Haagerup property for locally compact and classical quantum groups

based on joint work with M. Daws, P. Fima and S. White

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Equivalent definitions of the Haagerup property

A locally compact group G has the Haagerup property (HAP) if the following equivalent properties hold:

- G admits a mixing unitary representation which weakly contains the trivial representation;
- there exists a normalised sequence of continuous, positive definite functions on G vanishing at infinity which converges uniformly to 1 on compact subsets of G;
- there exists a proper, continuous conditionally negative definite function on G;
- G admits a proper continuous affine action on a real Hilbert space.

General notations

$\mathbb{G}$ – a locally compact quantum group à la Kustermans-Vaes

$L^\infty(\mathbb{G})$ – a von Neumann algebra equipped with the *coproduct*

$$\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$$

carrying all the information about $\mathbb{G}$

$\mathrm{C}_0(\mathbb{G})$ – the corresponding (reduced) C^*-object, $\mathrm{C}_b(\mathbb{G}) := \mathrm{M}(\mathrm{C}_0(\mathbb{G}))$

$\mathrm{C}_0^u(\mathbb{G})$ – the universal version of $\mathrm{C}_0(\mathbb{G})$

$L^2(\mathbb{G})$ – the GNS Hilbert space of the *right invariant Haar weight* on $\mathbb{G}$

$W^{\mathbb{G}} \in B(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ – the multiplicative unitary associated to $\mathbb{G}$:

$$\Delta(f) = W^{\mathbb{G}}(f \otimes 1)(W^{\mathbb{G}})^*, \quad f \in L^\infty(\mathbb{G}).$$

Note the inclusions

$$\mathrm{C}_0(\mathbb{G}) \subset C_b(\mathbb{G}) \subset L^\infty(\mathbb{G}) = \mathrm{C}_0(\mathbb{G})''$$

Dual quantum groups

Each LCQG $\mathbb{G}$ admits the dual LCQG $\widehat{\mathbb{G}}$.

$L^\infty(\widehat{\mathbb{G}})$, $\mathrm{C}_0(\widehat{\mathbb{G}})$ – subalgebras of $B(L^2(\mathbb{G}))$

$W^{\mathbb{G}} \in \mathrm{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{G}))$ and

$$W^{\widehat{\mathbb{G}}} = (\sigma(W^{\mathbb{G}}))^*$$

In particular for G – locally compact group

$$L^\infty(\widehat{G}) = \mathrm{VN}(G)$$

$$\mathrm{C}_0(\widehat{G}) = C_r^*(G), \quad C_0^u(\widehat{\mathbb{G}}) = C^*(G)$$

Further properties of LCQGs

Definition

A locally compact quantum groups $\mathbb{G}$ is

- **compact** if $C_0(\mathbb{G})$ is unital (equivalently the Haar weights are finite);
- **discrete** if $\hat{\mathbb{G}}$ is compact;
- **unimodular** if the left and right Haar weights coincide;
- **amenable** if $L^\infty(\mathbb{G})$ admits a bi-invariant mean;
- **coamenable** if the universal and reduced algebras $C_0(\mathbb{G})$ and $C_0^u(\mathbb{G})$ are naturally isomorphic.

Some examples of locally compact quantum groups

- locally compact groups (all coamenable);
- duals of locally compact groups (all amenable);
- quantum deformations of classical Lie groups: for example $SU_q(2)$, quantum $ax + b$, $E_q(2)$ (amenable and coamenable);
- quantum symmetry groups: quantum permutation groups S_n^+, quantum automorphism groups of Wang $\mathbb{G}_{\text{aut}}(M_n)$, quantum orthogonal groups O_n^+ (mostly non-coamenable).

Representations of LCQGs

Definition

A (unitary) representation of $\mathbb{G}$ on a Hilbert space H is a unitary $U \in M(C_0(\mathbb{G}) \otimes K(\mathsf{H}))$ such that

$$(\Delta \otimes \iota)(U) = U_{13} U_{23}.$$

The operators $(\iota \otimes \omega_{\xi,\eta})(U) \in C_b(\mathbb{G})$, where $\xi, \eta \in \mathsf{H}$, are called *coefficients* of U.

Definition

A representation U of $\mathbb{G}$ is *mixing* if all its coefficients belong to $C_0(\mathbb{G})$. It *weakly contains the trivial representation/has almost invariant vectors* if there exists a net of unit vectors $(\xi_i)_{i \in I}$ such that for all $a \in \mathrm{C}_0(\mathbb{G})$

$$U(a \otimes \xi_i) - a \otimes \xi_i \longrightarrow 0.$$

The multiplicative unitary $W^{\mathbb{G}}$ plays the role of the left regular representation of $\mathbb{G}$ on $L^2(\mathbb{G})$; it is mixing.

Definition via representations

Definition

A locally compact quantum group $\mathbb{G}$ has HAP if it admits a mixing representation weakly containing the trivial representation.

Proposition

If $\hat{\mathbb{G}}$ is coamenable, then $\mathbb{G}$ has HAP. In particular, amenable discrete quantum groups have HAP. $\mathbb{G}$ is compact if and only if it has both HAP and Property (T).

This follows as coamenability is equivalent to the weak containment property of the left regular representation.

'Typicality' of mixing representations

Assume that $C_0(\mathbb{G})$ is separable ('$\mathbb{G}$ is second countable') and fix an infinite dimensional separable Hilbert space H. Then $\mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$ is a Polish space with a natural ('point-weak' topology).

Theorem

A locally compact quantum group $\mathbb{G}$ has HAP if and only if the set of mixing representations is dense in $\mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$.

The proof follows that in the classical case.

Definition via ‘positive definite functions’

What should continuous positive definite functions on $\mathbb{G}$ be? There are at least two possible points of view

- via Bochner's theorem, states on $C_0^u(\widehat{\mathbb{G}})$;
- elements in $C_b(\mathbb{G})$ yielding ‘completely positive multipliers’ on $L^\infty(\widehat{\mathbb{G}})$.

For a study of related notions see a recent paper of M.Daws and P.Salmi in Journal of Functional Analysis.

Theorem

Let $\mathbb{G}$ be a locally compact quantum group. Then the following conditions are equivalent:

- $\mathbb{G}$ has HAP;
- there exists a net of states $(\mu_i)_{i\in\mathcal{I}}$ on $C_0^u(\widehat{\mathbb{G}})$ such that the net $((\mathrm{id}\otimes\mu_i)(W))_{i\in\mathcal{I}}$ is an approximate identity in $C_0(\mathbb{G})$;
- there is a net $(a_i)_{i\in\mathcal{I}}$ in $C_0(\mathbb{G})$ of representing elements of completely positive multipliers which forms an approximate identity for $C_0(\mathbb{G})$.

Corollary for subgroups

Let $\mathbb{G}$, $\mathbb{H}$ be locally compact quantum groups. We say that $\mathbb{H}$ a closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz if there is a surjective *-homomorphism $\pi : C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$ such that

$$(\pi \otimes \pi) \circ \Delta_{\mathbb{G}} = \Delta_{\mathbb{H}} \circ \pi.$$

Corollary

If $\mathbb{G}$ has HAP and is coamenable, and $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$ (in the sense of Woronowicz), then $\mathbb{H}$ has HAP.

Cnd functions, and convolution semigroups of states

From now on we pass to discrete quantum groups.

Classical Schönberg correspondence says that the conditionally negative definite functions are 'generators of families of positive definite functions':

$$\psi \longleftrightarrow \exp(-t\psi).$$

Let $\mathbb{G}$ be a discrete quantum group. Then the algebra $C(\widehat{\mathbb{G}})$ contains a natural dense Hopf *-algebra, $\mathrm{Pol}(\widehat{\mathbb{G}})$. States on $\mathrm{Pol}(\widehat{\mathbb{G}})$ are in one-to-one correspondence with the states on $C^u(\widehat{\mathbb{G}})$.

Definition

A convolution semigroup of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ is a family $(\mu_t)_{t\geq 0}$ of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ such that

1. $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta_{\hat{\mathbb{G}}}, \quad t, s \geq 0;$
2. $\mu_t(a) \stackrel{t\to 0^+}{\longrightarrow} \mu_0(a) := \epsilon(a), \quad a \in \mathrm{Pol}(\widehat{\mathbb{G}}).$

Conditionally negative definite functions versus generating functionals

The following theorem is essentially due to M. Schürmann.

Theorem (Quantum Schönberg correspondence)

Each convolution semigroup of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$ possesses a *generating functional* $L : \mathrm{Pol}(\widehat{\mathbb{G}}) \to \mathbb{C}$:

$$L(a) = \lim_{t \to 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \ \ a \in \mathrm{Pol}(\hat{\mathbb{G}}).$$

The functional L is selfadjoint, vanishes at 1 and is positive on the kernel of the counit; in turn each functional enjoying these properties generates a convolution semigroup of states.

Thus – conditionally negative functions on $\mathbb{G}$ correspond to **generating functionals** on $\mathrm{Pol}(\widehat{\mathbb{G}})$.

Definition via conditionally negative definite functions

Theorem

Let $\mathbb{G}$ be a discrete quantum group. The following are equivalent:

1. $\mathbb{G}$ has HAP;
2. there exists a convolution semigroup of states $(\mu_t)_{t\geq 0}$ on $C_0^u(\widehat{\mathbb{G}})$ such that each $a_t := (\mathrm{id} \otimes \mu_t)(W)$ is an element of $c_0(\mathbb{G})$, and a_t tend strictly to 1 as $t \to 0^+$;
3. $\widehat{\mathbb{G}}$ admits a symmetric proper generating functional.

Definition via proper affine actions on Hilbert spaces?

There is one remaining 'classical' equivalent definitions of HAP – it is the one related to proper affine actions of G on Hilbert spaces.
But we do not even quite know what it means that $\mathbb{G}$ acts on a Hilbert space in the affine way...

We thus need to view affine isometric actions simply as cocycles for unitary representations.

Definition via proper cocycles

$\mathbb{G}$ - discrete quantum group.

Unitary representations of $\mathbb{G}$ on H correspond to unital *-representations of $\mathrm{Pol}(\widehat{\mathbb{G}})$ on H.

Definition

If $\pi : \mathrm{Pol}(\widehat{\mathbb{G}}) \to B(\mathsf{H})$ is a unital *-homomorphism, then we say that $c : \mathrm{Pol}(\widehat{\mathbb{G}}) \to \mathsf{H}$ is a cocycle for π if

$$c(ab) = \pi(a)c(b) + c(a)\epsilon(b), \quad a, b \in \mathrm{Pol}(\hat{\mathbb{G}})$$

The next theorem is inspired by the ideas introduced by R.Vergnioux.

Theorem

A discrete quantum group $\mathbb{G}$ has HAP if and only if it admits a proper real cocycle.

HAP via the approximation property for the von Neumann algebra

Recall that a vNa M with a faithful normal tracial state τ has the **von Neumann algebraic Haagerup approximation property** if there exists a family of unital completely positive τ-preserving normal maps $(\Phi_i)_{i\in\mathcal{I}}$ on M such that each of the respective induced maps T_i on $L^2(M,\tau)$ is compact and for each $x \in M$

$$\Phi_i(x) \xrightarrow{i\in\mathcal{I}} x$$

σ-weakly.

P.Jolissaint showed that this property does not depend on the choice of τ.

Theorem (M. Choda)

A discrete group Γ has HAP if and only if $\mathrm{VN}(\Gamma)$ has the von Neumann algebraic Haagerup approximation property.

vNa HAP in the quantum context

Theorem

Let $\mathbb{G}$ be a discrete unimodular quantum group. Then $\mathbb{G}$ has HAP if and only if $L^\infty(\hat{\mathbb{G}})$ has the von Neumann algebraic Haagerup approximation property.

Proof.

Follows the classical idea of Choda: if $\mathbb{G}$ has HAP, we have good positive definite functions, so constructing multipliers out of them (see M.Junge + M.Neufang + Z.J.Ruan, later also M.Daws) yields the approximation property for $L^\infty(\hat{\mathbb{G}})$ (this does not use the unimodularity).
The other direction is based on 'averaging' approximating maps on $L^\infty(\hat{\mathbb{G}})$ into multipliers. Here unimodularity seems crucial.

□

Corollary

Let $\mathbb{G}$ be a discrete unimodular quantum group. Then $\mathbb{G}$ has HAP if and only if $\mathrm{C}(\widehat{\mathbb{G}})$ has (a stronger version of) the C^*-algebraic Haagerup approximation property of Dong.

Examples

The following (non-amenable) discrete quantum groups are now known to have HAP:

- duals of quantum orthogonal and unitary groups, $\widehat{O_N^+}$, $\widehat{U_N^+}$ (M.Brannan, 2012);
- duals of Wang's quantum automorphism groups (M.Brannan, 2013);
- duals of quantum reflection groups (F.Lemeux, 2013).

All these cases are unimodular, and the HAP is proved via the von Neumann algebraic Haagerup approximation property.

Further examples

The last two examples are even more recent:

- duals of Wang's 'free orthogonal quantum groups' $\widehat{O_F}$ (K.De Commer, A.Freslon, M.Yamashita, 2013);
- $\widetilde{SU_q(1,1)}$ (M.Caspers, 2013)

Here HAP is proved via the construction of suitable completely positive multipliers. The second is not discrete.

Go to summary

Free product of discrete quantum groups

Definition

Let $\mathbb{G}_1$, $\mathbb{G}_2$ be discrete quantum groups. The free product of $\mathbb{G}_1$ and $\mathbb{G}_2$ is the discrete quantum group $\mathbb{G}_1 * \mathbb{G}_2$ 'defined' by the equality

$$\mathrm{C}(\widehat{\mathbb{G}_1 * \mathbb{G}_2}) = \mathrm{C}(\widehat{\mathbb{G}}_1) \star \mathrm{C}(\widehat{\mathbb{G}}_2)$$

(universal C^*-product with amalgamation over units).

The last formula is of course inspired by the equality

$$\mathrm{C}^*(\Gamma_1 * \Gamma_2) \approx \mathrm{C}^*(\Gamma_1) \star C^*(\Gamma_2).$$

We have also

$$L^\infty(\widehat{\mathbb{G}_1 * \mathbb{G}_2}) = L^\infty(\widehat{\mathbb{G}}_1) \star L^\infty(\widehat{\mathbb{G}}_2)$$

(von Neumann algebraic free product, with respect to Haar states).

Easy case - unimodular discrete quantum groups

Theorem

Let $\mathbb{G}_1$, $\mathbb{G}_2$ be discrete unimodular quantum groups with HAP. Then $\mathbb{G}_1 \star \mathbb{G}_2$ has HAP.

Proof.

Note that $\mathbb{G}_1 \star \mathbb{G}_2$ is unimodular and use the result of F.Boca showing that the free product of von Neumann algebras with the Haagerup approximation property has the Haagerup approximation property. □

Free products of the convolution semigroups of states

If $\mathbb{G}_1$, $\mathbb{G}_2$ are discrete quantum groups with HAP, but are not unimodular, we cannot use the von Neumann characterization.

Lemma

Let $\mathbb{G}_1$, $\mathbb{G}_2$ be discrete quantum groups, with respective convolution semigroups of states $(\phi_t)_{t\geq 0}$ and $(\omega_t)_{t\geq 0}$ on $\mathrm{Pol}(\widehat{\mathbb{G}_1})$ and $\mathrm{Pol}(\widehat{\mathbb{G}_2})$. Then $(\phi_t \diamond \omega_t)_{t\geq 0}$ is a convolution semigroup of states on $\mathrm{Pol}(\widehat{\mathbb{G}})$, where $\mathbb{G} = \mathbb{G}_1 * \mathbb{G}_2$.

The symbol $\diamond$ denotes here the *conditionally free product of states* à la Boca or Bożejko (with respect to the Haar states on $\widehat{\mathbb{G}_1}$ and $\widehat{\mathbb{G}_2}$).

Theorem

Let $\mathbb{G}_1$, $\mathbb{G}_2$ be discrete quantum groups with HAP. Then $\mathbb{G}_1 * \mathbb{G}_2$ has HAP.

Summary

Let $\mathbb{G}$ be a discrete, 'second countable' unimodular quantum group. The following conditions are equivalent (and can be used as the definition of HAP):

- $\mathbb{G}$ admits a mixing representation weakly containing the trivial representation;
- the set of mixing representations is dense in $\mathrm{Rep}_{\mathbb{G}}(\mathsf{H})$;
- $c_0(\mathbb{G})$ admits an approximate unit built of 'positive definite functions';
- $\hat{\mathbb{G}}$ admits a symmetric proper generating functional;
- $\mathbb{G}$ admits a real proper cocycle;
- $L^{\infty}(\hat{\mathbb{G}})$ has the von Neumann algebraic Haagerup approximation property.

Open questions

- how to characterise the HAP for discrete non-unimodular $\mathbb{G}$ via the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$?
- how to define the Haagerup approximation property for a von Neumann algebra with a state?
- what are the right versions of the conditionally negative definite function / cocycle characterization of HAP for non-discrete locally compact quantum groups?
- can one characterise HAP for $\mathbb{G}$ via existence of suitable actions of $\mathbb{G}$ on some C^*-algebras?

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