$ is the required dense $G_\delta$ subset of $X$. \qed

Exercise 2.16. Let $(X, \tau)$ be a separable metrizable space and $(Y, \rho)$ a pseudometric space. Suppose that $f : X \to Y$ is a fragmented onto map. Then $Y$ is separable. Hint: use the idea of the Cantor-Bendixon theorem.

Proof. Assume (to the contrary) that the pseudometric space $(Y, \rho)$ is not separable. Then there exist an $\epsilon > 0$ and an uncountable subset $H$ of $Y$ such that $\rho(h_1, h_2) > \epsilon$ for all distinct $h_1, h_2 \in H$. Choose a subset $A$ of $X$ such that $f(A) = H$ and $f$ is bijective on $A$. Since $X$ is second countable the uncountable subspace $A$ of $X$ (in its relative topology) is a disjoint union of a countable set and a nonempty closed perfect set $M$ comprising the condensation points of $A$ (this follows from the proof of the Cantor-Bendixon theorem; see e.g. [?]). By fragmentability there exists an open subset $O$ of $X$ such that $O \cap M$ is nonempty and $f(O \cap M)$ is $\epsilon$-small. By the property of $H$ the intersection $O \cap M$ must be a singleton, contradicting the fact that no point of $M$ is isolated. \qed

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