ISOMETRIC DILATIONS OF REPRESENTATIONS OF PRODUCT SYSTEMS OF C^*-CORRESPONDENCES OVER $\mathbb{N}_0^k$

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Classical multi-dimensional dilation theory ([SzF]) for Hilbert space operators is concerned with dilating tuples of contractions to tuples of isometries or unitaries, preserving some specific properties of the original family. Celebrated examples of S. Parrott, N. Varopoulos and others show that commuting isometric dilations of more than two commuting contractions need not exist. In general we cannot expect a characterisation of these tuples for which the commuting dilation exists, however if one requests a specific form of the dilation then precise answers can be obtained. In particular the existence of so-called regular or *-regular dilations (i.e. dilations satisfying additional conditions with respect to products of the original contractions and their adjoints, see [Bre], [Tim]) can be detected via simple conditions corresponding to positive-definiteness of certain operator-valued functions associated with the initial tuple.

In this talk we discuss analogous results for dilations of representations of product systems of representations of product systems of C^*-correspondences over $\mathbb{N}_0^k$. For us a C^*-correspondence E over a C^*-algebra A is a C^*-Hilbert module (a right module over A with A-valued scalar product) equipped additionally with the structure of a left module over A. Formally: there is a nondegenerate $*$-homomorphism $\phi : \mathsf{A} \to \mathcal{L}(E)$.

Definition. A product system of C^*-correspondences over $\mathbb{N}_0^k$, denoted by $\mathbb{E}$, is a family of k C^*-correspondences $\{E_1, \ldots, E_k\}$ over a C^*-algebra A together with the unitary isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ $(i > j)$ satisfying a natural associativity condition.

For the purpose of this talk we think of each E_i describing the type of the i-th element of the tuple we intend to dilate (a contraction, a row contraction, a family of contractions associated to a graph) and of $t_{i,j}$ as means of encoding the commutation relations between different elements of the tuple.

Definitions of completely contractive representations of $\mathbb{E}$ and their isometric dilations can be found in [Sol] or [Sk]. In [Sol] Solel characterised the existence of regular isometric dilation of a given representation via Brehmer-type conditions. Here we focus on **-regular* dilations. Contrary to the classical context of dilating commuting tuples of contractions regular dilations cannot be straightforwardly transformed into *-regular ones. We still however have the following result:

Theorem. *A minimal isometric dilation $\overrightarrow{V}$ of a representation $\overrightarrow{T}$ of $\mathbb{E}$ is *-regular if and only if it is doubly commuting.*

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Under a certain technical condition on the product system $\mathbb{E}$ and the assumption that a given representation $\overrightarrow{T}$ of $\mathbb{E}$ on a Hilbert space H satisfies a so-called Popescu condition (or condition 'P'), *-regular isometric dilation of $\overrightarrow{T}$ can be constructed via so-called generalised Poisson transform. Generalised Poisson transform is a completely positive map $R_{\overrightarrow{T}} : \mathcal{T}_{\mathbb{E}} \to B(\mathsf{H})$, where $\mathcal{T}_{\mathbb{E}}$ is the Toeplitz-type algebra associated with $\mathbb{E}$. The map $R_{\overrightarrow{T}}$ was earlier studied in a simpler context for example in [Pop]. Sufficient conditions for the construction of $R_{\overrightarrow{T}}$ are summarised in the following theorem.

Theorem. *Let $\mathbb{E}$ have a normal ordering property and let $\overrightarrow{T}$ be a representation of $\mathbb{E}$ on H satisfying the Popescu condition. Then there exists a unique continuous linear map $R_{\overrightarrow{T}} : \mathcal{T}_{\mathbb{E}} \to B(\mathsf{H})$ satisfying*

$$R_{\overrightarrow{T}}(L_e L_f^*) = T(n)(e)(T(m)(f))^*, \quad n, m \in bn_0^k, e \in \mathbb{E}(n), f \in \mathbb{E}(m).$$

The map $R_{\overrightarrow{T}}$ is completely positive and contractive, unital if $\mathcal{T}_{\mathbb{E}}$ is unital.

The Stinespring dilation for $R_{\overrightarrow{T}}$ provides in a natural way an isometric dilation for $\overrightarrow{T}$.

In the second part of the talk we present applications of the above results to families of contractions associated to a given higher-rank graph Λ ([KuPa], [Rae]). There is a natural way of associating to Λ a product system $\mathbb{E}(\Lambda)$ ([RaS]) and it can be showed that $\mathbb{E}(\Lambda)$ has the normal ordering property if and only if Λ is *finitely aligned* ([Rae]) and if and only if $\mathbb{E}(\Lambda)$ is *compactly aligned* ([Fow]). We have the following result:

Theorem. *There is a 1-1 correspondence between (completely contractive) representations of $\mathbb{E}(\Lambda)$ on a Hilbert space H and* Λ-contractions *in $B(\mathsf{H})$. The representation is isometric if and only if the corresponding Λ-contraction forms a* Toeplitz family*, isometric and doubly commuting if and only if the corresponding Λ-contraction forms a* Toeplitz-Cuntz-Krieger family.

The above correspondence can be used to transform our and Solel's results on dilation of representations of product systems to the context of dilating Λ-contractions. As an example we present the following theorem, first proved in [SkZ]:

Theorem. *Let Λ be finitely aligned and let $\mathcal{V}$ be a Λ-contraction on a Hilbert space H which satisfies the Popescu condition. Then there exists a Hilbert space $\mathsf{K} \supset \mathsf{H}$ and a Λ-contraction $\mathcal{W}$ on K consisting of partial isometries forming a Toeplitz-Cuntz-Krieger family such that*

$$W_\lambda^*|_{\mathsf{H}} = V_\lambda^*, \quad \lambda \in \Lambda.$$

One may assume that $\mathsf{K} = \overline{\mathrm{Lin}}\{W_\lambda \mathsf{H} : \lambda \in \Lambda\}$; under this assumption the family $\mathcal{W}$ is unique up to unitary equivalence.

Most of the results presented in the talk can be found in [Sk].

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