CHRISTENSEN-EVANS THEOREM ON QUASI-INNERNESS OF π-DERIVATIONS — SEMINAR NOTES

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ABSTRACT. The notes are concerned with the famous Christensen-Evans theorem characterising generators of norm continuous completely positive semigroups on C^*-algebras. A full(ish) proof is given.

In these notes we are interested in properties of generators of semigroups of operators on operator algebras satisfying certain algebraic properties. Let us start with quoting the fundamental result on generators of homomorphisms:

Theorem 0.1. *Let B be a Banach algebra, $\{P_t : t \geq 0\}$ a c_0-semigroup on B. The following are equivalent:*

(i) *each P_t is a homomorphism;*
(ii) *the generator L of $\{P_t : t \geq 0\}$ is a derivation, that is $Dom(L)$ is a subalgebra of B and for all $a, b \in Dom(L)$*
$$L(ab) = aL(b) + L(a)b.$$

Characterisation of all derivations is a difficult (if not hopeless) task. Further we are interested in the bounded case, but instead of looking at homomorphisms we will extend the context to (completely) positive semigroups.

Further A will usually denote a C^*-algebra, and M will be used for von Neumann algebras. The notation $\overline{E}$ is used for the ultraweak closure of a set $E \subset B(\mathsf{h})$ (or the usual norm closure if E is a subset of h).

1. ALGEBRAIC PROPERTIES OF GENERATORS OF NORM CONTINUOUS (COMPLETELY) POSITIVE SEMIGROUPS

Let us start discussing the basic algebraic properties of generators of positive semigroups. Fix for this section a C^*-algebra A.

Lemma 1.1. *Let $L : \mathsf{A} \to \mathsf{A}$ be bounded and real (the latter means that $L(a^*) = L(a)^*$ for all $a \in \mathsf{A}$). The following are equivalent:*

(i) *the semigroup generated by L is positive (i.e. consists of positive operators);*
(ii) *L is conditionally positive: for all $a, b \in \mathsf{A}$ if $ab = 0$ then $b^*L(a^*a)b \geq 0$;*

Additionally, if A is unital they are also equivalent to the condition

(iii) *for all $a \in \mathsf{A}_h$*
$$L(a^2) + aL(1)a \geq L(a)a + aL(a).$$

Proof. The implication (i)⇒(ii) is trivial; (ii)⇒(i) may be proved via looking at the resolvent family. To deduce (iii) from (i) one considers the (unital) semigroup generated by $L' = L + \{-\frac{1}{2}L(1), \cdot\}$. The Kadison-Schwarz inequality for the latter yields the desired inequality. Finally the implication (iii)⇒(ii) may be deduced by working with states on A and applying several times Schwarz inequality. □

Definition 1.2. Let B be a C^*-algebra. A linear map $T : \mathsf{A} \to \mathsf{B}$ is called completely positive if for each $n \in \mathbb{N}$ the map $T^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{B})$ defined by $T^{(n)}[a_{ij}]_{i,j=1}^n = [T(a_{ij})]_{i,j=1}^n$ is positive. It is called conditionally completely positive if for all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in \mathsf{A}, b_1, \ldots, b_n \in \mathsf{B}$ such that $\sum_{i=1}^n a_i b_i = 0$ the following inequality holds:

$$\sum_{i,j=1}^{n} b_i^* T(a_i^* a_j) b_j \geq 0. \tag{1}$$

One can show that in fact $T : \mathsf{A} \to \mathsf{B}$ is completely positive if and only if condition (1) holds for arbitrary $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}, b_1, \ldots, b_n \in \mathsf{B}$.

Definition 1.3. Let X be a set, h a Hilbert space. A map $K : X \times X \to B(\mathsf{h})$ is called a positive-definite kernel (on X) if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ the matrix $[K(x_i, x_j)]_{i,j=1}^n \in M_n(B(\mathsf{h}))$ is positive.

Lemma 1.4. *Let $L : \mathsf{A} \to \mathsf{A}$ be bounded and real. The following are equivalent:*

(i) *L is conditionally completely positive;*

(ii) *for all $a \in \mathsf{A}$ the map $K_a : \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ defined by ($s, t \in \mathsf{A}$)*

$$K_a(s, t) = L(s^* a^* a t) + s^* L(a^* a) t - L(s^* a^* a) t - s^* L(a^* a t)$$

is a positive-definite kernel;

(iii) *the map $K : (\mathsf{A} \times \mathsf{A}) \times (\mathsf{A} \times \mathsf{A}) \to \mathsf{A}$ defined by ($s_1, s_2, t_1, t_2 \in \mathsf{A}$)*

$$K((s_1, s_2), (t_1, t_2)) = L(s^* t_1^* t_2 t) + s_1^* L(s_2^* t_1) t_2 - L(s_1^* s_2^* t_1) t_2 - s_1^* L(s_2^* t_1 t_2)$$

is a positive-definite kernel.

Proof. Here (iii)⇒(ii) is trivial, (ii)⇒(i) is a simple computation + the existence of approximate identity, and (i)⇒(iii) is obtained by 'doubling the dimension' and judicious choice of elements used in the conditional complete positivity condition. □

Theorem 1.5. *Let $L : \mathsf{A} \to \mathsf{A}$ be bounded and real. The following are equivalent:*

(i) *the semigroup generated by L is completely positive (i.e. consists of positive operators);*

(ii) *L is conditionally completely positive.*

Proof. As in Lemma 1.1, (i)⇒(ii) is trivial. Going to the double dual allows to assume that A is unital, and then it is enough to compares condition (iii) of Lemma 1.1 applied to the semigroup $P_t^{(n)} : M_n(\mathsf{A}) \to M_n(\mathsf{A})$ with the condition (ii) of Lemma 1.4. □

The imprtance of the notion of a positive definite kernel lies in the following construction:

Theorem 1.6 (Minimal Kolmogorov Construction)**.** *Let X be a set, h a Hilbert space and $K : X \times X \to B(\mathsf{h})$ a positive-definite kernel. Then there exist (unique up to unitary equivalence) a Hilbert space k and a map $\lambda : X \to B(\mathsf{h}; \mathsf{k})$ such that*

(i) *$K(x, y) = \lambda(x)^* \lambda(y)$ for all $x, y \in X$;*

(ii) *$\mathsf{k} = \overline{Lin\{\lambda(x)\xi : x \in X, \xi \in \mathsf{h}\}}$.*

The theorem above allows for the first algebraic characterisation of the generators of completely positive semigroups and in particular provides the connection with derivations on A.

Lemma 1.7. *Let* $\mathsf{A} \subset B(\mathsf{h})$ *and suppose that* $L : \mathsf{A} \to \mathsf{A}$ *is bounded, real, and conditionally completely positive. There exist a representation of* A*, denoted further by* $\{\pi, \mathsf{k}\}$ *and a linear map* $V : \mathsf{A} \to B(\mathsf{h}; \mathsf{k})$ *such that for all* $a, b, c \in \mathsf{A}$

(i) $L(abc) + aL(b)c - aL(bc) - L(ab)c = V(a^*)^*\pi(b)V(c), \quad a, b, c \in \mathsf{A}$;
(ii) $V(ab) = \pi(a)V(b) + V(a)b, \quad a, b \in \mathsf{A}$;
(iii) $\mathsf{k} = \overline{Lin\{\pi(a)V(b)\xi : a, b \in \mathsf{A}, \xi \in \mathsf{h}\}}$

Proof. The starting point is the Kolmogorov construction applied to the positive definite kernel from Lemma 1.4 (iii). This provides a Hilbert space k and a Kolmogorov map $\lambda : \mathsf{A} \times \mathsf{A} \to B(\mathsf{h}; \mathsf{k})$. If A is unital, maps V and π are given by the formulas:

$$\pi(a)\lambda(a_1, a_2) = \lambda(aa_1, a_2), \quad V(a) = \lambda(a, 1)$$

$(a, a_1, a_2 \in \mathsf{A})$; otherwise limit constructions exploiting either the multiplier algebra or approximate unit are used. The algebraic properties of π and V follow essentially from the uniqueness of the minimal Kolmogorov construction. □

All the results above are essentially algebraic, and may be found (mostly with proofs) in [EvL].

2. Automatic continuity properties of derivations on C^*-algebras

In this section we discuss basic continuity properties of derivations on C^*- and W^*-algebras.

Theorem 2.1 ([Ri1])**.** *Let* A *be a* C^**-algebra and* X *a Banach* A*-bimodule. If a linear map* $\delta : \mathsf{A} \to X$ *is a derivation, then it is continuous.*

Proof. Define $J = \{a \in \mathsf{A} : t \to \delta(at) \text{ is continuous}\} = \{a \in \mathsf{A} : t \to a\delta(t) \text{ is continuous}\}$. It is a twosided ideal, and $\delta|_J$ is continuous. Consider the algebra A/J. It has to be finite-dimensional - otherwise it would contain an infinite-dimensional commutative C^*-algebra and therefore would contain a selfadjoint operator with infinite spectrum. Basic considerations based on the spectral theorem would allow then to produce contradiction. □

A Banach A-bimodule X is called a dual bimodule if X is a dual Banach space and the maps $m \to ma$, $m \to am$ are weak*-continuous for all $a \in \mathsf{A}$. If additionally $\mathsf{A} \subset B(\mathsf{h})$ and the maps $a \to ma$, $a \to am$ are ultraweak-weak*-continuous for each $m \in X$, X is called normal.

Theorem 2.2 ([Ri1])**.** *Let* (π, h) *be a faithful representation of a* C^**-algebra* A *and let* X *be a dual normal* $\overline{\pi(\mathsf{A})}$*-bimodule (*$\overline{\pi(\mathsf{A})}$ *denotes the ultraweak closure of* $\pi(\mathsf{A})$*). If a linear map* $\delta : \pi(\mathsf{A}) \to X$ *is a derivation, then it is ultraweakly-weak* continuous.*

Proof. The proof is based on considering the universal representation π_u of A and using two fundamental facts from the theory of universal representations: each two-sided ideal in $\overline{\pi_u(\mathsf{A})}$ corresponds to a projection in the centre of $\overline{\pi_u(\mathsf{A})}$ and each continuous functional on $\pi_u(\mathsf{A})$ is ultraweakly continuous. These after certain technical manipulations allow to reduce the problem to the norm continuity, which was established in the previous theorem. □

Lemma 2.3 ([Ri2]). *Suspose that there exists a subgroup of the unitary group of* A*, which is amenable and whose norm closed linear span is the whole* A *Then every derivation of* A *with values in a dual* A*-bimodule* X *is inner.*

Proof. Suppose that V is a subgroup with the properties described above. Amenability and the fact that M is dual allows to define an 'invariant' contractive linear mapping $\overline{\mu} : C_b(V;X) \to X$. Then if $\delta : V \to X$ is a derivation, define $f : V \to X$ by $f(v) = v^*\delta(v)$ $(v \in V)$. The element $m = \overline{\mu}(f)$ implements δ. □

Corollary 2.4. *Suppose that* A *is either commutative or (isomorphic to) the* I_∞ *factor. Then every derivation of* A *with values in a dual* A*-bimodule* X *is inner.*

Proof. If A is commutative, its unitary group is clearly amenable. If A is a hyperfinite factor, an approximation argument is needed. □

3. Property D and innerness of derivations on properly infinite von Neumann algebras

Before we pass to the formulation and proof of the C-E Theorem in the final section, we need to discuss a series of results due to E. Christensen closely related to certain perturbation theorems for maps on C^*-algebras. The techniques here are typical for the theory of von Neumann algebras.

Definition 3.1. Let $M \subset B(\mathsf{h})$ be a von Neumann algebra. M has (Schwartz) property P if for any $x \in B(\mathsf{h})$

$$\overline{\text{co}}\{u^*xu : u \in \mathcal{U}(M)\} \cap M' \neq \emptyset.$$

M has property D_0 if there exists $k > 0$ such that for all $x \in B(\mathsf{h})$

$$\|\delta_x|_M\| \leq d(x, M') \leq k\|\delta_x|_M\|,$$

where

$$\delta_x(y) = xy - yx, \quad y \in B(\mathsf{h}).$$

Finally M has property D if $M \otimes I_{l^2} \subset B(\mathsf{h} \otimes l^2)$ has property D_0.

The lemma below explains the immediate connection between the properties above and the derivations.

Lemma 3.2. *Let* $M \subset B(\mathsf{h})$ *be a von Neumann algebra. If every derivation on* M *with values in* $B(\mathsf{h})$ *is inner then* M *has property* D_0.

Proof. Consider the vector space $B(\mathsf{h})/M'$ with the norm $|||[x]||| = \|\delta_x|_M\|$. Then $(B(\mathsf{h})/M', |||\cdot|||)$ is isometrically isomorphic with the space of all inner derivations $\delta : M \to B(\mathsf{h})$. If the latter coincides with the space of all derivations, it is complete. Therefore the norm $(|||\cdot|||)$ has to be equivalent to the canonical norm on $B(\mathsf{h})/M'$. □

Recall two following properties of properly infinite von Neumann algebras. Every properly infinite vN algebra contains a copy of the I_∞ factor; more specifically it is isomorphic to a tensor product of the I_∞ factor with another von Neumann algebra (this can be deduced from Proposition V.1.22 in [Tak] and Proposition 4.12 of [StZ]). Moreover if M is properly infinite then every normal functional on M' is a vector functional (Theorem 8.16 of [StZ]). This(*) allows to obtain one more automatic continuity results. Recall that if N is a von Neumann algebra then the ultrastrong topology on N is the one induced by the family of seminorms $x \to |\phi(x^*x)|^{\frac{1}{2}}, \phi \in N_*$.

Lemma 3.3. *Let $M \subset B(\mathsf{h})$ be a properly infinite von Neumann algebra. Every derivation $\delta : M \to B(\mathsf{h})$ is ultrastrongly-ultrastrongly continuous.*

Proof. We can assume that δ is nonzero. As M is assumed to be properly infinite, it contains a copy of the I_∞ factor, call it R. Using Corollary 2.4 we may assume that δ vanishes on R. Further R contains an infinite family of isometries with orthogonal ranges: there exists a sequence $(v_i)_{i=1}^\infty$ such that $v_i \in R$, $v_i^* v_i = I$, $v_i^* v_j = 0$ for all $i, j \in \mathbb{N}, i \neq j$. Exploiting the isometries above one can show that for any $n \in \mathbb{N}, x_1, \ldots, x_n \in M$

$$\| \sum_{j=1}^n x_j^* x_j \| \geq \|\delta\|^{-2} \| \sum_{j=1}^n \delta(x_j)^* \delta(x_j) \|.$$

Let now $\phi \in B(\mathsf{h})_*^+$ be nonzero and define $S_1 = \{x \in M : \phi(\delta(x)^*\delta(x)) = 1\}$, $S_2 = \mathrm{co}\{x^*x : x \in S_1\}$. Then for any $y \in S_2$ there is $\|y\| \geq \|\phi\|^{-1}\|\delta\|^{-2}$ and a Hahn-Banach type argument yields the existence of a hermitian functional $\psi \in M^*$ such that $\psi(y) \geq 1$ whenever $y \in S_2$ (hermitianity may be obtained as S_2 contains only selfadjoint operators). Let ω be the positive part of ψ. It is now easy to check that $\phi(\delta(x)^*\delta(x)) \leq \omega(x^*x)$ for all $x \in M$. Let $\omega = \omega_n + \omega_s$ denote the decomposition of ω into normal and singular parts. Using the fact that the kernel of each singular positive functional contains a family of projections increasing to I (Theorem 3.8 in [Tak]) and Cauchy-Schwarz inequality we can deduce that actually $\phi(\delta(x)^*\delta(x)) \leq \omega_n(x^*x)$ for all $x \in M$. This suffices to conclude the proof. □

Lemma 3.4. *If $M \subset B(\mathsf{h})$ is the I_∞ factor then M has property P.*

Proof. Each type I_n factor (an isomorphic copy of M_n) has property P, as its unitary group is compact and averaging any operator on h with respect to the action of the unitary group (equipped with its Haar measure) provides an element in the commutant of M. To obtain the conclusion it suffices therefore to show that the property P is stable under taking unions (in the von Neumann category). This follows from a straightforward weak*-compactness argument. □

Lemma 3.5. *If $M \subset B(\mathsf{h})$, $x \in B(\mathsf{h})$ and $(x \cup x^* \cup M)'$ is properly infinite, then $\|\delta_x|_M\| = 2d(x, M')$.*

Proof. Write $B = (M' \cup x \cup x^*)''$. Then each normal functional on B is a vector functional. Moreover if $f \in B_*$ vanishes on M' one can assume that $f(\cdot) = \langle \xi, \cdot \eta \rangle$ for some $\xi, \eta \in \mathsf{h}$ such that $p\xi = 0$, $p\eta = \eta$, where $p \in M$ denotes the projection onto the subspace $[M'\eta]$. Exploiting the (Hahn-Banach type) formula

$$d(x, M') = \sup\{|f(x)| : f \in B_*, \|f\| = 1, f|_{M'} = 0\}$$

yields the desired conclusion. □

Lemma 3.6. *If $M \subset B(\mathsf{h})$ has property P then for all $x \in B(\mathsf{h})$*

$$\|\delta_x|_M\| \leq 2d(x, M') \leq 2\|\delta_x|_M\|.$$

Proof. Fix $x \in B(\mathsf{h})$. For any $u \in \mathcal{U}(M)$ there is

$$\|u^*xu - x\| = \|\delta_x(u)\| \leq \|\delta_x|_M\|.$$

Therefore for all $z \in \overline{\text{co}}\{u^*xu : u \in \mathcal{U}(M)\}$ there is $\|z-x\| \leq \|\delta_x|_M\|$ and the result follows. □

The following result is key to proving the crucial statement of this section, Theorem 3.10.

Theorem 3.7. *Let $M \subset B(\mathsf{h})$ be a von Neumann algebra. If M is properly infinite then for all $x \in B(\mathsf{h})$*

$$\|\delta_x|_M\| \leq 2d(x, M') \leq 3\|\delta_x|_M\|.$$

In particular M has property D.

Proof. Let C be a copy of the I_∞ factor and $M = C\overline{\otimes}D$ for some other von Neumann algebra D. Arguing as in the last lemma we see that there is $y \in \overline{\text{co}}\{u^*xu : u \in \mathcal{U}(M)\} \cap (C\overline{\otimes}I)'$ such that $\|x-y\| \leq \|\delta_x|_M$. Further for any $u \in \mathcal{U}(C)$ and $a \in M$

$$\|u^*xua - au^*xu\| = \|xuau^* - uau^*x\| \leq \|\delta_x|_M\|\|a\|.$$

This means that $\|\delta_{u^*xu}|_M\| \leq \|\delta_x|_M\|$, so also $\|\delta_y|_M\| \leq \|\delta_x|_M\|$. Note now that as $y \in (C\overline{\otimes}I)' = I\overline{\otimes}D$, the algebra $(M \cup y \cup y^*)'$ is equal to $C\overline{\otimes}N$, where N is a von Neumann subalgebra of D. By Lemma 3.5

$$d(y, M') = \frac{1}{2}\|\delta_y|_M\| \leq \frac{1}{2}\|\delta_x|_M\|,$$

so that

$$d(x, M') \leq \|x-y\| + d(y, M') \leq \frac{3}{2}\|\delta_x|_M\|.$$

□

Lemma 3.8. *Let $M \subset B(\mathsf{h})$ have property D_0 and let $\delta : M \to B(\mathsf{h})$ be a derivation. There exists a projection $p \in M'$ such that*

$$\{x \in M' : x\delta \text{ is inner }\} = M'p.$$

Moreover if δ is hermitian then for any $q, s \in P_{M'}$ the derivation $q\delta s$ is inner if and only if $q \geq p^\perp$ and $s \geq p^\perp$.

Proof. For the first part note that the set $I := \{x \in M' : x\delta \text{ is inner }\}$ is a left ideal of M'. Whenever (q_α) is an increasing set of projections, its supremum is in I - this is where the property D_0 is used, as we need a common bound on operators implementing $q_\alpha\delta$. Put $p = \sup\{q \in P_{M'} : q \in I\}$. Note that if $x \in I$ then its right support $r(x) = \chi_{\mathbb{R}\setminus\{0\}}(x^*x)$ is also in I (this exploits the approximation by polynomials in x^*x with the vanishing constant term). This suffices to deduce that $I = M'p$.

For the second part note that if δ is hermitian and $x \in M'$ then $x\delta$ is inner if and only if δx^* is inner, so that $\{x \in M' : \delta x \text{ is inner }\} = pM'$. The rest is easy. □

The next lemma shows that if an algebra has property D_0 then every derivation on A induced by a commutator with an unbounded operator is actually inner (i.e. the unbounded operator may be replaced by a bounded one).

Lemma 3.9. *Let $\mathsf{A} \subset B(\mathsf{h})$ be a unital C^*-algebra with property D_0 and let $T : D_T \to \mathsf{h}$ be a closed densely defined operator such that its domain D_T is left invariant by operators in A. Suppose that for each $a \in \mathsf{A}$ the operator $aT - Ta$ is bounded. Then there exists $x \in B(\mathsf{h})$ such that for all $a \in \mathsf{A}$*

$$\overline{aT - Ta} = ax - xa.$$

Proof. Note first that the map $\delta : \mathsf{A} \to B(\mathsf{h})$ defined by

$$\delta(a) = \overline{aT - Ta}, \quad a \in \mathsf{A},$$

is a derivation, so in particular by Theorem 2.1 it is bounded.

Let S be a closed densely defined operator on a Hilbert space k. Then the operators $(I+S^*S)^{-1}$, $S(I+S^*S)^{-1}$ are well defined contractions. It is clear that $\mathrm{Ran}(I+S^*S)^{-1} \subset \mathrm{Dom}(S^*S)$. Moreover it can be shown that $\mathrm{Ran}(\overline{I+S^*S)^{-1}S^*}) \subset \mathrm{Dom}(S)$. Finally the projection on the graph of S in $\mathsf{k} \oplus \mathsf{k}$ is given by the following formula:

$$P_S = \begin{bmatrix} (I+S^*S)^{-1} & \overline{(I+S^*S)^{-1}S^*} \\ S(I+S^*S)^{-1} & S\overline{(I+S^*S)^{-1}S^*} \end{bmatrix}.$$

The last formula follows from the fact that the graph of S is the orthogonal complement of the image of the graph of S^* via the unitary $U : \mathsf{k} \oplus \mathsf{k} \to \mathsf{k} \oplus \mathsf{k}$, $U(k_1, k_2) = (k_2, -k_1)$. Moreover $S|_{\mathrm{Dom}(S^*S)}$ is closable and $\overline{S}|_{D_{S^*S}} = S$. Full proofs of all these statements can be found for example in [RS-N].

Let us return to the context of our lemma. Let $[T]$ denote the range projection of T, let $T = V(T^*T)^{\frac{1}{2}}$ be the polar decomposition of T. For each $s > 0$ write P_{sT} for the orthogonal projection onto the graph of sT and let $K_s = sT(I + s^2T^*T)^{-1}$. Further $\overline{K_sT^*T}|_{\mathrm{Dom}(T^*T)} = K_sT^*T$, $\overline{K_sT^*TT^*TK_s}$ and $\overline{K_sT^*T}|_{\mathrm{Dom}(T)} = \overline{K_sT^*T}$. The first one is easy to see, the second follows from the fact that K_s is a function of a selfadjoint operator T^*T and the third is a consequence of $\overline{T}|_{D_{T^*T}} = T$ and obvious closability of K_sT^*T. Note also the following equalities:

$$\begin{aligned} I_{\mathsf{h}} - K_s &= s^2T^*TK_s, \\ \overline{VK_sV^*T} &= V\overline{K_sV^*T} = V\overline{K_s(T^*T)^{\frac{1}{2}}} = V(T^*T)^{\frac{1}{2}}K_s = TK_s, \\ [T] - VK_sV^* &= V(I_{\mathsf{h}} - K_s)V^* = V(s^2T^*TK_s)V^* \\ &= s^2V(T^*T)^{\frac{1}{2}}\overline{K_s(T^*T)^{\frac{1}{2}}}V^* = s^2T\overline{K_sT^*}. \end{aligned}$$

Together with the described earlier formula for the projection on the graph of a closed densely defined operator they imply the following

$$\begin{aligned} P_{sT} &= \begin{bmatrix} K_s & s\overline{K_sT^*} \\ sTK_s & s^2T\overline{K_sT^*} \end{bmatrix}, \\ I_{\mathsf{h}\oplus\mathsf{h}} - P_{sT} &= \begin{bmatrix} s^2T^*TK_s & -s\overline{K_sT^*} \\ -sTK_s & (I_{\mathsf{h}} - [T]) + VK_sV^* \end{bmatrix}. \end{aligned}$$

Let $\mathsf{A}^{(2)} := \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathsf{A}\}$. For all $X_a = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in \mathsf{A}^{(2)}$ we have (again exploiting the formulas listed before)

$$\begin{aligned} ((I - P_{sT})XP_{sT})_{11} &= s^2T^*TK_saK_s - s\overline{K_sT^*}asTK_s = s\overline{K_sT^*}(sTa - asT)K_s, \\ ((I - P_{sT})XP_{sT})_{12} &= s^2T^*TK_sas\overline{K_sT^*} - s\overline{K_sT^*}as^2T\overline{K_sT^*} \\ &= s\overline{K_sT^*}(sTa - asT)s\overline{K_sT^*}, \\ ((I - P_{sT})XP_{sT})_{21} &= -sTK_saK_s + (I_{\mathsf{h}} - [T] + VK_sV^*)asTK_s \\ &= VK_sV^*(asT - sTa)K_s + (I_{\mathsf{h}} - [T])(asT - sTa)K_s, \\ ((I - P_{sT})XP_{sT})_{22} &= -sTK_sas\overline{K_sT^*} + (I_{\mathsf{h}} - [T] + VK_sV^*)as^2T\overline{K_sT^*} \\ &= VK_sV^*(asT - sTa)s\overline{K_sT^*} + (I_{\mathsf{h}} - [T])(asT - sTa)s\overline{K_sT^*}. \end{aligned}$$

This implies that for all $i, j = 1, 2$

$$\|((I - P_{sT})XP_{sT})_{ij}\| \leq 2s\|\delta(a)\| \leq 2s\|\delta\|\|X_a\|.$$

Further

$$\|\delta_{P_{sT}}|_{\mathsf{A}^{(2)}}\| = \sup\{\|(I_{\mathsf{h}\oplus\mathsf{h}} - P_{sT})XP_{sT}\| : X \in \mathsf{A}^{(2)}\} \leq 4s\|\delta\|.$$

Analysing the $2,1$-coefficient of $P_{sT}X - XP_{sT}$ we obtain that

$$\|\delta_{TK_s}|_{\mathsf{A}}\| \leq 4\|\delta\|.$$

As A was assumed to have property D_0 there exists $C > 0$ such that for all $s > 0$ there exists $x_s \in B(\mathsf{h})'$ such that $\|x_s\| \leq C$, $x_s + TK_s \in \mathsf{A}'$. Choose a subnet $\{s_\iota : \iota \in \mathcal{J}\}$ convergent to 0 and such that $\{x_{s_\iota} : \iota \in \mathcal{J}\}$ weak* converges to an operator $x \in B(\mathsf{h})$. Then for any $\xi \in \mathrm{Dom}_T$, $\eta \in \mathsf{H}$, and a unitary $u \in \mathsf{A}$

$$\langle (T + x)\xi, \eta\rangle = \lim_{\iota\in\mathcal{J}} \langle (TK_{s_\iota} + x_{s_\iota})u\xi, u\eta\rangle = \langle (T + x)u\xi, u\eta\rangle.$$

This implies that

$$xa - ax = \overline{aT - Ta}, \quad a \in \mathsf{A}.$$

□

We are ready to formulate and prove the main theorem of the section.

Theorem 3.10 ([Ch$_2$])**.** *Let $M \subset B(\mathsf{h})$ be a properly infinite von Neumann algebra. Every derivation $\delta : M \to B(\mathsf{h})$ is inner.*

Proof. Using Lemma 2.4 we can assume that $\delta|_{Z(M)} = 0$. Considering separately $\delta + \delta^\dagger$ and $i(\delta - \delta^\dagger)$ we can assume that δ is hermitian. As M has property D_0, exploiting Lemma 3.8 we can further assume that $\{x \in M' : x\delta \text{ is inner }\} = \{0\}$. Finally considering the diagonal derivation δ^∞ on $M \otimes I_{l^2}$ we can also assume that M' is also properly infinite.

Suppose first that M has a cyclic vector $\xi \in \mathsf{h}$. By Lemma 3.3 the map $\Phi : M \to B(\mathsf{h}) \otimes M_2$ given by

$$\Phi(r) = \begin{bmatrix} r & 0 \\ \delta(r) & r \end{bmatrix}, \quad r \in M,$$

is an ultrastrongly continuous homomorphism. This means that there exists $\psi \in M_*$ such that for all $r \in M$

$$\|r\xi\|^2 + \|\delta(r)\xi\|^2 = \left\| \begin{bmatrix} r & 0 \\ \delta(r) & r \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\|^2 \leq \psi(r^*r). \tag{2}$$

Let $\eta \in \mathsf{h}$ be such that $\psi(r) = \langle \eta, r\eta\rangle$ $(r \in M)$. Put $C = \Phi(M)$ and let

$$L = \overline{\left\{ \begin{bmatrix} r & 0 \\ \delta(r) & r \end{bmatrix} \begin{bmatrix} \xi \\ 2\eta \end{bmatrix} : r \in M \right\}}.$$

Further let $q \in B(\mathsf{h})$ be the orthogonal projection onto $L \cap (0 \oplus \mathsf{h})$. One can show that $q \in M'$. Put

$$G = \begin{bmatrix} I & 0 \\ 0 & q^\perp \end{bmatrix} L.$$

Then G is the graph of a closed densely defined operator T, which satisfies the following property: if $(\gamma_1, \gamma_2) \in L$ then $\gamma_1 \in \mathrm{Dom}(T)$ and $T\gamma_1 = q^\perp\gamma_2$. If h is nontrivial then q cannot be equal to I_h. Indeed, as the inequality (2) implies that for all $r \in M$

$$\|\delta(r)\xi + 2r\eta\| \geq \|r\xi\|,$$

so that for all $(\gamma_1, \gamma_2) \in L$ there is $\|\gamma_1\| \geq \|\gamma_2\|$. But if $q = I_{\mathsf{h}}$ then $L = \mathsf{h} \oplus \mathsf{h}$ and we reach contradiction.

Define a subalgebra of $B(\mathsf{h} \oplus \mathsf{h})$ by

$$D = \left\{ \begin{bmatrix} r & 0 \\ q^{\perp}\delta(r) & r \end{bmatrix} : r \in M \right\}.$$

Then G is left invariant by D so that for any $\gamma \in \mathrm{Dom}(T)$ and $r \in M$

$$r\gamma \in \mathrm{Dom}(T). \quad \gamma = q^{\perp}\delta(x)\gamma) + xT\gamma.$$

Lemma 3.9 implies therefore that $q^{\perp}\delta$ is inner and as $q^{\perp} \neq 0$ we have reached the contradiction unless $\mathsf{h} = \{0\}$. In the latter case δ is obviously inner.

It remains to show that the cyclicity assumption may be dropped. Let $\mathcal{P}$ denote the family of all projections $p \in P_{M'}$ such that p is σ-finite with respect to M' and M'_p is properly infinite. Given $p, q \in \mathcal{P}$ the projection $p \vee q$ belongs to $\mathcal{P}$, moreover for any σ-finite projection $s \in M'$ there is $p \in \mathcal{P}$ such that $p \geq s$.

Consider $p \in \mathcal{P}$. As M_p is properly infinite (consider the tensor decomposition involving the I_∞ factor), M'_p has a faithful vector state ω_ξ. Vector ξ is then cyclic for M_p and the derivation $\delta^p : M_p \to B(p\mathsf{h})$ given by $\delta^p(rp) = p\delta(r)p$ $(r \in M)$ is implemented by $x_p \in B(p\mathsf{h})$, $\|x_p\| \leq 2\|\delta\|$ (seee Lemma 3.7). Considering all x_p as acting on $B(\mathsf{h})$ we obtain a bounded family indexed by $\mathcal{P}$. Further as we can find in $\mathcal{P}$ a net strongly convergent to I and

$$p\delta(r)p = x_p r - r x_p, \quad p \in \mathcal{P}, r \in M,$$

a standard weak* approximation argument ends the proof.

□

4. Christensen-Evans Theorem - formulation and proof

Theorem 4.1 (Christensen-Evans Theorem). *Let $\mathsf{A} \subset B(\mathsf{h})$, $\{\pi, \mathsf{k}\}$ be a representation of A and $V : \mathsf{A} \to B(\mathsf{h}; \mathsf{k})$ a linear map such that for all $a, b \in \mathsf{A}$*

(i) $V(a)^* V(b) \in \overline{\mathsf{A}}$
(ii) $V(ab) = \pi(a)V(b) + V(a)b$.

Then there exists an operator $r \in \overline{Lin\{V(a)b : a, b \in \mathsf{A}\}}$ such that

$$V(a) = ra - \pi(a)r, \quad a \in \mathsf{A}.$$

Proof. We may assume that $I_{\mathsf{h}} \in \overline{\mathsf{A}}$ (otherwise consider what happens only on the 'support' of $\overline{\mathsf{A}}$). Let $\delta : \hat{\mathsf{A}} := (\mathrm{id}_{\mathsf{A}} \oplus \pi)(\mathsf{A}) \to B(\mathsf{h} \oplus \mathsf{k})$ be given by the formula

$$\delta\left(\begin{bmatrix} a & 0 \\ 0 & \pi(a) \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ V(a) & 0 \end{bmatrix}, \quad a \in \mathsf{A}.$$

Let $p = P_{\mathsf{h}} \in B(\mathsf{h} \oplus \mathsf{k})$, $\mathsf{B} = C^*(\hat{\mathsf{A}}, \delta(\hat{\mathsf{A}}), p)$, $\mathsf{M} = \overline{\hat{\mathsf{A}}}$, $\mathsf{N} = \overline{\mathsf{B}}$. To prove the theorem it is enough to establish that δ is implemented by some $x \in (I-p)\mathsf{N}p$ - this statement follows from the easily established fact that N can be explicitly described as a certain algebra of matrices whose bottom left corners lie in $\overline{V(\mathsf{A})\mathsf{A}}$. By Lemma 2.3 one can assume that $\delta|_{Z(M)} = 0$. This allows to exploit the unique decomposition of M into finite and properly infinite parts and consider both cases separately.

Case 1: M finite. Denote the unitary group of M by $U(M)$ and let $C = \overline{co}\{u^*\delta(u) : u \in U(M)\}$. For each $u \in U(M)$ let $T_u : C \to C$ be an affine map given by $T(c) = u^* c u + u^* \delta(u)$. It is easy to check that $\{T_u, u \in U(M)\}$ is a group.

As C is weak*-compact, and the existence of the center-valued trace on M allows to prove that the group in question is noncontracting, Ryll-Nardzewski fixed point theorem implies that there is $c_0 \in C$ which is a fixed point for all T_u. The element c_0 may be easily shown to implement δ.

Case 2: M properly infinite. By Theorem 3.10 there exists $y \in B(\mathsf{h} \oplus \mathsf{k})$ implementing δ. It remains to show that one can replace it with an operator in N. The proof is based on arguments of [Kad]. Note first that the implementing operator may be chosen so that it has *totally minimal norm*, that is for all $e \in Z(N)$ the operator ye has minimal norm among those which implement the derivation $m \to \delta(m)e$. This exploits first the weak-operator lower semicontinuity of the norm (so that for a given inner derivation there is an operator implementing it which has a minimal norm), and then uses the weak*-limit argument for a net of operators, each of which has minimal norm with respect to a given decomposition of 1_N into finite central projections.

In the next step we observe that as $p \in M'$ if y has totally minimal norm, we can aditionally assume that $y = p^{\perp}yp$. Further let $e \in P_{N'}$. There exists a family $\{e_j : j \in \mathcal{J}\}$ of pairwise orthogonal projections in N' such that $\sum_{j\in\mathcal{J}} e_j = c(e)$ and $e_j \preceq e$ for each $j \in \mathcal{J}$. Considering the element $y_e = \sum_{j\in\mathcal{J}} v_j y v_j^*$, where each v_j is a partial isometry with the initial space contained in $e\mathsf{h}$ and target space equal to $e_j\mathsf{h}$, we can show that y_e implements the derivation $m \to \delta(m)e$. Now the total minimal norm assumption implies that $\|yc(e)\| \leq \|y_e\|$ and so

$$\|ye\| = \|eye\| = \|yc(e)\|.$$

It is easy to check that $\delta : N' \to M'$. Suppose that $y \notin N$. There must then exist $e \in N'$ such that $[y, e] \neq 0$. One can assume that $M' \ni e^{\perp}ye \neq 0$. The operator $ey^*e^{\perp}ye$ is a nonzero element of $pM'p = (M')_p = (M_p)' = (N_p)' = N'_p$. This implies the existence of (a unique) $x \in N'_+$ such that $xp = ey^*e^{\perp}ye$and $xc(p) = x$. Let $r > 0$ and $f \in P_{N'}$ be such that $x \geq rf$. Then $f \leq c(p)$ and $ep \geq fp$. This suffices to deduce that

$$\|fy^*eyf\| = \|eyf\|^2 \geq \|yf\|^2 = \|p(fy^*eyf)p + pfxfp\| > \|p(fy^*eyf)p\| = \|fy^*eyf\|,$$

which yields a contradiction.

□

Remark 4.2. The analysis of the proof allows to deduce the following result: suppose that $\mathsf{A} \subset B(\mathsf{h})$ is a C^*-algebra, p is a projection in A', $\delta : \mathsf{A} \to p^{\perp}B(\mathsf{h})p$ is a derivation and for all $a, b \in \mathsf{A}$, $\delta(a)^*\delta(b) \in p\overline{\mathsf{A}}p$. Then δ is implemented by an operator in $\overline{\mathrm{Lin}\{\delta(\mathsf{A})\mathsf{A}\}}$.

The theorem above leads to the characterisation of the structure of generators of norm continuous completely positive semigroups.

Corollary 4.3. *Let* A *be as above and* $L : \mathsf{A} \to \mathsf{A}$ *be bounded, real. The following are equivalent:*

(i) L *is conditionally completely positive;*
(ii) *there exists a completely positive map* $\Psi : \mathsf{A} \to \overline{\mathsf{A}}$ *and an operator* $k \in \overline{\mathsf{A}}$ *such that*

$$L(a) = \Psi(a) + k^*a - ak, \quad a \in \mathsf{A}.$$

Proof. The implication (ii)$\Longrightarrow$(i) is trivial. If (i) holds, Lemma 1.7 yields π and V which satisfy the assumptions of Theorem 4.1 (Kaplansky theorem is used to obtain 'minimality'). Suppose that r implements V and let $\Psi = r^*\pi(\cdot)r$. If A is unital, put $h = \frac{1}{2}(L(I) - \psi(1))$. Then $\delta := L - \Psi - \{h, \cdot\}$ is a selfadjoint derivation and as such is implemented by a skewadjoint element $g \in \overline{\mathsf{A}}$ (use again Theorem 4.1). Putting $k = h - g$ ends the proof.

For A nonunital one once again uses approximate units and weak*-limit points of appropriate nets in $B(\mathsf{h})$. □

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