ENTROPY AND INVARIANT ABELIAN SUBALGEBRAS FOR ENDOMORPHISMS OF CUNTZ ALGEBRAS

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Topological entropy for a continuous transformation of a compact space (see [Wal]) is a numerical invariant which in a sense measures the degree of 'mixing' or 'chaotic' behaviour of the dynamical system in question. In [Voi] it was extended by Voiculescu to automorphisms of (nuclear) C^*-algebras, with the definition based on the growth of sizes of suitable completely positive approximations.

Let A be a nuclear C^*-algebra, $\alpha \in \mathrm{End}(\mathsf{A})$ and let $\mathrm{ht}\,\alpha$ denote the Voiculescu's topological entropy of α. The usual method of computing $\mathrm{ht}\,\alpha$ is based on two steps. First one produces an explicit or semi-explicit approximating net for A through matrix algebras whose rank can be controlled and thus provides an estimate of the Voiculescu entropy from above. Then, to obtain a lower bound, one looks for α-invariant commutative C^*-subalgebras $\mathsf{C} \subset \mathsf{A}$ in order to exploit the monotonicity of entropy with respect to passing to subalgebras and the fact that $\alpha|_{\mathsf{C}}$ is induced by a homeomorphism T of the spectrum of C and it was shown in [Voi] that $\mathrm{ht}\,\alpha|_{\mathsf{C}} = h_{\mathrm{top}}(T)$. Note that the general difficulty in understanding how the positive Voiculescu entropy is produced is reflected in the fact that there is still no direct proof of the inequality $\mathrm{ht}\,\alpha|_{\mathsf{C}} \geq h_{\mathrm{top}}(T)$, the corresponding argument in [Voi] exploits the properties of dynamical state entropy and classical variational principle. Other connections between the appearance of a non-zero noncommutative entropy and commutativity can be seen in [HSt] (where the occurrence of maximal entropy for a system of subalgebras is related to existence of suitable maximally abelian subalgebras) and in [BDS] (where free shifts are shown to have zero Voiculescu entropy).

On the other hand we have the following result.

Theorem ([Sk$_1$]). *There exist pairs* (A, α) *(certain bitstream shifts) such that*

$$\mathrm{ht}\,\alpha > \mathrm{ht}_c\,\alpha := \sup\{\mathrm{ht}\,\alpha|_{\mathsf{C}} : \mathsf{C} \text{ is an } \alpha - \text{invariant commutative subalgebra of } \mathsf{A}\}.$$

The above discussion leads to two natural questions related to the computations of the Voiculescu entropy:

- given an endomorphism of a C^*-algebra what are the (maximal) abelian subalgebras it leaves globally invariant?
- what other techniques, not based on the existence of invariant abelian subalgebras, can be used to compute the lower bounds for the Voiculescu entropy?

Below we present some results related to these questions in the context of the endomorphisms of Cuntz algebras.

Let $\mathcal{O}_N$ denote the Cuntz algebra generated by N-isometries $S_1, \ldots, S_N$ whose range projections are orthogonal and sum to 1 ([Cu$_1$]). We use the symbol μ to denote a $\{1, \ldots, N\}$-valued multiindex and let $S_\mu := S_{\mu_1} S_{\mu_2} \ldots S_{\mu_k}$, if the length of μ, denoted by $|\mu|$, is k. The Cuntz algebra contains a so-called *diagonal masa* (maximal abelian subalgebra) $\mathcal{C}_N := \overline{\mathrm{Lin}}\{S_\mu S_\mu^*\}$, isomorphic to the algebra of continuous functions on a Cantor set (equivalently, a full Markov shift on N letters). Moreover, if we write $\mathcal{F}_N^k = \mathrm{Lin}\{S_\mu S_\nu^* : |\mu| = |\nu| \leq k\} \approx M_N^{\otimes k}$, $\mathcal{F}_N = \lim_{k\to\infty} \mathcal{F}_N^k$, we obtain natural inclusions

$$\mathcal{C}_N = \bigotimes_{n=1}^{\infty} D_N \subset \bigotimes_{n=1}^{\infty} M_N = \mathcal{F}_N \subset \mathcal{O}_N.$$

By 'changing coordinates' in M_N and replacing diagonals D_N by $\mathcal{U}^* D_N \mathcal{U}$ ($\mathcal{U} \in M_N$ - a unitary) we can construct other, so-called *standard masas* in $\mathcal{O}_N$.

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In [Cu$_2$] it was shown that there is a bijective correspondence between unitaries in $\mathcal{O}_N$ and unital endomorphisms of $\mathcal{O}_N$, given by the formulas

$$\rho_U(S_i) = US_i, \quad i = 1, \ldots, N$$

and

$$U_\rho = \sum_{i=1}^{n} \rho(S_i) S_i^*.$$

This correspondence makes the endomorphisms of $\mathcal{O}_N$ particularly amenable to study (see [CoS] and references therein). From the point of view of the entropy we have the following result.

Theorem ([SkZ])**.** *Let $k \in \mathbb{N}$ and $U \in \mathcal{U}(\mathcal{F}_N^k)$. Then*

$$\mathrm{ht}(\rho_U) \leq (k-1) \log N.$$

The above estimate is, perhaps surprisingly, analogous to the bounds on index appearing in the work of Doplicher, Longo, Roberts, Conti, Pinzari and others.

The canonical shift, implemented by the *flip unitary* $F = \sum_{i,j=1}^N S_i S_j S_i^* S_j^* \in \mathcal{F}_N^2$ gives an easy example of the bound being achieved (its Voiculescu entropy, equal to $\log N$, was computed in [Cho], using the fact that the canonical shift leaves $\mathcal{C}_N$ invariant and the corresponding restriction is dual to the usual shift transformation on the full shift of N-letters). Note that for *Bogolyubov automorphisms*, i.e. automorphisms associated with unitaries $U \in \mathcal{F}_N^1 \approx M_N$ we have $\mathrm{ht}\,\rho_U = 0$. The standard masas can be alternatively described as images of $\mathcal{C}_N$ with respect to Bogolyubov automorphisms.

In [SkZ] we present an example of an endomorphism ρ of $\mathcal{O}_2$ induced by a unitary in $\mathcal{F}_2^2$, which leaves the diagonal masa invariant, but $\mathrm{ht}\rho = \log 2$, and $\mathrm{ht}\rho|_{\mathcal{C}_2} = 0$. In fact ρ leaves all standard masas invariant, and in some of them reduces again to the (dual of the) classical full shift.

In [HSS] we analyse in detail the endomorphisms which 'look the same' in all standard masas, i.e. commute with all Bogolyubov automorphisms. Moreover we develop there several sufficient (and necessary) conditions for an endomorphism to preserve a given standard masa. Here we just sample some of the interesting examples:

- if $U \in \mathcal{N}_{\mathcal{C}_N} := \{U \in U(\mathcal{O}_N) : U\mathcal{C}_N U^* = \mathcal{C}_N\}$, then ρ_U leaves $\mathcal{C}_N$ invariant;
- there exists a unitary $U \notin \mathcal{N}_{\mathcal{C}_2}$ such that ρ_U leaves $\mathcal{C}_2$ invariant;
- there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ which leaves $\mathcal{C}_2$ invariant, but no other standard masa;
- there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ which leaves invariant each standard masa, but does not commute with all Bogolyubov automorphisms.

Combined results of [HSS] and [Sk$_2$] show also that there exists $\rho \in \mathrm{End}(\mathcal{O}_2)$ (in fact originally studied in [Izu] in relation to Watatani indices of the subalgebras of Cuntz algebras) which leaves no standard masa invariant, but whose Voiculescu entropy is non-zero. The related entropy computation is in fact very easy, and uses the fact that ρ^2 has a simple form. It can be however related to the following general result.

Let H be a (finite-dimensional) Hilbert space. A *multiplicative unitary* is a unitary V on $\mathsf{H} \otimes \mathsf{H}$ satisfying the following relation (on $\mathsf{H} \otimes \mathsf{H} \otimes \mathsf{H}$) (in the leg notation, so for example $V_{12} := V \otimes I_{\mathsf{H}}$):

$$V_{12} V_{13} V_{23} = V_{23} V_{12}.$$

It is called *irreducible* if it cannot be non-trivially written as '$V_1 \otimes I_{\mathsf{H}_1}$' for some other multiplicative unitary V_1.

Theorem ([Sk$_2$])**.** *Let V be an irreducible multiplicative unitary on $\mathsf{H} \otimes \mathsf{H}$, where $\mathsf{H} \approx \mathbb{C}^N$; view V as a matrix in $M_N \otimes M_N$ and further via the usual isomorphism $M_N \otimes M_N \approx \mathcal{F}_N^2$ as a unitary in $\mathcal{O}_N$. Let F be the flip unitary in $M_N \otimes M_N$. The topological entropy of $\rho_{VF} \in \mathrm{End}(\mathcal{O}_N)$ is equal to $\log N$.*

The proof of the above result is based on the von Neumann algebraic techniques: one first identifies a certain extension of ρ_{VF} with a canonical endomorphism of the Longo type, then passes to certain finite von Neumann subalgebras, views the respective restricted endomorphism as the Ocneanu canonical shift for the tower of subfactors and finally uses some computations of the CNT entropy ([NSt]) in terms of the index due to Hiai ([Hia]).

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