QUANTUM SYMMETRY GROUPS AND RELATED TOPICS WINTER SCHOOL ON OPERATOR SPACES, NONCOMMUTATIVE PROBABILITY AND QUANTUM GROUPS MÉTABIEF, DECEMBER 2014

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Abstract. Groups first entered mathematics in their geometric guise, as collections of all symmetries of a given object, be it a finite set, a polygon, a metric space or a differential manifold. Original definitions of quantum groups, also in the analytic context, had rather algebraic character. In these lectures we describe several examples of quantum symmetry groups of a given quantum (or classical) space. The theory is based on the concept of actions of (compact) quantum groups on C^*-algebras and viewing symmetry groups as universal objects acting on a given structure. Initiated by Wang in 1990s, in recent years it has been developing rapidly, exhibiting connections to combinatorics, free probability and noncommutative geometry. In these lectures we will present both older and newer research developments regarding quantum symmetry groups, discussing both the general theory and specific examples.

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Plan of lectures

Lecture 1 **Compact quantum groups and their actions**: motivation behind noncommutative mathematics, definition, basic properties and first examples of compact quantum groups; compact quantum group actions, their continuity and nondegeneracy, invariant states, ergodicity.

Lecture 2 **Quantum symmetry groups of finite structures**: categories of quantum groups or semigroups acting on a given C^*-algebra and preserving some additional structure, free permutation groups, quantum symmetry groups of graphs, Wang and Van Daele's universal compact quantum groups.

Lecture 3 **Quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations**: quantum symmetry groups of C^*-algebras equipped with orthogonal filtrations with the existence proof, examples related to group C^*-algebras.

Lecture 4 **Further examples, connections to liberated quantum groups**: projective limits of quantum symmetry groups, quantum symmetry groups of Bratteli diagrams, relation between quantum symmetry groups and liberated quantum groups.

Lecture 5 **Other structures related to quantum symmetry groups and open problems**: quantum homogeneous spaces inside S_N^+, quantum partial permutations, open problems.

All algebras and vector spaces in these lectures will be over $\mathbb{C}$. The algebraic tensor product of spaces/algebras will be denoted by $\odot$, with the symbol $\otimes$ reserved for the tensor product of maps and the *minimal/spatial* tensor product of C*-algebras. The algebraic dual of a (finite-dimensional) vector space V will be denoted V'. Inner products will always be linear in the second variable, and Σ will be the symbol reserved for the tensor flip. We will often use the 'leg' notation for operators acting on tensor products: so that for example if A, B, C are unital algebras and $T \in L(A \odot C)$, we write $T_{13} \in L(A \odot B \odot C)$ for an operator formally defined as $(\mathrm{id}_A \otimes \Sigma)(T \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \Sigma)$. The linear span of a subset X of a vector space will be denoted by $\mathrm{Lin}\, X$, and a closed linear span of a subset X of a normed space by $\overline{\mathrm{Lin}}\, X$.

1. Compact quantum groups and their actions

In this lecture we discuss possible approaches to the concept of symmetry groups, define compact quantum groups and discuss their actions on classical and quantum spaces.

1.1. **The notion of symmetry groups.**

The concept of a *group* first appeared in mathematics, in the 19th century in the work of Abel and Galois (with some earlier developments due to Euler, Bezout and Lagrange), as a name for a collections of symmetries of some structure: a set of solutions of a given equation, a figure on the plane, a fixed finite set. Symmetries of a given structure X are viewed as transformations of X preserving its relevant properties; so for example if X is a metric space then it is natural to require that the transformations do not change the metric, and we land with the concept of isometries of X. It was soon noted that so understood symmetries have natural properties: they can be composed in an associative manner, there always exists a trivial symmetry, and each symmetry transformation admits an inverse transformation, which is also a symmetry. Thus an abstract notion of a group was born in the late 19th century (in the finite case due to Cayley, and soon later generalised by Weber and van Dyck) and has remained a cornerstone of mathematics ever since.

In hindsight, one can define the symmetry groups of a structure X abstractly as follows: consider all the groups acting on X (in a manner preserving the relevant features of X). These form a category; and the symmetry group of X, say Sym_X, is a universal object of this category. Thus Sym_X is a group acting on X, and every action of a group G on X can be viewed simply as a homomorphism from G to Sym_X. This viewpoint will be indispensable for these lectures, where we will study analogous concepts for *quantum groups*. Here the notion of individual, point transformations will be completely absent and the categorical approach becomes the only possible way to define symmetry groups. As we will only work with (quantum) symmetries of finite or compact structures, it will be natural to restrict attention to *compact quantum groups*.

1.2. **Compact quantum groups – definition and basic facts.**

The starting point of noncommutative/quantum generalisations of classical mathematics is based on the fundamental result of Gelfand and Najmark. Recall that a C*-algebra A is a Banach *-algebra (i.e. a *-algebra equipped with a submultiplicative norm, with respect to which it is a Banach space) satisfying the C*-*condition*:

$$\|x^*x\| = \|x\|^2, \quad x \in \mathsf{A}.$$

Two main motivating examples of C*-algebras are $\mathrm{B}(\mathsf{H})$, the algebra of all bounded operators on a Hilbert space H equipped with an operator norm, and $\mathrm{C}(X)$, the algebra of continuous functions on a compact space X equipped with the supremum norm.

Theorem 1.1 (Gelfand-Najmark)**.** *Every commutative unital C^*-algebra* A *is (isometrically) isomorphic to the algebra* $\mathrm{C}(X_{\mathsf{A}})$ *for a unique (up to a homeomorphism) compact space* X_{A}*. Given two compact spaces* X_1 *and* X_2 *and a continuous map* $T : X_1 \to X_2$ *the map* $\alpha_T : \mathrm{C}(X_2) \to \mathrm{C}(X_1)$ *given by the formula*

$$\alpha_T(f) = f \circ T, \quad f \in C(X_2),$$

is a unital **-homomorphism. Moreover every unital* **-homomorphism between commutative* C^**-algebras arises in this way.*

Note the inversion of arrows:

$$\begin{array}{llcll} T: & X_1 & \longrightarrow & X_2 & \\ & \mathrm{C}(X_1) & \longleftarrow & C(X_2) & :\alpha_T \end{array}$$

The compact space X_{A} is the character space of the C^*-algebra A; this identification explains also why the second part of the above theorem, concerning the morphisms, is true.

The above facts, inspiring most of the noncommutative mathematics, allow us to view unital C^*-algebras as the algebras of continuous functions on 'compact quantum spaces' (we will sometimes use a suggestive notation $\mathsf{A} = \mathrm{C}(\mathbb{X})$ to stress that we think of a C^*-algebra A as the algebra of continuous functions on the 'virtual' space $\mathbb{X}$). In this correspondence the role of maps between spaces is taken by unital *-homomorphisms between C^*-algebras. The state space of a C^*-algebra A (i.e. the space of all positive norm 1 functionals on A) will be denoted by $S(\mathsf{A})$ – it is a noncommutative counterpart of the set of all regular probability measures on the underlying space.

In these lectures we are interested in the study of *compact groups*; it turns out that it is easier to first 'quantise' the notion of a compact *semigroup.* For that we need to understand how the notion of multiplication transfers to the setting introduced above. A compact semigroup S is a compact topological space together with a continuous map $M : S \times S \to S$ which is associative. The dual transformation $\alpha_M : \mathrm{C}(S) \to \mathrm{C}(S \times S)$ is a unital *-homomorphism. It is not difficult to see, using the Stone-Weierstrass theorem, that the algebra $\mathrm{C}(S \times S)$ is isomorphic to the algebra $\mathrm{C}(S) \otimes \mathrm{C}(S)$, where the symbol $\otimes$ denotes the *spatial/minimal* tensor product of C^*-algebras (for this notion and other basic facts related to C^*-algebras which will be used without further comment we refer to [Mur]; an exhaustive treatment of C^*-algebraic tensor products and corresponding extensions of linear maps can be found in [BrO]). Thus we may view α_M as a map taking values in $\mathrm{C}(S) \otimes \mathrm{C}(S)$; this, via another application of the Stone-Weierstrass theorem, allows us to encode the associativity of M via *coassociativity* of α_M, i.e. the condition

$$(\alpha_M \otimes \mathrm{id}) \circ \alpha_M = (\mathrm{id} \otimes \alpha_M) \circ \alpha_M.$$

Definition 1.2. A unital C^*-algebra A is called an algebra of functions on a compact quantum semigroup, if it is equipped with the comultiplication, i.e. a unital *-homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ which is coassociative:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta.$$

The question of finding an appropriate definition of a compact quantum *group* is far subtler. One could attempt to try to dualise in a similar manner the existence of the neutral element and the inverse map, but this leads to a rather restrictive theory (the respective issues are related to the lack of *coamenability* and the *Kac property*, which we will explain later). The solution, found by Woronowicz, is based on the following fact.

Proposition 1.3. *Let S be a compact semigroup. If S satisfies the cancellation laws, i.e. for each $g_1, g_2, h \in S$ either of the equalities $g_1h = g_2h$ or $hg_1 = hg_2$ implies that $g_1 = g_2$, then S is in fact a group – the multiplication in S admits a neutral element and inverses.*

Exercise 1.1. Prove the above proposition (first reduce the case to abelian, considering a closed subsemigroup of S generated by a single element, and then consider non-empty closed ideals in S).

The full proof of the above result, together with a very gentle introduction to the theory of compact quantum groups may be found in the survey [MVD]. We are now ready for the main definition of this lecture.

Definition 1.4 (Woronowicz)**.** A unital C*-algebra A equipped with a unital *-homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ which is coassociative:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta)\Delta$$

and satisfies the quantum cancellation rules:

$$\overline{\mathrm{Lin}}\,\Delta(\mathsf{A})(1 \otimes \mathsf{A}) = \overline{\mathrm{Lin}}\,\Delta(\mathsf{A})(\mathsf{A} \otimes 1) = \mathsf{A} \otimes \mathsf{A}$$

is called an algebra of continuous functions on a compact quantum group.

We usually write $\mathsf{A} = \mathrm{C}(\mathbb{G})$ and informally call $\mathbb{G}$ a *compact quantum group*. The reader may have noticed the use of the indefinite article above: this will be explained later on. The following result is straightforward (recall that *-homomorphisms between C*-algebras are contractions).

Proposition 1.5. *Let $\mathbb{G}$ be a compact quantum group and let $\mathrm{C}(\mathbb{G})$ be an algebra of continuous functions on $\mathbb{G}$. Then the dual space $\mathrm{C}(\mathbb{G})^*$ equipped with the convolution multiplication*

$$\mu \star \nu := (\mu \otimes \nu) \circ \Delta, \quad \mu, \nu \in \mathrm{C}(\mathbb{G})^*$$

is a Banach algebra.

The notation and terminology reflect the fact that if G is a classical compact group then $\mathrm{C}(G)^*$ can be identified via Riesz theorem with the set of all regular measures on G and the multiplication $\star$ is the usual convolution of measures. Note that a convolution of states on $\mathrm{C}(\mathbb{G})$ is a state.

Convolution multiplication can be defined also in the context of compact quantum semigroups. The next key result requires however the quantum group structure and in a sense justifies Definition 1.4.

Theorem 1.6 ([Wo$_2$])**.** *Let $\mathbb{G}$ be a compact quantum group. Then $\mathrm{C}(\mathbb{G})$ admits a (unique)* Haar state*: a state $h \in \mathrm{C}(\mathbb{G})^*$ such that*

$$(h \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes h) \circ \Delta = h(\cdot)1_{\mathsf{A}};$$

equivalently, for any $\omega \in \mathrm{C}(\mathbb{G})^*$

$$h \star \omega = \omega \star h = \omega(1)h.$$

In general the Haar state need not be faithful (see however Theorem 1.9 below).

Let $n \in \mathbb{N}$. A unitary matrix $U = (u_{ij})_{i,j=1}^n \in M_n(\mathrm{C}(\mathbb{G}))$ is called a (finite-dimensional) *unitary representation* of $\mathbb{G}$ if

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad i, j = 1, \ldots, n.$$

Any linear combination of the elements u_{ij} appearing above is called a *coefficient* of U. Each unitary representation as above can be identified with a unitary operator in $B(\mathsf{H}_U) \otimes \mathrm{C}(\mathbb{G})$, where H_U is an n-dimensional Hilbert space. Then the formula displayed above can be rewritten as the following equality in $B(\mathsf{H}_U) \otimes \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$:

$$(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}. \tag{1.1}$$

Two unitary representations U, V are said to be *equivalent* if there exists a unitary $T \in B(\mathsf{H}_U; \mathsf{H}_V)$ such that $U = (T^* \otimes \mathrm{id})V(T \otimes \mathrm{id})$; U is said to be *contained in* V if H_U is a subspace of H_V and $U = (P_{\mathsf{H}_U} \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})V(P_{\mathsf{H}_U} \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})$. Finally U is *irreducible* if it contains no proper (i.e. different from U) non-zero representation. We denote by $\mathrm{Irr}(\mathbb{G})$ the set of all equivalence classes of irreducible representations of $\mathbb{G}$. Note that the equality (1.1) makes sense also for $U \in B(\mathsf{H}_U) \otimes \mathrm{C}(\mathbb{G})$, where H_U is an infinite-dimensional Hilbert space, so can be used to define representations on infinite-dimensional Hilbert spaces. The *Peter-Weyl theory* for compact quantum groups shows that any irreducible representation of $\mathbb{G}$ studied in this a priori more general context must in fact be finite-dimensional and moreover any representation, finite-dimensional or not, decomposes into a direct sum of irreducible ones. We will later also need the notion of a *fundamental* representation of $\mathbb{G}$: it is a finite-dimensional unitary representation of $\mathbb{G}$ such that its coefficients generate $\mathrm{C}(\mathbb{G})$ as a C*-algebra. A compact quantum group is said to be *a compact matrix quantum group* if it admits a fundamental representation.

We will often choose without further comment for each $\beta \in \mathrm{Irr}(\mathbb{G})$ a representative $U^\beta \in M_{n_\beta}(\mathrm{Pol}(\mathbb{G}))$.

The following result shows that big parts of the study of compact quantum groups can be conducted in the purely algebraic context.

Theorem 1.7 ([Wo$_2$])**.** *Let $\mathbb{G}$ be a compact quantum group. The linear span of all coefficients of finite dimensional unitary representations of $\mathbb{G}$ is a dense unital *-subalgebra of $\mathrm{C}(\mathbb{G})$, which turns out to have the structure of a Hopf *-algebra with the coproduct inherited from $\mathrm{C}(\mathbb{G})$. Moreover the collection $\{u^\beta_{ij} : \beta \in \mathrm{Irr}(\mathbb{G}), i, j = 1, \ldots, n_\beta\}$ forms a linear basis of $\mathrm{Pol}(\mathbb{G})$. The Haar state is faithful on $\mathrm{Pol}(\mathbb{G})$: if $a \in \mathrm{Pol}(\mathbb{G})$ and $h(a^*a) = 0$, then $a = 0$.*

The Hopf *-algebra $\mathrm{Pol}(\mathbb{G})$ carries all the essential information on $\mathbb{G}$. In particular one can always associate to it a C*-algebra viewed as an algebra of continuous functions on $\mathbb{G}$. There are at least two such canonical constructions.

Proposition 1.8. *Let $\mathbb{G}$ be a compact quantum group. The universal C^*-algebraic completion of $\mathrm{Pol}(\mathbb{G})$ is the completion of $\mathrm{Pol}(\mathbb{G})$ with respect to the norm given by the formula*

$$\|x\|_u := \sup\{\|\pi(x)\| : \pi : \mathrm{Pol}(\mathbb{G}) \to B(\mathsf{H}) \text{ is a (cyclic) unital } {}^*\text{−homomorphism } \}, \quad x \in \mathrm{Pol}(\mathbb{G}).$$

We will denote it by $\mathrm{C}_u(\mathbb{G})$ and call it the universal algebra of continuous functions on $\mathbb{G}$*. It admits a natural coproduct $\Delta_u : \mathrm{C}_u(\mathbb{G}) \to \mathrm{C}_u(\mathbb{G}) \otimes \mathrm{C}_u(\mathbb{G})$ defined by the linear continuous extension of the prescription*

$$\Delta_u(u^\beta_{ij}) = \sum_{k=1}^{n} u^\beta_{ik} \otimes u^\beta_{kj}, \quad \beta \in \mathrm{Irr}(\mathbb{G}), i, j = 1, \ldots, n_\beta.$$

Note that already in the formulation above we used the fact that the displayed formula indeed defines a *norm* on $\mathrm{Pol}(\mathbb{G})$. As $\mathrm{Pol}(\mathbb{G})$ is a Hopf*-algebra, it in particular admits a *counit*, a character $\epsilon : \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ such that $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}_{\mathrm{Pol}(\mathbb{G})}$. It is easy to see that the counit admits a continuous extension to a character on $\mathrm{C}_u(\mathbb{G})$ and that it still satisfies the equality above, with Δ replaced by its universal version.

An alternative construction leads to the *reduced* C^*-algebra.

Theorem 1.9. *Let $\mathbb{G}$ be a compact quantum group and let $\mathrm{C}(\mathbb{G})$ be an algebra of continuous functions on $\mathbb{G}$. Denote by $(\pi_h, L^2(\mathbb{G}), \Omega_h)$ the GNS representation of $\mathrm{C}(\mathbb{G})$ with respect to the Haar state. Then the unital C^*-algebra $\pi_h(\mathrm{C}(\mathbb{G}))$, with the coproduct Δ_r determined by the condition*

$$\Delta_r \circ \pi_h = (\pi_h \otimes \pi_h) \circ \Delta$$

has the structure of an algebra of continuous functions on a compact quantum group; it is called the reduced algebra of continuous functions on $\mathbb{G}$ *and sometimes denoted* $\mathrm{C}_r(\mathbb{G})$.

Note that the Haar state of $\mathrm{C}_r(\mathbb{G})$ is naturally given by the vector state associated to the GNS vector Ω_h. It is always faithful.

Exercise 1.2. Show that $\mathrm{C}_r(\mathbb{G})$ is isomorphic to the GNS-completion of $\mathrm{Pol}(\mathbb{G})$ with respect to the Haar state restricted to that *-algebra.

The faithfulness of the Haar state on $\mathrm{Pol}(\mathbb{G})$ implies that we can always view $\mathrm{Pol}(\mathbb{G})$ as a subalgebra of both $\mathrm{C}_u(\mathbb{G})$ and $\mathrm{C}_r(\mathbb{G})$; we will do it without further comments. Abstract considerations imply that $\mathrm{C}_r(\mathbb{G})$ is a quotient of $\mathrm{C}_u(\mathbb{G})$; the canonical quotient map $\Lambda_{\mathbb{G}} : \mathrm{C}_u(\mathbb{G}) \to \mathrm{C}_r(\mathbb{G})$ is called the *reducing morphism.* We say that $\mathbb{G}$ is *coamenable* if the reducing morphism is injective (i.e. $\mathrm{C}_u(\mathbb{G})$ and $\mathrm{C}_r(\mathbb{G})$ are canonically isomorphic). An obvious class of examples of coamenable compact quantum groups is given by *finite quantum groups*, i.e. those $\mathbb{G}$ for which $\mathrm{C}(\mathbb{G})$ is finite-dimensional.

Finally note that the Hopf *-algebras arising as $\mathrm{Pol}(\mathbb{G})$ for a compact quantum group $\mathbb{G}$ have an abstract characterisation as *CQG-algebras*, i.e. Hopf*-algebras spanned by their unitary corepresentations ([DiK]). In these lectures we will sometimes ignore the distinction between different possible completions of $\mathrm{Pol}(\mathbb{G})$; on some occasions however it plays an important role and we will then use the notations $\mathrm{C}_r(\mathbb{G})$ and $\mathrm{C}_u(\mathbb{G})$.

1.3. **First examples of compact quantum groups.**

Example 1.10. Let G be a compact group. The algebra $\mathrm{C}(G)$ equipped with the coproduct $\Delta : \mathrm{C}(G) \to \mathrm{C}(G) \otimes \mathrm{C}(G) \approx \mathrm{C}(G \times G)$ given by

$$\Delta(f)(s,t) = f(s \cdot t), \quad s, t \in G.$$

Then $\mathrm{Pol}(G)$ is the algebra spanned by the coefficients of finite-dimensional unitary representations of G, and the Haar state is the usual Haar integral on G. Classical compact groups are automatically coamenable.

Example 1.11. Let Γ be a discrete group. Then $\mathbb{C}[\Gamma]$, the group ring of Γ, equipped with the coproduct given by the linear extension of

$$\Delta(\gamma) = \gamma \otimes \gamma, \quad \gamma \in \Gamma,$$

is a CQG-algebra. The corresponding compact quantum group is denoted by $\hat{\Gamma}$ and should be viewed as a 'Pontriagin' dual of Γ [1]. We further have $\mathrm{C}_r(\hat{\Gamma}) = \mathrm{C}^*_r(\Gamma)$, the reduced group

[1] In fact the world of topological quantum groups admits a perfect generalization of the idea of Pontriagin duality of locally compact *abelian* groups – to formulate it one however needs to pass to the framework of *locally* compact quantum groups of Kustermans and Vaes, [KuV].

C*-algebra of Γ, and $\mathrm{C}_u(\hat{\Gamma}) = \mathrm{C}^*_u(\Gamma)$, the universal group C*-algebra of Γ. The Haar state on each of these algebras arises as a (linear, continuous) extension of the formula ($\gamma \in \Gamma$, and e denotes the neutral element of Γ).

$$h(\gamma) = \begin{cases} 1 & \gamma = e \\ 0 & \gamma \neq e \end{cases}. \tag{1.2}$$

The compact quantum group $\hat{\Gamma}$ is coamenable if and only if Γ is *amenable* – which of course motivates the terminology.

Exercise 1.3. Describe irreducible representations of $\hat{\Gamma}$.

The last two classes of examples are in a sense of a classical nature, although the second one exhibits many of the noncommutative features of the general theory. In particular in each case the relevant quantum group is *of Kac type*, that is the Haar state is tracial. This is no longer the case in the following example, Woronowicz's quantum version of $SU(2)$ ([Wo$_1$]).

Example 1.12. Let $q \in [-1, 0) \cup (0, 1)$. Consider the universal unital C*-algebra $\mathrm{C}(SU_q(2))$ generated by two elements α, γ satisfying the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1 \tag{1.3}$$

$$\gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha. \tag{1.4}$$

The formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

determine uniquely the coproduct $\Delta : \mathrm{C}(SU_q(2)) \to \mathrm{C}(SU_q(2)) \otimes \mathrm{C}(SU_q(2))$ and Woronowicz showed in [Wo$_1$] that this gives $\mathrm{C}(SU_q(2))$ the structure of an algebra of continuous functions on a compact quantum group, called $SU_q(2)$. Further $SU_q(2)$ is coamenable (for a more general version of this result see [Ba$_1$] or the Appendix of [FST]), not of Kac type, and $\mathrm{Pol}(SU_q(2))$ is the universal unital *-algebra generated by elements satisfying the relations (1.3)-(1.4).

It turns out that similar deformations exist for all compact semisimple connected Lie groups (see [KoS]).

Exercise 1.4. Prove that if we put $q = 1$ then the construction above leads to the C*-algebra $\mathrm{C}(SU(2))$, where $SU(2)$ is the group of 2 by 2 unitary matrices of determinant 1 and verify that the algebraic coproduct introduced above coincides with the one arising via Example 1.10.

We will see many more examples of compact quantum groups in the following lectures. Verifying that a given compact quantum semigroup is in fact a compact quantum group is often non-trivial, and the following result of Woronowicz is a crucial tool (in particular it offers the quickest way to show that $SU_q(2)$ is a compact quantum group).

Theorem 1.13 ([Wo$_1$])**.** *Suppose that* A *is a unital* C*-*algebra equipped with a coassociative coproduct* $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$. *Let* $n \in \mathbb{N}$ *and let* $U = (u_{ij})_{i,j=1}^n \in M_n(\mathsf{A})$ *be a unitary matrix satisfying the following conditions:*

(i) $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad i, j = 1, \ldots, n;$
(ii) *the* *-*algebra* $\mathcal{A}$ *generated by the set* $\{u_{ij} : i, j = 1, \ldots, n\}$ *is dense in* A;
(iii) *there exists a linear antimultiplicative map* $S : \mathcal{A} \to \mathcal{A}$ *such that* $S \circ^* \circ S \circ^* = \mathrm{id}_{\mathcal{A}}$ *and* $S(u_{ij}) = u_{ji}^*$ *for all* $i, j = 1, \ldots n$.

Then $\mathsf{A} = \mathrm{C}(\mathbb{G})$ *for a certain compact matrix quantum group* $\mathbb{G}$ *and* U *is a fundamental representation of* $\mathbb{G}$*.*

1.4. **Morphisms between compact quantum groups.**

Definition 1.14. Let $\mathbb{G}_1$, $\mathbb{G}_2$ be two compact quantum groups. By a morphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ is understood a unital *-homomorphism $\gamma : \mathrm{C}_u(\mathbb{G}_2) \to \mathrm{C}_u(\mathbb{G}_1)$ such that

$$\Delta_1 \circ \gamma = (\gamma \otimes \gamma) \circ \Delta_2,$$

where Δ_1, Δ_2 denote the respective (universal) coproducts.

Note the usual inversion of arrows. There is a one-to-one correspondence between morphisms from $\mathbb{G}_1$ to $\mathbb{G}_2$ and Hopf*-algebra morphisms from $\mathrm{Pol}(\mathbb{G}_2)$ to $\mathrm{Pol}(\mathbb{G}_1)$, given by the natural restriction/continuous-extension procedure. Note that it is not the case that all such morphisms lead to maps between reduced C*-algebras; in particular the counit of $\mathrm{Pol}(\mathbb{G})$, which can be viewed as a morphism from the trivial group $\{e\}$ to $\mathbb{G}$, extends continuously to $\mathrm{C}_r(\mathbb{G})$ if and only if $\mathbb{G}$ is coamenable ([BMT]).

Definition 1.15. Let $\mathbb{G}_1$, $\mathbb{G}_2$ be two compact quantum groups. We say that $\mathbb{G}_1$ is a quantum subgroup of $\mathbb{G}_2$ if there exists a morphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ such that its associated unital *-homomorphism $\gamma : \mathrm{C}_u(\mathbb{G}_2) \to \mathrm{C}_u(\mathbb{G}_1)$ is surjective.

Note that the above condition is easily seen to be equivalent to the surjectivity of the restriction map $\gamma : \mathrm{Pol}(\mathbb{G}_2) \to \mathrm{Pol}(\mathbb{G}_1)$.

The following result was shown in [Wan$_1$] (see also [BhGS]). For the description of inductive limits of C*-algebras we refer to [Bla] and [Mur].

Lemma 1.16. *Suppose that* $(\mathbb{G}_n)_{n=1}^{\infty}$ *is a sequence of compact quantum groups, that for each* $n, m \in \mathbb{N}$, $n < m$ *there exists a compact quantum group morphism from* $\mathbb{G}_m$ *to* $\mathbb{G}_n$ *(given by a unital* **-homomorphism* $\pi_{n,m} : \mathrm{C}_u(\mathbb{G}_n) \to \mathrm{C}_u(\mathbb{G}_n)$ *and the compatibility conditions*

$$\pi_{k,m} \circ \pi_{n,k} = \pi_{n,m}, \qquad n < k < m,$$

hold. Then the inductive limit of the sequence $\big(\mathrm{C}_u(\mathbb{G}_n)\big)_{n=1}^{\infty}$ *of* C**-algebras admits a canonical structure of the algebra of continuous functions on a compact quantum group. Denote the resulting compact quantum group by* $\mathbb{G}_\infty$ *and let for each* $n \in \mathbb{N}$ *the associated morphism from* $\mathbb{G}_\infty$ *to* $\mathbb{G}_n$ *be denoted by* $\pi_{n,\infty}$*. Then* $\mathbb{G}_\infty$ *has the following universal property: for any compact quantum group* $\mathbb{H}$ *such that there exists a family of (compatible in a natural sense) morphisms from* $\mathbb{H}$ *to* $\mathbb{G}_n$*, given by maps* $\gamma_n : \mathrm{C}_u(\mathbb{G}_n) \to \mathrm{C}_u(\mathbb{H})$ *there exists a unique morphism from* $\mathbb{H}$ *to* $\mathbb{G}_\infty$ *(described by a map* $\gamma : \mathrm{C}_u(\mathbb{G}_\infty) \to \mathrm{C}_u(\mathbb{H})$*) such that* $\gamma \circ \pi_{n,\infty} = \gamma_n$*. We will sometimes write*

$$\mathbb{G}_\infty = \varprojlim \mathbb{G}_n.$$

1.5. **Actions of compact quantum groups.** Classically a (left, continuous) action of a compact group G on a compact space X is a continuous map $\alpha : X \times G \to X$ such that for each $g \in G$ the associated map $\alpha_g : X \to X$ ($\alpha_g(x) := \alpha(x, g), x \in X$) is a homeomorphism of X and the mapping $G \to \mathrm{Homeo}(X)$, $g \mapsto \alpha_g$, is a homomorphism. In the quantum world we as usual invert the arrows.

Definition 1.17. Let $\mathbb{G}$ be a compact quantum group and let B be a unital C*-algebra. We say that a compact quantum group $\mathbb{G}$ acts on B if there exists a unital *-homomorphism $\alpha : \mathsf{B} \to \mathsf{B} \otimes \mathrm{C}(\mathbb{G})$, called the (left, continuous) action of $\mathbb{G}$ on B, such that

(i) $(\alpha \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})})\alpha = (\mathrm{id}_{\mathsf{B}} \otimes \Delta)\alpha$;
(ii) $\overline{\mathrm{Lin}}\, \alpha(\mathsf{B})(1 \otimes C(\mathbb{G})) = \mathsf{B} \otimes C(\mathbb{G})$.

The first condition displayed above is often called the *action equation* and corresponds classically to the fact that the map $g \mapsto \alpha_g$ is a homomorphism. The second condition is known as Podleś/nondegeneracy condition, and corresponds to the requirement that each α_g is a homeomorphism of X. It first appeared in the PhD thesis of Podleś ([Po$_1$], see also [Po$_2$]). If we use the suggestive notion $\mathsf{B} = \mathrm{C}(\mathbb{X})$, then we could also informally write $\alpha : \mathrm{C}(\mathbb{X}) \to \mathrm{C}(\mathbb{X} \times \mathbb{G})$. To sustain this analogy, if B is commutative, so isomorphic to $\mathrm{C}(X)$ for some compact space X, we often speak simply of an action of $\mathbb{G}$ on X.

Podleś, and independently Boca (see respectively [Po$_2$] and [Boc]) showed that actions of compact quantum groups have always purely algebraic 'cores', in the sense described by the following theorem.

Theorem 1.18. *Let $\mathbb{G}$ be a compact quantum group acting on a unital C^*-algebra B via a unital *-homomorphism $\alpha : \mathsf{B} \to \mathsf{B} \otimes \mathrm{C}(\mathbb{G})$. Define for each $\beta \in \mathrm{Irr}_{\mathbb{G}}$ a continuous linear functional $\phi_\beta : \mathrm{C}(\mathbb{G}) \to \mathbb{C}$ determined by the conditions*

$$\phi_\beta(u^{\beta'}_{i,j}) = \delta_{\beta,\beta'}\delta_{i,j}, \quad \beta' \in \mathrm{Irr}_{\mathbb{G}}, i, j = 1, \ldots, n_{\beta'}$$

and define $E_\beta : \mathsf{B} \to \mathsf{B}$ as $E_\beta := (\mathrm{id}_{\mathsf{B}} \otimes \phi_\beta) \circ \alpha$. Then the following conditions hold:
(i) *the space $\mathcal{B} := \bigoplus_{\beta \in \mathrm{Irr}_{\mathbb{G}}} E_\beta(\mathsf{B})$ (the algebraic direct sum) is a dense unital *-subalgebra of B;*
(ii) *the restriction $\alpha|_{\mathcal{B}}$ takes values in $\mathcal{B} \odot \mathrm{Pol}(\mathbb{G})$ and is a coaction of the Hopf *-algebra $\mathrm{Pol}(\mathbb{G})$ on the *-algebra $\mathcal{B}$.*

Again, the existence of $\mathcal{B}$, sometimes called the *Podleś algebra*, allows us to construct reduced/universal versions of the action ([Li]). For more information on this we refer to [So$_4$]. In particular it is proved in that paper that the Podleś algebra in many cases coincides with the space $\{b \in \mathsf{B} : \alpha(b) \in \mathsf{B} \odot \mathrm{Pol}(\mathbb{G})\}$.

For an action as above and $\beta \in \mathrm{Irr}_{\mathbb{G}}$ we can define the following subspace of $\mathrm{Pol}(\mathbb{G})$: $W_\beta = \{(f \otimes \mathrm{id})\alpha(v) : f \in \mathsf{B}^*, v \in E_\beta(\mathcal{B})\}$. The algebra generated by all W_β inside $\mathrm{Pol}(\mathbb{G})$ is a Hopf *-algebra, which we will denote $R_\alpha(\mathbb{G})$. If $R_\alpha(\mathbb{G})$ is dense in $\mathrm{C}(\mathbb{G})$ (equivalently, $R_\alpha(\mathbb{G})$ is equal to $\mathrm{Pol}(\mathbb{G})$, see [DiK]), we say that the action α of $\mathbb{G}$ on B is *faithful*.

It turns out that in the construction of quantum symmetry groups a crucial role is played by the quantum version of an invariant measure.

Definition 1.19. Let $\mathbb{G}$ be a compact quantum group acting on a unital C^*-algebra B via a unital *-homomorphism $\alpha : \mathsf{B} \to \mathsf{B} \otimes \mathrm{C}(\mathbb{G})$. We say that the action α preserves a state $\omega \in S(\mathsf{B})$ if

$$(\omega \otimes \mathrm{id}_{C(\mathbb{G})}) \circ \alpha = \omega(\cdot)1_{C(\mathbb{G})}.$$

Exercise 1.5. Verify that the coproduct defines an action of $\mathbb{G}$ on $\mathrm{C}(\mathbb{G})$, preserving the Haar state. Interpret this action in the case where G is a classical compact group.

An action α of $\mathbb{G}$ on B is said to be *ergodic* if the *fixed point algebra* of α,

$$\mathrm{Fix}\, \alpha := \{b \in \mathsf{B} : \alpha(b) = b \otimes 1_{\mathrm{C}(\mathbb{G})}\},$$

is one-dimensional (i.e. equal to $\mathbb{C}1_{\mathsf{B}}$). Note that the fixed point algebra is the image of the *conditional expectation* (i.e. a completely positive norm one projection) $(\mathrm{id}_{\mathsf{B}} \otimes h) \circ \alpha$, where h denotes the Haar state of $\mathbb{G}$.

Exercise 1.6. Verify the last statement.

We finish this lecture by the following proposition due to Sołtan.

Proposition 1.20. *Let* B *be a finite-dimensional* C^**-algebra. Every action of a compact quantum group on* B *preserves some faithful state on* B*.*

2. Quantum symmetry groups of finite structures

The second lecture introduces the categorical approach to quantum group actions on a given finite-dimensional C^*-algebra, defines free permutation groups, quantum symmetry groups of finite graphs and universal quantum groups of Wang and Van Daele.

2.1. **Category of compact quantum groups acting on a given structure.** Consider a unital C^*-algebra B. We want to consider the *category of compact quantum group actions on* B, which we will denote by $\mathcal{C}_{\mathsf{B}}$. The objects in $\mathcal{C}_{\mathsf{B}}$ are pairs $(\mathbb{G}, \alpha)$, where $\mathbb{G}$ is a compact quantum group and α is an action of $\mathbb{G}$ on B. A morphism in $\mathcal{C}_{\mathsf{B}}$ from $(\mathbb{G}_1, \alpha_1)$ to $(\mathbb{G}_2, \alpha_2)$ is a morphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ intertwining the respective actions, i.e. a unital *-homomorphism $\gamma : \mathrm{C}_u(\mathbb{G}_2) \to \mathrm{C}_u(\mathbb{G}_1)$ such that

$$\alpha_1 = (\mathrm{id} \otimes \gamma) \circ \alpha_2.$$

A careful reader will have noticed that the displayed formula above formally speaking mixes the universal and reduced context; indeed, formally speaking we should understand the equality only as valid on respective Podleś algebras (the distinction will not be a problem in this lecture, as we will consider only finite-dimensional B, we will however come back to it later on). We say that $(\mathbb{G}_u, \alpha_u)$ is a *(universal) final object in* $\mathcal{C}_{\mathsf{B}}$ if for any object $(\mathbb{G}', \alpha')$ in $\mathcal{C}_{\mathsf{B}}$ there exists a unique morphism γ from $(\mathbb{G}', \alpha')$ to $(\mathbb{G}_u, \alpha_u)$. If it exists, we will call $\mathbb{G}_u$ the *quantum symmetry group of* B and denote by $\mathsf{QSYM}_{\mathsf{B}}$. The usual abstract categorical nonsense guarantees that if the quantum symmetry group of B exists, it is unique up to an isomorphism.

If we in addition consider a state ω on B, we can define in an obvious way the category $\mathcal{C}_{\mathsf{B},\omega}$ of all compact quantum group actions on B preserving the state ω. It is a full subcategory of $\mathcal{C}_{\mathsf{B}}$; in case it admits a final object we will call the underlying compact quantum group the *quantum symmetry group of* (B, ω) and denote it by $\mathsf{QSYM}_{\mathsf{B},\omega}$.

Exercise 2.1. Fix B and ω as above and suppose that $\mathcal{C}_{\mathsf{B}}$ admits a final object. Show that if a compact quantum group $\mathbb{G}$ admits a faithful action on B preserving the state ω, then it is a quantum subgroup of $\mathsf{QSYM}_{\mathsf{B},\omega}$.

In the rest of this lecture we will discuss some cases in which the quantum symmetry groups exist and some where they do not.

2.2. **Wang's free permutation groups.** The history of quantum symmetry groups starts in a sense with the fundamental paper [Wan$_2$], where Wang established the existence of the quantum (or free) permutation groups. Before we formulate the existence result we need to introduce the desired quantum group. The following result is Theorem 3.1 a of [Wan$_2$]; its proof is based on Theorem 1.13.

Proposition 2.1. *Let* $n \in \mathbb{N}$ *and consider the universal unital* C^**-algebra* $\mathsf{A}_s(n)$ *generated by a family* $(p_{ij} : i, j = 1, \ldots, n)$ *of the orthogonal projections such that for each* $i = 1, \ldots, n$

$$\sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} p_{ji} = 1. \tag{2.1}$$

The algebra $\mathsf{A}_s(n)$*, together with the coproduct* $\Delta : \mathsf{A}_s(n) \to \mathsf{A}_s(n) \otimes \mathsf{A}_s(n)$ *determined by the formula*

$$\Delta(p_{ij}) = \sum_{k=1}^{n} p_{ik} \otimes p_{kj}, \quad i, j = 1, \ldots, n.$$

is the (universal) algebra of continuous functions on a compact quantum group of Kac type, denoted S_n^+ *and called the free permutation group on n-elements.*

Note that the matrix $(p_{ij})_{i,j=1}^n \in M_n(\mathsf{A}_s(n))$ is a *magic unitary*, i.e. a unitary matrix whose entries are orthogonal projections, and moreover it is a fundamental representation of S_n^+. It is easy to verify, using the universal properties, that S_n^+ acts on $\mathbb{C}^n$ via the map

$$\alpha(\delta_i) = \sum_{j=1}^{n} \delta_j \otimes p_{ji}, \quad i = 1, \ldots, n. \tag{2.2}$$

Exercise 2.2. Prove that $\mathsf{A}_s(n)$ is commutative for $n = 1, 2, 3$ and noncommutative as soon as $n \geq 4$. Show that the classical group S_n is a quantum subgroup of S_n^+.

We are ready for the main result of this subsection.

Theorem 2.2. *Fix* $n \in \mathbb{N}$ *and consider the category of all compact quantum groups acting on an n-element set, i.e. the category* $\mathcal{C}_\mathsf{B}$ *for* $\mathsf{B} = \mathbb{C}^n$*. The category* $\mathcal{C}_\mathsf{B}$ *admits a final object; the quantum symmetry group of* $\mathbb{C}^n$ *is the free permutation group.*

Proof. Let $\mathbb{G}$ be a compact quantum group and let $\alpha' : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathrm{C}(\mathbb{G})$ be an action of $\mathbb{G}$ on the algebra $\mathbb{C}^n$, say given by the formulas

$$\alpha'(\delta_i) = \sum_{j=1}^{n} \delta_j \otimes x_{ji}, \quad i = 1, \ldots, n.$$

Simple computations using the fact that each of the elements $\delta_i \in \mathbb{C}^n$ is an orthogonal projection and that $\sum_{i=1}^n \delta_i = 1$ show that each of the elements $x_{ji} \in \mathrm{C}(\mathbb{G})$ is an orthogonal projection and that we have for each $i = 1, \ldots n$ the equality $\sum_{i=1}^n x_{ji} = 1$. To show the existence of a unital *-homomorphism from $\mathsf{A}_s(n)$ to $\mathrm{C}(\mathbb{G})$ we need to show that the matrix $(x_{ij})_{i,j=1}^n$ is a magic unitary, i.e. establish the other equality featuring in (2.1). To that end it suffices to show that α must preserve the counting measure of $\mathbb{C}^n$. By Proposition 1.20 there exists a faithful state ρ on $\mathbb{C}^n$ which is preserved by α. This means that there exists a sequence $(c_i)_{i=1}^n$ of strictly positive numbers summing to 1 such that for each $j \in \{1, \ldots, n\}$ we have $\sum_{j=1}^n c_j x_{ji} = c_i 1$. Relabelling the elements if necessary we can assume that there exists $k \in \{1, \ldots, n\}$ such that $c_1 = \cdots = c_k$ and if $l \in \{k+1, \ldots, n\}$ then $c_l > c_1$. If $k = n$ then we are done, as then ρ corresponds to the normalised counting measure. Consider then the case when $k < n$. As each x_{ij} is a projection, the equality

$$c_1 1 = \sum_{j=1}^{n} c_j x_{j1}$$

implies that $x_{l1} = 0$ if $l > k$. Similarly $x_{li} = 0$ for each $i \leq k$, $l > k$. This means also that

$$1 = \sum_{j=1}^{k} x_{ji}, \quad i \leq k.$$

But then

$$\sum_{j=1}^{k}\sum_{i=1}^{n} x_{ji} = k1 = \sum_{i=1}^{k}\sum_{j=1}^{k} x_{ji},$$

and as we are dealing with the sums of positive operators we must actually have $x_{li} = 0$ for $i > k$, $l \leq k$. This means that the matrix $(x_{ji})_{i,j=1}^n$ is in fact a block-diagonal matrix which has a magic unitary as a top-left $k \times k$ block. An obvious finite induction (working in the next step with $c_{k+1} = \cdots = c_{k+l} < c_{k+l+1}$) shows that the whole matrix is a magic unitary and thus the action preserves the counting measure.

Thus the universal property of $\mathsf{A}_s(n)$ provides the existence of a unique unital *-homomorphism $\gamma : \mathsf{A}_s(n) \to \mathrm{C}(\mathbb{G})$ such that $\gamma(p_{ij}) = x_{ij}$ for all $i, j = 1, \ldots, n$. It is easy to see that it intertwines the respective actions (see (2.2)) and thus defines a unique morphism from $(\mathbb{G}, \alpha')$ to (S_n^+, α). □

We established in the above proof that S_n^+ is also the quantum symmetry group of the pair $(\mathbb{C}^n, \omega)$, where ω is the counting measure. In fact Wang showed in [Wan$_2$] only the latter result – the proof given here comes from a recent article [BhSS]. For more information on quantum permutation groups we refer to the survey [BBC$_1$].

The terminology 'free permutation group' originates from its many relations to the free probability and also to the fact that the algebra $\mathsf{A}_s(n)$ carries many features similar to that of the group C*-algebra of the free group. A *quantum permutation group* is a compact quantum group which is a quantum subgroup of S_n^+ for some $n \in \mathbb{N}$. For many years it remained an open problem whether the quantum version of the Cayley's theorem holds, i.e. whether every finite quantum group is a quantum subgroup of a quantum permutation group. Recently it was solved in the negative in [BBN].

2.3. **Wang's quantum automorphism groups of matrix algebras.** The next simplest class of examples of C*-algebras, after the algebras $\mathbb{C}^n$ studied in the last subsection, are given by matrix algebras M_n. Here however the situation is more complicated, as already Wang noticed the following fact (see Theorem 6.1 (2) of [Wan$_2$]).

Proposition 2.3. *Let $n \geq 2$ and let $\mathsf{B} = M_n$. Then the category $\mathcal{C}_\mathsf{B}$ does not admit a final object.*

The reason behind the last fact informally can be explained by stating that the universal *quantum family of automorphisms* of M_n is only a compact quantum semigroup. Indeed, the category of compact quantum semigroup actions on M_n admits a final object. For more information on these topics and explanation of the concept of *quantum families* we refer to [So$_1$] and [So$_2$].

It is however also possible to formulate a positive result in this context (again proved via an application of Theorem 1.13).

Theorem 2.4 ([Wan$_2$])**.** *Let B be a finite-dimensional C*-algebra and let $\omega \in \mathsf{B}^*$ be a faithful state. Then the category $\mathcal{C}_{\mathsf{B},\omega}$ admits a final object.*

The quantum symmetry group of (M_n, ω) is usually denoted by $\mathsf{QAUT}(M_n, \omega)$. Its universal C*-algebra may be described explicitly via generators and relations. In particular Sołtan showed in [So$_3$] that $\mathsf{QAUT}(M_2, \omega_q)$, where ω_q is the state on M_2 given by the density matrix with eigenvalues $1/(1+q^2)$ and $q^2/(1+q^2)$, is isomorphic to $SO_q(3)$.

2.4. **Quantum symmetry groups of finite graphs.** After the existence of the universal compact quantum group acting on a finite set was established, it became natural to look for its quantum subgroups corresponding to quantum symmetry groups of finite sets equipped with some additional structure. The following concept was introduced by Bichon in [Bic] and later studied by Banica and Bichon (see [BB_1] and references therein).

Definition 2.5. Let $\mathcal{G}$ be a finite, non-directed graph (without multiple edges) with an associated adjacency matrix $D \in M_{|\mathcal{G}|}(\{0,1\})$. An action of a quantum group $\mathbb{G}$ on $\mathcal{G}$ is an action α of $\mathbb{G}$ on the algebra $\mathbb{C}^{|\mathcal{G}|}$ such that the associated magic unitary matrix $U := (u_{ij})_{i,j=1}^{|\mathcal{G}|}$, defined as usual by the formula

$$\alpha(\delta_i) = \sum_{j=1}^{n} \delta_j \otimes u_{ji}, \;\; i = 1, \ldots, |\mathcal{G}|,$$

commutes with D:

$$DU = UD.$$

Exercise 2.3. Show that if G is a classical compact group and α is an action of G on the set $\mathcal{G}$, then the commutativity relation in the above definition corresponds to the fact that α preserves the graph structure.

With the above definition in hand it is easy to define the quantum symmetry group of a finite graph $\mathcal{G}$, denoted $\mathsf{QSYM}(\mathcal{G})$, and prove its existence. We leave the details, similar to these presented in the last two sections, to the reader. The key combinatorial/algebraic question related to the concept of the quantum symmetry group of a finite graph is the following.

Question 2.6. When does a finite graph $\mathcal{G}$ admit quantum symmetries? In other words, when is the algebra $\mathrm{C}(\mathsf{QSYM}(\mathcal{G}))$ commutative?

Banica and Bichon answered this question and computed explicitly the quantum symmetry groups for many small graphs. We refer to [BB_1] for the list of results, and here note only that for example cyclic graphs admit no quantum symmetries.

The concept of the quantum symmetry group of a finite graph can be extended to the quantum symmetry group of a finite metric space ([Ba_2]), simply by replacing the adjacency matrix in Definition 2.5 by the corresponding metric matrix. This can be also viewed as looking for a quantum symmetry group of a finite coloured graph (different distances correspond to different colours of edges).

2.5. **Universal quantum groups of Wang and Van Daele.** In the following lecture we will need one more construction due to Van Daele and Wang ([VDW]). The following result is Theorem 1.3 of their paper, recast in the language we are using in this course. Given a matrix $V = (v_{ij})_{i,j=1}^n$ with entries in a C^*-algebra we define new matrices $\bar{V} = (v_{ij}^*)_{i,j=1}^n$ and $V^t = (v_{ji})_{i,j=1}^n$ (recall that $V^* = (v_{ji}^*)_{i,j=1}^n$).

Theorem 2.7. *Let $n \in \mathbb{N}$ and let $Q \in M_n(\mathbb{C})$ be an invertible matrix. Denote by $\mathsf{A}_u(Q)$ the universal C^*-algebra generated by the elements $\{u_{ij} : i,j = 1,\ldots,n\}$ such that*

(i) *the matrix $U := (u_{ij})_{i,j=1}^n \in M_n(\mathsf{A}_u(Q))$ is unitary;*
(ii) *the matrix U satisfies the following commutation relations:*

$$U^t Q \bar{U} Q^{-1} = I_n = Q \bar{U} Q^{-1} U^t.$$

There exists a compact quantum group $U_n^+(Q)$ such that $\mathrm{C}_u(U_n^+(Q)) \approx \mathsf{A}_u(Q)$ and the matrix U above is a fundamental unitary representation of $U_n^+(Q)$.

In the case where $Q = I_n$ we denote $U_n^+(Q)$ simply by U_n^+ and call it the *free unitary group.* Van Daele and Wang observe that the quantum groups defined via Theorem 2.7 have the following universal property: whenever $\mathbb{G}$ is a compact matrix quantum group, it is a quantum subgroup of some $U_n^+(Q)$.

The construction above has also a 'self-adjoint' version.

Theorem 2.8. *Let $n \in \mathbb{N}$ and let $Q \in M_n(\mathbb{C})$ be an invertible matrix. Denote by $\mathsf{A}_o(Q)$ the universal* C^**-algebra generated by the* ***self-adjoint*** *elements $\{u_{ij} : i, j = 1, \ldots, n\}$ such that*

(i) *the matrix $U := (u_{ij})_{i,j=1}^n \in M_n(\mathsf{A}_u(Q))$ is unitary;*

(ii) *the matrix U satisfies the following commutation relations:*

$$U^t Q \bar{U} Q^{-1} = I_n = Q \bar{U} Q^{-1} U^t.$$

There exists a compact quantum group $O_n^+(Q)$ such that $\mathrm{C}_u(O_n^+(Q)) \approx \mathsf{A}_o(Q)$ and the matrix U above is a fundamental unitary representation of $O_n^+(Q)$.

In the case where $Q = I_n$ we denote $O_n^+(Q)$ simply by O_n^+ and call it the *free orthogonal group.* Both Theorem 2.7 and Theorem 2.8 follow from Theorem 1.13. Compact quantum groups O_n^+ and U_n^+ are of Kac type, for more information on the structure and dependence of the quantum groups on the matrix Q we refer to [Wan$_3$]

Exercise 2.4. Prove that the quantum groups O_2^+ and $SU_{-1}(2)$ are isomorphic.

2.6. **Dual free product of compact quantum groups.** The following definition/theorem is due to Wang. Note however that the original paper uses a different terminology.

Theorem 2.9 ([Wan$_1$])**.** *Let $\{\mathbb{G}_i : i \in \mathcal{I}\}$ be a family of compact quantum groups. Then the* C^**-algebraic (universal) free product $\star_{i\in\mathcal{I}} \mathrm{C}(\mathbb{G}_i)$ has a natural structure of an algebra of continuous functions on a compact quantum group, to be denoted $\hat{\star}_{i\in\mathcal{I}} \mathbb{G}_i$.*

The construction is dual to the usual free product of discrete groups: when the quantum groups in question are duals of classical discrete groups, $\mathbb{G}_1 = \widehat{\Gamma_1}$, $\mathbb{G}_2 = \widehat{\Gamma_2}$, then $\mathbb{G}_1 \hat{\star} \mathbb{G}_2 \approx \widehat{\Gamma_1 \star \Gamma_2}$. For the construction of the coproduct we refer to Wang's paper, but here note that it is easy to describe for compact matrix quantum groups: if $U_1 \in M_n(\mathrm{C}(\mathbb{G}_1))$ and $U_2 \in M_m(\mathrm{C}(\mathbb{G}_2))$ are respective fundamental representations, then $\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \in M_{n+m}(\mathrm{C}(\mathbb{G}_1) \star \mathrm{C}(\mathbb{G}_2))$ is the fundamental representation of $\mathbb{G}_1 \hat{\star} \mathbb{G}_2$.

Exercise 2.5. Show that if $n, m \in \mathbb{N}$ then $S_n^+ \hat{\star} S_m^+$ is a quantum subgroup of S_{n+m}^+.

3. Quantum isometry groups of C*-algebras equipped with orthogonal filtrations

In this lecture we will describe a construction of a quantum symmetry group of an infinite (but possessing certain 'compactness' aspects) structure. It comes from the article [BS$_3$] and was motivated on one hand by a definition of the quantum isometry group of a noncommutative compact manifold à la Connes, thus generalizing the classical notion of the isometry group of a compact Riemannian manifold, due to Goswami ([Gos]) and on the other by a specific example of the Goswami's approach appearing in the context of finitely generated discrete groups. The latter will be described in more detail in the second part of the lecture.

3.1. **Quantum actions preserving orthogonal filtrations.**

Definition 3.1. Let B be a unital C*–algebra equipped with a faithful state ω and with a family $(V_i)_{i\in\mathcal{I}}$ of finite-dimensional subspaces of B (with the index set $\mathcal{I}$ containing a distinguished element 0) satisfying the following conditions:

(i) $V_0 = \mathbb{C}1_{\mathsf{B}}$;
(ii) for all $i, j \in \mathcal{I}$, $i \neq j$, $a \in V_i$ and $b \in V_j$ we have $\omega(a^*b) = 0$;
(iii) the set $\mathrm{Lin}(\bigcup_{i\in\mathcal{I}} V_i)$ is a dense *-subalgebra of B.

If the above conditions are satisfied we say that the pair $(\omega, (V_i)_{i\in\mathcal{I}})$ defines an orthogonal filtration of B; sometimes abusing the notation we will omit ω and simply say that $(\mathsf{B}, (V_i)_{i\in\mathcal{I}})$ is a C*-algebra with an orthogonal filtration. The (dense) *-subalgebra spanned in B by $\{V_i : i \in \mathcal{I}\}$ will be denoted by $\mathcal{B}$.

Note that the existence of an orthogonal filtration does not imply that the C*-algebra A is AF (i.e. *approximately finite-dimensional*, see [Mur]), although unital separable AF C*-algebras admit orthogonal filtrations, as we will see below. Other examples of importance for us are the reduced group C*-algebras. In most examples we have in fact $V_i = V_i^*$ and ω is a trace. Note that B can be viewed as the completion of $\mathcal{B}$ in the GNS representation with respect to ω.

Definition 3.2. Let $(\mathsf{B}, \omega, (V_i)_{i\in\mathcal{I}})$ be a C*-algebra with an orthogonal filtration. We say that a quantum group $\mathbb{G}$ acts on B in a filtration preserving way if there exists an action α of $\mathbb{G}$ on B such that the following condition holds:

$$\alpha(V_i) \subset V_i \odot C(\mathbb{G}), \;\; i \in \mathcal{I}.$$

We will then write $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$.

Before we continue, we make one important observation. Let $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$. It is not difficult to check that in fact for each $i \in \mathcal{I}$ we have

$$\alpha(V_i) \subset V_i \odot \mathrm{Pol}(\mathbb{G}).$$

It is also easy to see that if $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$, then α preserves the state ω:

$$(\omega \otimes \mathrm{id}_{\mathrm{C}(\mathbb{G})}) \circ \alpha = \omega(\cdot)1_{\mathrm{C}(\mathbb{G})}. \tag{3.1}$$

Indeed, the conditions (i) and (iii) in Definition 3.1 imply immediately that $\omega(a) = 0$ for all $a \in \mathrm{Lin}\,(\cup_{i\in\mathcal{I}\setminus\{0\}} V_i)$. Hence the equality (3.1) holds on the dense subalgebra $\mathcal{B}$; as both sides of (3.1) are continuous, it must in fact hold everywhere.

As before, the morphisms in the category $\mathcal{C}_{\mathsf{B},\mathcal{V}}$ are compact quantum group morphisms which intertwine the respective actions. Let us now be very precise: this means that if $(\mathbb{G}_1, \alpha_1), (\mathbb{G}_2, \alpha_2) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$ then a morphism from $(\mathbb{G}_1, \alpha_1)$ to $(\mathbb{G}_2, \alpha_2)$ is a unital *-homomorphism $\gamma : \mathrm{C}_u(\mathbb{G}_2) \to \mathrm{C}_u(\mathbb{G}_1)$ such that

$$(\mathrm{id}_{\mathsf{A}} \otimes \gamma|_{\mathrm{Pol}(\mathbb{G}_2)}) \circ \alpha_2|_{\mathcal{B}} = \alpha_1|_{\mathcal{B}}.$$

Definition 3.3. We say that $(\alpha_u, \mathbb{G}_u)$ is a final object in $\mathcal{C}_{\mathsf{B},\mathcal{V}}$ if for any $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$ there exists a unique morphism γ from $(\alpha, \mathbb{G})$ to $(\alpha_u, \mathbb{G}_u)$.

To prove the existence result we need to complete the purely algebraic description of $\mathcal{C}_{\mathsf{B},\mathcal{V}}$.

Definition 3.4. We say that a compact quantum group $\mathbb{G}$ admits an algebraic action α_0 on $\mathcal{B}$ (the dense unital *-subalgebra of B), preserving the filtration $\mathcal{V}$ if $\alpha_0 : \mathcal{B} \to \mathcal{B} \odot \mathrm{Pol}(\mathbb{G})$ is a unital *-homomorphism such that

(i) $\alpha_0(\mathcal{B})(1 \odot \mathrm{Pol}(\mathbb{G}))) = \mathcal{B} \odot \mathrm{Pol}(\mathbb{G})$;
(ii) $(\alpha_0 \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{G}))})\alpha_0 = (\mathrm{id}_{\mathcal{B}} \otimes \Delta)\alpha_0$;
(iii) $\alpha_0(V_i) \subset V_i \odot \mathrm{Pol}(\mathbb{G})$, $i \in \mathcal{I}$.

We then write $(\alpha_0, \mathbb{G}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. The morphisms in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ are defined analogously to those in $\mathcal{C}_{\mathsf{B},\mathcal{V}}$, with all maps acting between the algebraic objects (so that $\gamma : \mathrm{Pol}(\mathbb{G}_2) \to \mathrm{Pol}(\mathbb{G}_1)$, etc.).

Lemma 3.5. *The categories $\mathcal{C}_{\mathsf{B},\mathcal{V}}$ and $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ are isomorphic.*

Proof. The discussion before the definition implies that if $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$, then $(\alpha_0 := \alpha|_{\mathcal{B}}, \mathbb{G}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. On the other hand Lemma 3.1 of [Cur] implies that if $(\alpha_0, \mathbb{G}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$, then, as the action α_0 preserves the (faithful) state $\omega|_{\mathcal{B}}$, and the corresponding GNS completion of $\mathcal{B}$ is isomorphic to B, α_0 extends to an action $\alpha : \mathsf{B} \to \mathsf{B} \otimes \mathrm{C}_r(\mathbb{G})$ (recall that $\mathrm{C}_r(\mathbb{G})$ denotes the 'reduced version' of $\mathrm{C}(\mathbb{G})$). It is then easy to check that $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$.

Given a morphism γ in $\mathcal{C}_{\mathsf{B},\mathcal{V}}$ between $(\alpha_1, \mathbb{G}_1)$ and $(\alpha_2, \mathbb{G}_2)$, we know that as it is a compact quantum group morphism, it restricts to a *-homomorphism between respective dense Hopf *-algebras. On the other hand an 'algebraic' compact quantum group morphism acting on the level of Hopf *-algebras extends uniquely to a unital *-homomorphism acting between their universal completions and preserving the respective coproducts. The facts that respective restrictions/extensions intertwine the respective actions follow directly from the definitions. □

In the next subsection we will often use for a given action the algebra $R_\alpha(\mathbb{G})$ defined in Lecture 1. Note that if $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$ then $R_\alpha(\mathbb{G})$ is the algebra generated by spaces $\{(f \otimes \mathrm{id})\alpha(v) : v \in V_i, f \in V_i'\}$, $i \in \mathcal{I}$, and the same description works with α replaced by α_0.

3.2. **Main existence result.** In this subsection we present the main existence result for the quantum symmetry group of an orthogonal filtration and sketch its proof. Full details can be found in [BS$_3$].

Theorem 3.6. *Let $(\mathsf{B}, \omega, (V_i)_{i\in\mathcal{I}})$ be a C^*-algebra with an orthogonal filtration. The category $\mathcal{C}_{\mathsf{B},\mathcal{V}}$ admits a final object; in other words there exists a universal compact quantum group $\mathbb{G}_u$ acting on B in a filtration preserving way. We call $\mathbb{G}_u$ the quantum symmetry group of $(\mathsf{B}, \omega, (V_i)_{i\in\mathcal{I}})$. The canonical action of $\mathbb{G}_u$ on B is faithful.*

Proof. Observe first that by Lemma 3.5 it suffices to show that the category $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ has a final object. Let us divide the proof into several steps.

I

Fix for each $i \in \mathcal{I}$ an orthonormal basis $\{e_1, \ldots, e_{k_i}\}$ in V_i with respect to the scalar product given by the state ω (i.e. $\omega(e_l^* e_m) = \delta_{lm}1$, $l, m = 1, \ldots, k_i$) and let $\{f_1, \ldots, f_{k_i}\}$ in V_i^* be an orthonormal basis for V_i^*. Consider the family $(e_l^*)_{l=1}^{k_i}$ of elements of V_i^*. This family is linearly independent, so there exists an invertible matrix $S^{(i)} \in M_{k_i}$ such that

$$e_l^* = \sum_{m=1}^{k_i} S^{(i)}_{lm} f_m \quad l = 1, \ldots, k_i. \tag{3.2}$$

Put $Q_i = \overline{S^{(i)}}(S^{(i)})^T \in GL_{k_i}(\mathbb{C})$ and let $\mathsf{D}_V = \star_{i\in\mathcal{I}} \mathsf{A}_u(Q_i)$, where for each $i \in \mathcal{I}$ the algebra $\mathsf{A}_u(Q_i)$ is considered with the canonical generating set $\{\mathcal{U}^{(i)}_{lm} : l, m = 1, \ldots, k_i\}$ (see Definition

2.7). Define the algebra D_V to be the universal algebra of continuous functions on the compact quantum group $\hat{\star}_{i\in\mathcal{I}} z, U^+(Q_i)$ (now see Definition 2.9); the corresponding algebraic free product $\mathrm{Pol}(\hat{\star}_{i\in\mathcal{I}} z, U^+(Q_i))$ will be denoted by $\mathcal{D}_V$.

II

In the second step we show that if α_0 is an algebraic action of a quantum group $\mathbb{H}$ on $\mathcal{B}$ and $(\alpha_0, \mathbb{H}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$, then the restriction of α_0 to a map on V_i determines in a natural way a representation of $\mathbb{H}$. We also prove that this representation is automatically unitary and construct a *-homomorphism from $\mathcal{D}_V$ to $R_{\alpha_0}(\mathbb{H})$.

Let $(\alpha_0, \mathbb{H}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. Fix $i \in \mathcal{I}$ (and skip it from most of the notation in the next paragraph). Condition (iii) in Definition 3.4 implies that there exists a matrix $U = (u_{lm})_{l,m=1}^k \in M_k(\mathrm{Pol}(\mathbb{H}))$ such that

$$\alpha_0(e_m) = \sum_{l=1}^k e_l \otimes u_{lm}, \quad m = 1, \ldots, k. \tag{3.3}$$

Due to the condition (3.1) U is an isometry; indeed,

$$\begin{aligned}
\delta_{lm} 1_{\mathrm{C}_r(\mathbb{H})} &= \omega(e_l^* e_m) 1_{\mathrm{C}_r(\mathbb{H})} = (\omega \otimes 1_{\mathrm{C}_r(\mathbb{H})})(\alpha_0(e_l)^* \alpha_0(e_m)) \\
&= (\omega \otimes 1_{\mathrm{C}_r(\mathbb{H})})\big((\sum_{p=1}^k e_p^* \otimes u_{pl}^*)(\sum_{p=1}^k e_q \otimes u_{qm})\big) = \sum_{p,q=1}^k \omega(e_p^* e_q) u_{pl}^* u_{qm} \\
&= \sum_{p=1}^k u_{pl}^* u_{pm} = (U^*U)_{lm}.
\end{aligned}$$

To show that U is actually a unitary, we need to employ the Podleś condition for the action (it is easier here to use Lemma 3.5 and pass to the 'analytic' version of α_0, to be denoted by α; we can also assume that we are dealing with the 'reduced' action $\alpha : \mathsf{B} \to \mathsf{B} \otimes \mathrm{C}_r(\mathbb{H})$). Suppose that U is not unitary. Viewing U as an operator on the Hilbert module $\mathcal{M} := \mathbb{C}^k \otimes \mathrm{C}_r(\mathbb{H})$, we see that (by Theorem 3.5 in [Lan]) there must exist some element in $\mathcal{M}$ which is not in the range of U; in fact its distance from the range of U must be strictly greater than some $\epsilon > 0$. In other words there is a sequence $b_1, \ldots, b_k$ of elements in $\mathrm{C}_r(\mathbb{H})$ such that for all possible sequences $c_1, \ldots, c_k$ of elements in $C_r(\mathbb{H})$ we have

$$b_l \neq \sum_{m=1}^k u_{lm} c_m, \quad \text{for some } l \in \{1, \ldots, m\}.$$

Consider now an element $\mathbf{b} = \sum_{l=1}^k e_l \otimes b_l \in V_i \odot \mathrm{C}_r(\mathbb{H}) \subset \mathsf{B} \otimes \mathrm{C}_r(\mathbb{H})$. The last displayed formula means precisely that $\mathbf{b} \notin \alpha(V_i)(1 \otimes C_r(\mathbb{H}))$. Moreover, the remark on the Hilbert module distance means that if $\mathbf{d} \in \alpha(V_i)(1 \otimes \mathrm{C}_r(\mathbb{H}))$ then

$$\|(\omega \otimes \mathrm{id}_{\mathrm{C}_r(\mathbb{H})})((\mathbf{b} - \mathbf{d})^*(\mathbf{b} - \mathbf{d}))\| > \epsilon^2.$$

Consider then any $c \in \mathrm{Lin}\left\{\bigcup_{j\in\mathcal{I}} V_j\right\}$, say $c = \sum_{j\in F} c_j$, where F is a finite subset of $\mathcal{I}$, and any family $(b_j)_{j\in F}$ of elements of $\mathrm{C}_r(\mathbb{H})$. Put $\mathbf{d} := \sum_{j\in F} \alpha(c_j)(1 \otimes b_j)$, $\mathbf{d}_i = \alpha(c_i)(1 \otimes b_i)$. Then

$$\|\mathbf{b} - \mathbf{d}\|^2 \geq \|(\omega \otimes \mathrm{id}_{\mathrm{C}_r(\mathbb{H})})((\mathbf{b} - \mathbf{d})^*(\mathbf{b} - \mathbf{d}))\| \geq \|(\omega \otimes \mathrm{id}_{\mathrm{C}_r(\mathbb{H})})((\mathbf{b} - \mathbf{d}_i)^*(\mathbf{b} - \mathbf{d}_i))\| > \epsilon^2.$$

It follows from this that $\mathbf{b} \notin \alpha(\mathsf{B})(1 \otimes C_r(\mathbb{H}))$, which is a contradiction.

Consider now V_i^*. As α_0 preserves also this set, the above proof shows that the matrix $W = (w_{ml})_{m,l=1}^k \in M_k(\mathrm{Pol}(\mathbb{H}))$ determined by the condition

$$\alpha(f_l) = \sum_{m=1}^{k} f_m \otimes w_{ml}, \quad l = 1, \ldots, k, \tag{3.4}$$

is also unitary. A comparison of the formulas (3.2)-(3.4) yields the following equality:

$$WS^T = S^T \bar{U},$$

so that the unitarity of W transforms into the following condition:

$$I = \bar{S}^{-1}\bar{U}^*\bar{S}S^T\bar{U}(S^T)^{-1} = S^T\bar{U}(S^T)^{-1}\bar{S}^{-1}\bar{U}^*\bar{S},$$

or, putting $Q = \bar{S}S^T \in GL_k(\mathbb{C})$,

$$I = \bar{U}^*Q\bar{U}Q^{-1} = \bar{U}Q^{-1}\bar{U}^*Q.$$

This means that the family $(U_{lm})_{l,m=1}^k$ satisfies the defining relations for the generators of Van Daele's and Wang's universal unitary algebra $\mathsf{A}_u(Q_i)$. Hence there exists a unique unital *-homomorphism $\pi_i : \mathsf{A}_u(Q_i) \to \mathrm{C}_r(\mathbb{H})$ such that

$$\pi_i(\mathcal{U}_{lm}) = u_{lm} \in \mathrm{Pol}(\mathbb{H}) \tag{3.5}$$

for $l, m = 1, \ldots, k$. It is easy to see that π_i intertwines respective coproducts; moreover π_i maps the *-algebra $\mathcal{B}_u(Q_i)$ spanned by the elements of type u_{lm} into $\mathrm{Pol}(\mathbb{H})$ (and even more specifically into $R_\alpha(\mathbb{H})$). Consider the algebraic free product of all the respective corestrictions of morphisms π_i:

$$\pi_{\alpha,\mathbb{H}} = \star_{i\in\mathcal{I}}\, \pi_i : \mathcal{D}_V \to R_\alpha(\mathbb{H}).$$

Note that the image of $\pi_{\alpha,\mathbb{H}}$ is actually equal to $R_\alpha(\mathbb{H}) = R_{\alpha_0}(\mathbb{H})$.

III

In the third step we introduce another class of *-homomorphisms which generalise actions in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ and establish some formulas satisfied by these homomorphisms.

In the rest of the proof we will only consider algebraic actions and denote them simply by α. We need to consider a larger class of *-homomorphisms from $\mathcal{D}_V$ into algebras of functions on compact quantum groups. This idea comes from [QS]. Denote the collection of all finite sequences $(\alpha_1, \mathbb{H}_1), (\alpha_2, \mathbb{H}_2), \cdots, (\alpha_k, \mathbb{H}_k) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ $(k \in \mathbb{N})$ by $T_\mathcal{C}$. For each such sequence $T \in T_\mathcal{C}$ consider the *-homomorphism $\alpha_T : \mathcal{B} \to \mathcal{B} \odot \mathrm{Pol}(\mathbb{H}_1) \odot \cdots \odot \mathrm{Pol}(\mathbb{H}_k)$ defined by

$$\alpha_T = (\alpha_1 \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{H}_2)} \otimes \cdots \otimes \mathrm{id}_{\mathrm{Pol}(\mathbb{H}_k)}) \cdots (\alpha_{k-1} \otimes \mathrm{id}_{R(\mathbb{H}_k)})\alpha_k$$

(α_T should be thought of as reflecting the composition of consecutive actions of $\mathbb{H}_1, \cdots, \mathbb{H}_k$ on $\mathcal{B}$ – note however it need not be an action of the group $\mathbb{H}_1 \times \cdots \times \mathbb{H}_k$). Similarly for a sequence T as above consider the mapping $\pi_T : \mathcal{D}_V \to \mathrm{Pol}(\mathbb{H}_1) \odot \cdots \odot \mathrm{Pol}(\mathbb{H}_k)$ given by

$$\pi_T = (\pi_{\alpha_1,\mathbb{H}_1} \otimes \cdots \otimes \pi_{\alpha_k,\mathbb{H}_k}) \circ \Delta_{k-1},$$

where $\Delta_k : \mathcal{D}_V \to \mathcal{D}_V^{\otimes k}$ is the usual iteration of the coproduct of $\mathcal{D}_V$ (and $\Delta_0 := \mathrm{id}_{\mathcal{D}_V}$). Note that if $T, S \in T_\mathcal{C}$ and TS denotes the concatenation of the sequences, we have formulas

$$\alpha_{TS} = (\alpha_T \otimes \mathrm{id})\alpha_S, \tag{3.6}$$

$$\pi_{TS} = (\pi_T \otimes \pi_S) \circ \Delta. \tag{3.7}$$

Define a linear map $\beta : \mathcal{B} \to \mathcal{B} \odot \mathcal{D}_V$ via the linear extension of the formula (considering separately each $i \in \mathcal{I}$ and the orthogonal basis $e_1, \ldots, e_{k_i} \in V_i$):

$$\beta(e_l) = \sum_{m=1}^{k_i} e_m \otimes \mathcal{U}_{ml}, \quad l = 1, \ldots, k_i.$$

Observe that although β need not be a *-homomorphism, it is unital and moreover is a coalgebra morphism:

$$(\beta \otimes \mathrm{id}_{\mathcal{D}_V})\beta = (\mathrm{id}_{\mathcal{B}} \otimes \Delta)\beta \tag{3.8}$$

(it is enough to check the above equality on all the elements e_l, where it is elementary). Moreover we have

$$\beta(\mathcal{B})(1 \odot \mathcal{D}_V) = \mathcal{B} \odot \mathcal{D}_V; \tag{3.9}$$

indeed, it is enough to show that the left hand side contains any element of the form $e_l^{(i)} \odot 1$, where $e_l^{(i)}$ is one of the basis elements of V_i. The latter elements can be obtained from the expressions of the type

$$\sum_{m=1}^{k_i} \beta(e_m^{(i)})(\mathcal{U}_{lm}^{(i)})^*.$$

Further we have for each $T \in T_C$

$$\alpha_T = (\mathrm{id}_{\mathcal{B}} \otimes \pi_T) \circ \beta. \tag{3.10}$$

Indeed, if the length of the sequence T is 1, then the formula above follows directly from the definition of $\pi_{\alpha,\mathbb{H}}$ for $(\alpha, \mathbb{H}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. Further, for any two sequences $T, S \in T_C$ for which (3.10) holds we have (using (3.6), (3.7) and (3.8))

$$\begin{aligned}\alpha_{TS} &= (\alpha_T \otimes \mathrm{id})\alpha_S = (((\mathrm{id}_{\mathcal{B}} \otimes \pi_T) \circ \beta) \otimes \mathrm{id}) \circ (\mathrm{id}_{\mathcal{B}} \otimes \pi_S) \circ \beta \\ &= (\mathrm{id}_{\mathcal{B}} \otimes \pi_T \otimes \pi_S) \circ (\beta \otimes \mathrm{id}_{\mathcal{D}_V}) \circ \beta = (\mathrm{id}_{\mathcal{B}} \otimes \pi_T \otimes \pi_S) \circ (\mathrm{id}_{\mathcal{B}} \otimes \Delta)\beta \\ &= (\mathrm{id}_{\mathcal{B}} \otimes \pi_{TS})\Delta,\end{aligned}$$

so (3.10) follows by induction for all sequences in T_C.

IV

Here we define the compact quantum group $\mathbb{G}$ which will turn out to be our universal object.

Let $I_0 = \bigcap_{T \in T_{\mathcal{C}}} \mathrm{Ker}\, \pi_T$ (the class of objects in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ need not be a set, but we can get around this problem in the usual way, identifying isomorphic objects and bounding the dimension of the algebras considered). Then I_0 is a two-sided *-ideal in $\mathcal{D}_V$. We will show that it is also a Hopf *-ideal, i.e. that if $q : \mathcal{D}_V \to \mathcal{D}_V / I_0$ is the canonical quotient map, then $(q \otimes q)\Delta(I) = \{0\}$. To this end it suffices (via the usual application of slice functionals) to show that if S, T are sequences in $T_{\mathcal{C}}$ then for each $b \in I$ we have

$$(\pi_T \otimes \pi_S)\Delta(b) = 0. \tag{3.11}$$

This however follows from (3.7). Thus the unital *-algebra $\mathcal{D}_V/_{I_0}$ is in fact a CQG algebra. Denote the corresponding compact quantum group by $\mathbb{G}$ (so that $\mathrm{Pol}(\mathbb{G}) = \mathcal{D}/_{I_0}$) and the quotient *-homomorphism from $\mathcal{D}$ onto $\mathcal{D}_V/_{I_0}$ by q. The construction above shows that

$$(q \otimes q) \circ \Delta_{\mathcal{D}_V} = \Delta_{\mathbb{G}} \circ q. \tag{3.12}$$

V

In this step we show that the quantum group $\mathbb{G}$ acts on $\mathcal{B}$ in a $\mathcal{V}$-preserving way.

Let $\alpha_u : \mathcal{B} \to \mathcal{B} \odot \mathrm{Pol}(\mathbb{G})$ be given by

$$\alpha_u = (\mathrm{id}_{\mathcal{B}} \otimes q) \circ \beta. \tag{3.13}$$

We want to show that α_u is a *-homomorphism. To this end it suffices to show that if $a, b \in \mathcal{B}$ then

$$\beta(a^*) - \beta(a)^* \in \mathcal{B} \odot I_0, \quad \beta(ab) - \beta(a)\beta(b) \in \mathcal{B} \odot I_0,$$

or, in other words, that for all $T \in T_{\mathcal{C}}$

$$(\mathrm{id}_{\mathcal{B}} \otimes \pi_T)(\beta(a^*) - \beta(a)^*) = 0, \quad (\mathrm{id}_{\mathcal{B}} \otimes \pi_T)(\beta(ab) - \beta(a)\beta(b)) = 0.$$

The above formulas are however equivalent (by the fact that $\mathrm{id}_{\mathcal{B}} \otimes \pi_T$ is a *-homomorphism and by (3.10)) to the formulas

$$\alpha_T(a^*) - \alpha_T(a)^* = 0, \quad \alpha_T(ab) - \alpha_T(a)\alpha_T(b) = 0,$$

which are clearly true as each α_T is defined as a composition of *-homomorphisms. The fact that α_u satisfies condition (ii) in Definition 3.4 follows by putting together (3.13), (3.8) and (3.12). Condition (iii) in Definition 3.4 can be checked directly. Finally the nondegeneracy condition (i) is a consequence of (3.13), (3.9) and the fact that $q : \mathcal{D} \to \mathrm{Pol}(\mathbb{G})$ is a surjective homomorphism.

Hence $(\alpha_u, \mathbb{G}) \in \mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. The fact that the action α_u is faithful follows from the construction.

VI

Finally we show that the pair $(\alpha_u, \mathbb{G})$ is the final object in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$.

Consider any object $(\alpha, \mathbb{H})$ in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$. Recall the map $\pi_{\mathsf{B},\mathbb{H}} : \mathcal{D}_V \to \mathrm{Pol}(\mathbb{H})$. The kernel of $\pi_{\mathsf{B},\mathbb{H}}$ is contained in I_0; hence there exists a unique map $\pi' : \mathcal{D}_V/_{I_0} \to \mathrm{Pol}(\mathbb{H})$ such that $\pi_{\mathsf{B},\mathbb{H}} = \pi' \circ q$. Using the fact that $\pi_{\mathsf{B},\mathbb{H}}$ intertwines the coproducts of $\mathcal{D}_V$ and $\mathrm{Pol}(\mathbb{H})$ together with the formula (3.12) we obtain that $\pi' : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H})$ is a morphism of compact quantum groups. Similarly we compute

$$(\mathrm{id}_{\mathcal{B}} \otimes \pi')\alpha_u = (\mathrm{id}_{\mathcal{B}} \otimes \pi')(\mathrm{id}_{\mathcal{B}} \otimes q)\beta = (\mathrm{id}_{\mathcal{B}} \otimes \pi_{\alpha,\mathbb{H}}) \circ \beta = \alpha,$$

where the last equality follows from (3.10). Thus π' is a desired morphism in $\mathcal{C}^{alg}_{\mathsf{B},\mathcal{V}}$ between $(\alpha, \mathbb{H})$ and $(\alpha_u, \mathbb{G})$. Its uniqueness can be easily checked using the fact that the elements of the type $q(\mathcal{U}^{(i)}_{lm})$, $i \in \mathcal{I}$, $l, m = 1, \ldots, k_i$ generate $\mathrm{Pol}(\mathbb{G})$ as a *-algebra. □

The first part of the proof was inspired by the arguments in Section 4 of [Gos]. Here however we avoid any references to the *Dirac operator* and work directly with the filtration of the underlying C*-algebra. Soon after the article [BS$_3$] appeared, the construction was generalised by Thibault de Chanvalon to the context of orthogonal filtrations of Hilbert modules, see [TDC].

The following corollary mirrors a similar observation for quantum symmetry groups of finite spaces.

Corollary 3.7. *Let $(\mathsf{B}, \omega, (V_i)_{i\in\mathcal{I}})$ be a* C*-*algebra with an orthogonal filtration and let $(\alpha_u, \mathbb{G}_u)$ be the universal object in $\mathcal{C}_{\mathsf{B},\mathcal{V}}$. If $(\alpha, \mathbb{G}) \in \mathcal{C}_{\mathsf{B},\mathcal{V}}$ and the action α is faithful, then the morphism $\pi_\alpha : \mathrm{Pol}(\mathbb{G}_u) \to \mathrm{Pol}(\mathbb{G})$ constructed in Theorem 3.6 is surjective. In other words, $\mathbb{G}$ is a quantum subgroup of $\mathbb{G}_u$.*

Proof. It suffices to observe that it follows from the construction in the proof of Theorem 3.6 that the image of the morphism γ_α contains $R_\alpha(\mathbb{G})$. □

In some cases certain properties of the universal quantum symmetry group $\mathbb{G}_u$ follow directly from certain properties of the filtration.

Theorem 3.8. *Let $(\mathsf{B}, \omega, (V_i)_{i\in\mathcal{I}})$ be a* C*-*algebra with an orthogonal filtration and let $\mathbb{G}$ be its quantum symmetry group, with a corresponding action $\alpha : \mathsf{B} \to \mathsf{B} \otimes C(\mathbb{G})$. The following implications hold:*

(i) *if ω is a trace then $\mathbb{G}_u$ is a compact quantum group of Kac type;*
(ii) *if there exists a finite set $F \subset \mathcal{I}$ such that the union of subspaces $\bigcup_{i\in F} V_i$ generates B as a* C*-*algebra, then $\mathbb{G}_u$ is a compact matrix quantum group;*
(iii) *if ω is a trace, and there exists $i \in \mathcal{I}$ such that V_i generates B as a* C*-*algebra, and $\{e_1, \cdots, e_k\}$ is an orthonormal basis of V_i with respect to the scalar product determined by ω (so that $\omega(e_l^* e_m) = \delta_{lm}1$ for $l, m = 1, \ldots, k$), then the matrix $U = (u_{lm})_{l,m=1}^k$ of elements of* C$(\mathbb{G})$ *determined by the condition*

$$\alpha(e_m) = \sum_{l=1}^{k} e_l \otimes u_{lm}, \quad j = 1, \ldots, k,$$

is a fundamental unitary representation of $\mathbb{G}$ (and $\bar{U}$ is also unitary). In particular $\mathbb{G}$ is a quantum subgroup of U_k^+.

Proof. It suffices to look at the proof of Theorem 3.6 and note that if ω is a trace and $\{e_1, \ldots, e_k\}$ is an orthonormal basis of V_i then $\{e_1^*, \ldots, e_k^*\}$ is an orthonormal basis of V_i^*, so that the matrix Q_i appearing in that proof is equal to I_k. □

In the following we will discuss several examples of quantum symmetry groups associated to orthogonal filtrations. Here we note one easy case, formulated as an exercise.

Exercise 3.1. Let B be a finite-dimensional C*-algebra with a faithful state ω. Prove that the quantum symmetry group $\mathsf{QAUT}(\mathsf{B}, \omega)$ may be viewed as the quantum symmetry group of a C*-algebra equipped with an orthonormal filtration.

3.3. **Filtrations related to discrete groups – general framework.** Let Γ be a discrete group. As before, the elements of the reduced group C*-algebra will be denoted in the same way as the elements of Γ; in particular we identify the group ring $\mathbb{C}[\Gamma]$ as a subalgebra of $C^*_r(\Gamma)$ via $\mathbb{C}[\Gamma] = \mathrm{span}\{\gamma : \gamma \in \Gamma\}$. The canonical trace on $C^*_r(\Gamma)$ is given by the continuous extension of the formula:

$$\tau(\gamma) = \begin{cases} 1 & \text{if} \quad \gamma = e \\ 0 & \text{if} \quad \gamma \neq e \end{cases}$$

(compare this formula to that in (1.2)). We will consider below partitions of Γ into finite sets, always assuming that $\{e\}$ (where e denotes the neutral element of Γ) is one of the sets in the partition. The following lemma is straightforward.

Lemma 3.9. *If $\mathcal{F} = (F_i)_{i\in\mathcal{I}}$ is a partition of Γ into finite sets and $V_i^{\mathcal{F}} := \mathrm{span}\{\gamma : \gamma \in F_i\} \subset \mathrm{C}^*_r(\Gamma)$ ($i \in \mathcal{I}$), then the pair $(\tau, (V_i^{\mathcal{F}})_{i\in\mathcal{I}})$ defines an orthogonal filtration of $\mathrm{C}^*_r(\Gamma)$.*

Definition 3.10. The quantum symmetry group of $(\mathrm{C}^*_r(\Gamma), \tau, (V_i^{\mathcal{F}})_{i\in\mathcal{I}})$, defined according to Theorem 3.6, will be called the quantum symmetry group of $\widehat{\Gamma}$ preserving the partition $\mathcal{F}$ and denoted $\mathsf{QSYM}_{\Gamma,\mathcal{F}}$.

For a discrete group Γ and any vector space $\mathcal{V}$ we will consider the linear maps $f_\gamma : \mathcal{V} \odot \mathbb{C}[\Gamma] \to \mathcal{V}$ ($\gamma \in \Gamma$) defined by the linear extension of the prescription

$$f_\gamma(\gamma' \otimes v) = \delta_{\gamma,\gamma'} v, \quad \gamma' \in \Gamma, v \in \mathcal{V}.$$

Definition 3.11. Let $\mathbb{G}$ be a compact quantum group and assume that $\alpha : \mathrm{C}^*_r(\Gamma) \to \mathrm{C}^*_r(\Gamma) \otimes \mathrm{C}(\mathbb{G})$ is an action of $\mathbb{G}$ on $\mathrm{C}^*_r(\Gamma)$. Let $\mathcal{F} = (F_i)_{i\in\mathcal{I}}$ be a partition of a discrete group Γ into finite sets. The action α is said to preserve $\mathcal{F}$ if

(i) $\alpha : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \odot \mathrm{C}(\mathbb{G})$;
(ii) for all $i \neq j \in \mathcal{I}$ and $\gamma \in F_i$ there is $f_\gamma|_{\alpha(F_j)} = 0$.

Proposition 3.12. *Let $\mathcal{F} = (F_i)_{i\in\mathcal{I}}$ be a partition of a discrete group Γ into finite sets. The quantum symmetry group of $\widehat{\Gamma}$ preserving the partition $\mathcal{F}$ is the universal compact quantum group $\mathbb{G}$ acting on $\mathrm{C}^*_r(\Gamma)$ via an $\mathcal{F}$ preserving action. Moreover if for some $i \in \mathcal{I}$ the subset $F_1 = \{\gamma_1, \ldots, \gamma_k\}$ generates Γ as a group, then the matrix $U = (u_{lm})_{l,m=1}^k \in M_k(C(\mathbb{G}))$ given by*

$$\alpha(\gamma_m) = \sum_{l=1}^{k} \gamma_l \otimes u_{lm}, \quad m = 1, \ldots, k,$$

is a fundamental unitary representation of $\mathbb{G}$.

Proof. The first part of the proposition follows from the comparison of the conditions defining respective classes of quantum group actions (the one in Definition 3.2 and the one in Definition 3.11). The second is a consequence of Theorem 3.8. □

A particular case of the above construction, motivated by the quantum isometry groups of Goswami, was studied earlier in detail in [BhS]. Let Γ be a finitely generated group with a fixed symmetric generating set S and the related word length $l : \Gamma \to \mathbb{N}_0$. Then the collection $\mathcal{F} = \{l^{-1}(\{n\}) : n \in \mathbb{N}_0\}$ is a partition of Γ. The corresponding quantum symmetry group $\mathsf{QSYM}_{\Gamma,\mathcal{F}}$ is called the *quantum isometry group* of $\widehat{\Gamma}$ and denoted $\mathsf{QISO}(\widehat{\Gamma})$ (or $\mathsf{QISO}(\widehat{\Gamma}, S)$). For the justification of this terminology we refer to [Gos] and [BhS] – a hint can be found in the following exercise.

Exercise 3.2. Consider the group $\mathbb{Z}$ with the usual generating set $\{-1, 1\}$. Compute the corresponding quantum isometry group $\mathsf{QISO}(\widehat{\Gamma})$. Note that we can view it as the universal compact group acting on $\widehat{\mathbb{Z}} = \mathbb{T}$ in a manner preserving the metric of $\mathbb{T}$!

3.4. **Quantum isometry group of the dual of the free group.** In this section we will fix $n \in \mathbb{N}$ and consider $\mathbb{F}_n$, the free group on n generators equipped with the standard symmetric generating set $S = \{s_1, s_1^{-1}, \ldots, s_n, s_n^{-1}\}$ and the word-length function l induced by S.

Define the matrix $\mathcal{Q}_n$ to be the $2n$ by $2n$ block-diagonal matrix with the only non-zero entries built of 2 by 2 blocks $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ put along the diagonal. The following result combines Theorem 5.1 of [BhS] and results of Sections 2 and 3 of [BS_1].

Theorem 3.13. *Consider the universal unital* C^**-algebra* $\mathsf{A}_h(n)$ *generated by the collection of elements* $(u_{i,j} : i, j = 1, \ldots, 2n)$ *satisfying the following properties:*

(i) *each* u_{ij} $(i, j = 1, \ldots, 2n)$ *is a partial isometry, i.e.* $u_{ij}u_{ij}^*u_{ij} = u_{ij}$*;*

(ii) *the matrix* $U = (u_{ij})_{i,j=1}^{2n}$ *is a unitary;*

(iii) *we have* $U = \mathcal{Q}_n \bar{U} \mathcal{Q}_n$*.*

Then $\mathsf{A}_h(n)$ *is* $\mathrm{C}(H_{n,0}^+)$ *for a certain compact quantum group* $H_{n,0}^+$*, and* $U \in M_{2n}(\mathsf{A}_h(n))$ *is a fundamental unitary representation of* $H_{n,0}^+$*. Moreover* $H_{n,0}^+$ *is the quantum isometry group* $\mathsf{QISO}(\widehat{\mathbb{F}_n})$*.*

Let us consider the case $n = 2$ in more detail. The fundamental unitary representation of $\mathsf{QISO}(\widehat{\mathbb{F}_2})$ introduced above takes the form

$$U = \begin{bmatrix} A & B & C & D \\ B^* & A^* & D^* & C^* \\ E & F & G & H \\ F^* & E^* & H^* & G^* \end{bmatrix},$$

where each of the operators $A, \ldots, H$ is a partial isometry and the defining relations of $\mathsf{A}_h(2)$ are equivalent to stating that

$$\begin{bmatrix} P_A & P_B & P_C & P_D \\ P_E & P_F & P_G & P_H \\ Q_B & Q_A & Q_D & Q_C \\ Q_F & Q_E & Q_H & Q_G \end{bmatrix},$$

where $P_A = AA^*, Q_A = A^*A$, etc., is a magic unitary. The action of $H_{2,0}^+$ on $\mathrm{C}_r^*(\mathbb{F}_2)$ is then determined by the following conditions (recall that s_1, s_2 denote the generators of $\mathbb{F}_2$)

$$\alpha(s_1) = s_1 \otimes A + s_1^{-1} \otimes B + s_2 \otimes C + s_2^{-1} \otimes D,$$

$$\alpha(s_2) = s_1 \otimes E + s_1^{-1} \otimes F + s_2 \otimes G + s_2^{-1} \otimes H.$$

As the notation above suggests, in fact we can view the quantum groups $H_{n,0}^+$ as a part of a two-parameter family $H_{n,m}^+$, where for example $H^+(0, m)$ is the m-th *quantum hyperoctahedral group*, i.e. the quantum symmetry group of the graph built of m connected pairs of points (m 'bars') – see [BS$_1$] and [BBC$_2$].

The free group $\mathbb{F}_n$ admits another natural length function, a so called *block length* b. The block length is defined in the following way: we view $\mathbb{F}_n$ as the free product of n copies of $\mathbb{Z}$, denote each of these copies by Γ_i $(i = 1, \ldots, n)$, write any element $\gamma \in \mathbb{F}_n$ as a reduced word in elements in each of the groups Γ_i, and declare the length of this word to be the block length of γ; thus

$$b(\gamma) = k$$

if

$$\gamma = \gamma_{i_1} \cdots \gamma_{i_k}, \quad i_j \in \{1, \ldots, n\}, \quad i_j \neq i_{j+1}, \quad \gamma_{i_j} \in \Gamma_{i_j} \setminus \{e\}.$$

The idea is that each element $\gamma_{i_j} \in \Gamma_{i_j}$ in the decomposition above corresponds to a block in γ. So for example

$$b(s_1^{k_1} s_2^{k_2} s_1^{k_3}) = 3, \text{ if only } k_1, k_2, k_3 \neq 0.$$

Consider the filtration of Γ given by the sets $F_{l,m} = \{\gamma \in \mathbb{F}_n : l(\gamma) = l, b(\gamma) = m\}$ $(l, m \in \mathbb{N}_0, l \leq m)$. It is clear that each $F_{l,m}$ is finite and closed under taking inverses. Write $\mathcal{F}_b := \{F_{l,m} : l, m \in \mathbb{N}_0, l \leq m\}$. Then we obtain the following result (Section 5 of [BS$_3$]).

Theorem 3.14. *Consider the universal unital* C^**-algebra* $\mathsf{A}_{hs}(n)$ *generated by the collection of elements* $(u_{i,j} : i, j = 1, \ldots, 2n)$ *satisfying the following properties:*

(i) *each* u_{ij} $(i, j = 1, \ldots, 2n)$ *is a normal partial isometry, i.e.* $u_{ij}u_{ij}^*u_{ij} = u_{ij}$ *and* $u_{ij}^*u_{ij} = u_{ij}u_{ij}^*$*;*

(ii) *the matrix* $U = (u_{ij})_{i,j=1}^{2n}$ *is a unitary;*

(iii) *we have* $U = \mathcal{Q}_n\bar{U}\mathcal{Q}_n$*.*

Then $\mathsf{A}_{hs}(n)$ *is* $\mathrm{C}(K_n^+)$ *for a certain compact quantum group* K_n^+*, and* $U \in M_{2n}(\mathsf{A}_{hs}(n))$ *is a fundamental unitary representation of* K_n^+*. Moreover* K_n^+ *is the quantum symmetry group* $\mathsf{QSYM}_{\mathbb{F}^n,\mathcal{F}_b}$*.*

It is easy to see that K_n^+ (as a universal quantum group acting on $\mathrm{C}_r^*(\mathbb{F}_n)$ in a way preserving both the word and block length) is a quantum subgroup of $H_{n,0}^+$.

For several more examples of computations of quantum isometry groups of duals of finitely generated discrete groups we refer for example to the articles [BhS], [L-DS], [TQi] and [SkS].

4. Further constructions, connections to the concept of the liberated quantum groups

In this lecture we describe projective limits of quantum symmetry groups, present their connections to Bratteli diagrams of AF algebras and discuss the example of the quantum symmetry group of (a particular presentation of) the Cantor set. We finish by introducing a connection of quantum symmetry groups with the so-called liberation procedure.

4.1. **Projective limits of quantum symmetry groups.** Suppose further that $(\mathsf{B}_n)_{n=1}^{\infty}$ is an inductive system of unital C^*-algebras, with the connecting unital *-homomorphisms $\rho_{n,m} \colon \mathsf{B}_n \to \mathsf{B}_m$. Assume that each B_n is equipped with a faithful state ω_n, an orthogonal filtration $\mathcal{V}_n$ (with respect to ϕ_n) and that we have the following compatibility condition: for each $n, m \in \mathbb{N}$, $n < m$, and each $V \in \mathcal{V}_n$ we have $\rho_{n,m}(V) \in \mathcal{V}_m$. Note that this implies the following two facts:

(i) for all n, m as above we have $\omega_m \circ \rho_{n,m} = \phi_m$,

(ii) each of the maps $\rho_{n,m}$ is injective (and thus so are the maps $\rho_{n,\infty}$).

If the above assumptions hold, we will say that $(\mathsf{B}_n, \mathcal{V}_n)_{n=1}^{\infty}$ is *an inductive system of* C^**-algebras equipped with orthogonal filtrations*. We can then speak about a natural inductive limit filtration $\mathcal{V}_\infty$ of the limit algebra B_∞, defined in the following way: a subspace $V \subset \mathsf{B}_\infty$ belongs to $\mathcal{V}_\infty$ if and only if there exists $n \in \mathbb{N}$ and $V_n \in \mathcal{V}_n$ such that $V = \rho_{n,\infty}(V_n)$. The arising filtration satisfies then the orthogonality conditions with respect to the inductive limit state $\omega_\infty \in \mathsf{B}_\infty^*$.

Note that there is one subtlety here: although we can always construct the inductive limit filtration, the inductive limit state need not be faithful on B_∞, so we need not be in the framework of Definition 3.1 – we only know that ω_∞ is faithful on the dense subalgebra $\bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \rho_{n,\infty}(V)$ of B_∞. Suppose in addition that ω_∞ is a trace (equivalently, each of the states ω_n is tracial). Then it is automatically faithful (as its null space, $\{a \in \mathsf{B}_\infty : \omega_\infty(a^*a) = 0\}$, is an ideal, cf. [Bla, Proposition II.8.2.4]).

The following result was shown in [SkS]; it is based on a straightforward diagram chasing.

Theorem 4.1. *Let* $(\mathsf{B}_n, \mathcal{V}_n)_{n=1}^{\infty}$ *be an inductive system of* C^**-algebras equipped with orthogonal filtrations and assume that each of the states defining the orthogonality of the filtrations* $\mathcal{V}_n$ *is* *tracial. Let $\mathcal{V}_\infty$ denote the orthogonal filtration of B_∞ arising as the inductive limit. Denote the respective quantum symmetry groups of $(\mathsf{B}_n, \mathcal{V}_n)$ and $(\mathsf{B}_\infty, \mathcal{V}_\infty)$ by $\mathbb{G}_n$ and $\mathbb{G}$. Then*

$$\mathbb{G} = \varprojlim \mathbb{G}_n.$$

The conditions in the above theorem are rather restrictive; we will present one application in the next subsection, but here would also like to note that in some cases one can identify the projective limit of quantum symmetry groups as a quantum symmetry group (of some new filtration) even if they are not satisfied. For an example of such a situation in the context of quantum isometry groups of finite (and infinite) symmetric groups we refer to Section 5 of [SkS].

4.2. **Quantum symmetry groups of Bratteli diagrams.** We will describe now a construction of a natural quantum symmetry group for an *AF* C*-algebra equipped with a faithful state, introduced first in [BhGS].

Let B denote a unital *AF* C*-algebra and let $(\mathsf{B}_n)_{n=1}^\infty$ be an increasing limit of finite-dimensional unital C*-subalgebras of B whose union is dense in B. In addition put $\mathsf{B}_0 = \mathbb{C}1_{blg}$. Further fix a faithful tracial state ω on B. In such a situation we can introduce a natural orthogonal filtration of B defining inductively $V_0 = \mathsf{B}_0$, $V_{n+1} = \mathsf{B}_{n+1} \ominus V_n := \{b \in \mathsf{B}_{n+1} : \omega(b^*v) = 0, v \in V_n\}$ for all $n \in \mathbb{N}$. It is easy to verify that $\mathcal{V} = (V_n)_{n\in\mathbb{N}_0}$ satisfies the conditions in Definition 3.1. Thus we can use Theorem 3.6 to define the quantum symmetry group $\mathsf{QSYM}_\mathcal{V}$; for historical reasons we call it the quantum isometry group of (B, ω) and denote $\mathsf{QISO}_{\mathsf{B},\omega}$.

Theorem 4.1[2] shows that $\mathsf{QISO}_{\mathsf{B},\omega} = \varprojlim \mathbb{G}_n$, where $\mathbb{G}_n$ denotes for each $n \in \mathbb{N}$ the quantum symmetry group of the orthogonal filtration $(V_k)_{k=0,\ldots,n}$ of the algebra B_n. In the particular case where B_n is commutative and the state is the canonical trace, Theorem 2.6 of [BhGS] shows that $\mathbb{G}_n$ is in fact isomorphic to the quantum symmetry group $\mathsf{QSYM}(\mathcal{G}_n)$, where $\mathcal{G}_n$ is the Bratteli diagram of B_n (in other words the Bratteli diagram of B 'cut' at the n-th level). Thus, slightly abusing the terminology, we can also call the $\mathsf{QISO}_{\mathsf{B},\omega}$ *the quantum symmetry group of the Bratteli diagram of* B.

Below we present one particular example related to the quantum isometry group of the 'middle-third' Cantor set.

Theorem 4.2 ([BhGS])**.** *Let $\mathrm{C}(\mathfrak{C})$ be the AF C*-algebra arising as a limit of the unital embeddings*

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \cdots .$$

Suppose that τ is the canonical trace on $\mathrm{C}(\mathfrak{C})$. Then $\mathsf{QISO}_{\mathrm{C}(\mathfrak{C}),\tau} = \varprojlim \mathbb{G}_n$, where $\mathrm{C}(\mathbb{G}_1) = \mathrm{C}(\mathbb{Z}_2)$ and for $n \in \mathbb{N}$ we have

$$\mathrm{C}(\mathbb{G}_{n+1}) = (\mathrm{C}(\mathbb{G}_n) \star \mathrm{C}(\mathbb{G}_n)) \oplus (\mathrm{C}(\mathbb{G}_n) \star \mathrm{C}(\mathbb{G}_n)).$$

Further $\mathrm{C}_u(\mathsf{QISO}_{\mathrm{C}(\mathfrak{C}),\tau})$ is the universal unital C*-algebra generated by the family of selfadjoint projections

$$\{p\} \cup \bigcup_{n\in\mathbb{N}} \{p_{m_1,\ldots m_n} : m_1, \ldots m_n \in \{1,2,3,4\}\}$$

subjected to the following relations:

$$p_1, p_2 \le p, \quad p_3, p_4 \le p^\perp,$$

[2]Note that although Theorem 4.1 was formulated only for tracial ω, here the assumptions guarantee that the state on the limit filtration is faithful, so that the claims of that theorem remain valid, see [BhGS].

$$p_{m_1,\ldots,m_n,1}, p_{m_1,\ldots,m_n,2} \leq p_{m_1,\ldots,m_n}, \quad p_{m_1,\ldots,m_n,3}, p_{m_1,\ldots,m_n,4} \leq p^{\perp}_{m_1,\ldots,m_n}$$
($n \in \mathbb{N}, m_1, \ldots m_n \in \{1, 2, 3, 4\}$).

The quantum group structure of $\mathsf{QISO}_{\mathrm{C}(\mathfrak{C}),\tau}$ can be read out for example from the description in the second part of the above theorem, we refer the reader to Theorem 3.1 of [BhGS] for the details. The inductive procedure leading to $\mathsf{QISO}_{\mathrm{C}(\mathfrak{C}),\tau}$ may be visualised by the sequence of pictures, representing consecutive subdivisions of a square. The fractal structure of the limiting algebra is apparent. Note also that the classical symmetry group of the tree-type graph we consider can be graphically interpreted as a one-dimensional version of the above two-dimensional picture (so that the classical symmetry group at the n-th level is simply equal to $\prod_{i=1}^{2^n} \mathbb{Z}_2$).

Exercise 4.1. Compute explicitly $\mathbb{G}_2$ for the example studied in Theorem 4.2.

We will note below one consequence of the above observation, showing that quantum group actions behave very differently from the classical ones. We first prove the relevant classical property.

Lemma 4.3. *Suppose that $(X, d_X), (Y, d_Y)$ are compact metric spaces and $T : X \times Y \to X \times Y$ is an isometry satisfying the following condition: $\alpha_T(C(X) \otimes 1_Y) \subset C(X) \otimes 1_Y$, where $\alpha_T : \mathrm{C}(X \times Y) \to \mathrm{C}(X \times Y)$ is given by the composition with T. Then T has to be a product isometry, i.e. $T = T_X \times T_Y$ where $T_X \in \mathrm{ISO}(X)$, $T_Y \in \mathrm{ISO}(Y)$.*

Proof. Denote the family of isometries of $X \times Y$ satisfying the conditions of the lemma by $\mathsf{ISO}_X(X \times Y)$. We claim that $\mathsf{ISO}_X(X \times Y)$ is a group. Recall that $\mathsf{ISO}(Z)$, the family of all isometries of a compact metric space (Z, d_Z), is a compact group when considered with the topology of uniform convergence (equivalently, pointwise convergence; equivalently, metric topology given by $d(T_1, T_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} d_Z(T_1(z_i), T_2(z_i))$, where $\{z_i : i \in \mathbb{N}\}$ is a countable dense subset of Z). It is easy to that $\mathsf{ISO}_X(X \times Y)$ is a unital closed subsemigroup of $\mathsf{ISO}(X \times Y)$. Thus it is a compact semigroup satisfying the cancellation properties and it has to be closed under taking inverses.

Suppose now that $T \in \mathsf{ISO}_X(X \times Y)$. Then if $f \in C(X)$ we have for all $x \in X, y, y' \in Y$

$$(f \otimes 1_Y)(T(x, y)) = \alpha_T(f \otimes 1_Y)(x, y) = \alpha_T(f \otimes 1_Y)(x, y') = (f \otimes 1_Y)(T(x, y')).$$

This is equivalent to the fact that T is given by the formula

$$T(x, y) = (h(x), g(x, y)), \quad x \in X, y \in Y,$$

for some transformations $h : X \to X$, $g : X \times Y \to Y$. The fact that T is an isometry implies in particular that for all $x, x' \in X$, $y \in Y$.

$$d_X(x, x') = d_X(h(x), h(x')) + d_Y(g(x, y), g(x', y)). \tag{4.1}$$

In particular $h : X \to X$ is a contractive transformation. As by the first part of the proof $T^{-1} \in \mathsf{ISO}_X(X \times Y)$, there exist transformations $h' : X \to X$, $g' : X \times Y \to Y$ such that

$$T^{-1}(x, y) = (h'(x), g'(x, y)), \quad x \in X, y \in Y,$$

It is easy to see that h' is the inverse transformation of h, and as by the same argument as above we see that h' is a contractive transformation, hence h has to be an isometry. This together with formula (4.1) implies that $g : X \times Y \to Y$ does not depend on the first coordinate, so that T must be a product isometry. In particular $\mathsf{ISO}_X(X \times Y) = \mathsf{ISO}_X \times \mathsf{ISO}_Y$. □

Theorem 4.2 shows that the result above has no counterpart for quantum group actions, even on classical spaces. We could think of 'elements' of $\mathbb{G}_2$ as quantum isometries acting on the Cartesian product of 2 two-point set, 'preserving' the first coordinate in the sense analogous to the one in the lemma above. If this forced elements of $\mathbb{G}_2$ to be product isometries, we would necessarily have $\mathbb{G}_2 = \mathbb{G}_1 \times \mathbb{G}_1$; in particular $\mathbb{G}_2$ would have to be a classical group.

4.3. **Liberated quantum groups.** A *liberation procedure* for classical compact groups, albeit not formally well-defined, has in recent years proved to be a very important concept, related to quantum symmetry groups. The starting point is the following straightforward observation.

Proposition 4.4. *Let $\mathbb{G}$ be a compact quantum group. Consider the commutator ideal $\mathcal{J} \subset$ $\mathrm{C}(\mathbb{G})$, i.e. the smallest closed two-sided ideal of $\mathrm{C}(\mathbb{G})$ containing all elements of the form $ab - ba$ with $a, b \in \mathrm{C}(\mathbb{G})$. Then the algebra $\mathrm{C}(\mathbb{G})/\mathcal{J}$ has the structure of an algebra of continuous functions on a compact group G, which we denote $\mathbb{G}_{\mathrm{clas}}$.*

Exercise 4.2. Prove the above proposition.

The key fact behind the liberation idea is the observation that sometimes one can 'reconstruct' in a natural way $\mathbb{G}$ from $\mathbb{G}_{\mathrm{clas}}$ by finding a suitable presentation of $\mathrm{C}(\mathbb{G}_{\mathrm{clas}})$ and dropping the commutation relation.

The liberation procedure can then be informally described as follows:

- take your favourite compact matrix group G with a fixed fundamental representation $U = (u_{ij})_{i,j=1}^n \in M_n(\mathrm{C}(G))$;
- describe properties of U via a family of relations $\mathcal{R}$ satisfied by its (mutually commuting!) entries;
- drop the commutativity assumption – i.e. consider the universal C*-algebra A generated by elements ($\tilde{u}_{ij} : i, j = 1, \ldots, n$) satisfying all the relations $\mathcal{R}$ apart from the commutativity requirement;
- show that $\mathsf{A} = \mathrm{C}(\mathbb{G})$ for a certain compact quantum group $\mathbb{G}$, $\tilde{U} = (\tilde{u}_{ij})_{i,j=1}^n$ is a fundamental representation of $\mathbb{G}$ and $G = \mathbb{G}_{\mathrm{clas}}$.

As examples of liberations of the type described above we mention the passages $S_n \to S_n^+$, $O_N \to O_n^+$, $U_n \to U_n^+$ or $H_n \to H_n^+$ (where we consider respectively classical and free permutation, orthogonal, unitary and hyperoctahedral groups). For example in the symmetric group case one first realises $\mathrm{C}(S_n)$ as the universal *commutative* C*-algebra generated by n^2 selfadjoint projections ($p_{ij} : i, j = 1, \ldots, n$) such that $(p_{ij})_{i,j=1}^n$ is a magic unitary.

It should be clear that the liberation procedure involves several ambiguities (the choice of a fundamental representation; and then the choice of relations describing it). It has however revealed many interesting connections to free probability, random matrix theory and quantum notions of independence. Probably most striking and initially motivating aspect of the liberation procedure is the relation between the representation theories of G and its liberated quantum partner $\mathbb{G}$, appearing on the level of combinatorial descriptions via categories of partitions (see [BSp]).

Here we only note that the symmetric group case mentioned above is an example of a situation in which the quantum symmetry group of a given structure X (in this case a finite set) is the liberation of the classical symmetry group of X.

Recall the matrix $\mathcal{Q}_n \in M_{2n}$ defined in the beginning of Subsection 3.4. The following theorem comes from [BS$_1$].

Theorem 4.5. *Let $n \in \mathbb{N}$. The classical version of $H^+(n,0)$ is $\mathbb{T}^n \rtimes H_n$; moreover as the algebra $\mathrm{C}(\mathbb{T}^n \rtimes H_n)$ is the universal commutative C^*-algebra generated by elements $(u_{i,j} : i,j = 1,\ldots,2n)$ such that*

(i) *each u_{ij} $(i,j=1,\ldots,2n)$ is a partial isometry, i.e. $u_{ij}u_{ij}^*u_{ij} = u_{ij}$;*
(ii) *the matrix $U = (u_{ij})_{i,j=1}^{2n}$ is a unitary;*
(iii) *we have $U = \mathcal{Q}_n \bar{U} \mathcal{Q}_n$.*

Thus we can view $H^+(n,0)$ as the liberation of $\mathbb{T}^n \rtimes H_n$

Exercise 4.3. Prove the above result.

Recalling that $\mathbb{T}^n \rtimes H_n$ is the usual isometry group of $\mathbb{T}^n$, Theorems 4.5 and 3.13 show that here the quantum isometry group of a structure $\mathbb{X}$ (i.e. $\widehat{\mathbb{F}_n}$) is the liberation of the classical symmetry group of X (i.e. $\mathbb{T}^n \approx \widehat{\mathbb{Z}_n}$), where $\mathbb{X}$ can be viewed as the liberation of X!

For a similar phenomenon related to viewing O_n^+ as the quantum isometry group of a free (liberated) sphere we refer to [BGo].

5. Other structures related to quantum symmetry groups and open problems

In this, last lecture we present certain examples of noncommutative C^*-algebras which are not algebras of continuous functions on compact quantum groups, but are closely related to quantum symmetry groups described earlier. They are in a sense of a 'rectangular' character. We end by presenting some open problems.

5.1. Quantum homogeneous spaces inside S_n^+ and O_n^+.

Definition 5.1. Let $\mathbb{G}$ be a compact quantum group with a quantum subgroup $\mathbb{H}$ and the associated surjective unital *-homomorphism $\pi : \mathrm{C}_u(\mathbb{G}) \to \mathrm{C}_u(\mathbb{H})$ intertwining the respective coproducts. The algebra of continuous functions on the quantum homogeneous space $\mathbb{G}/\mathbb{H}$ is defined as

$$\mathrm{C}_u(\mathbb{G}/\mathbb{H}) = \{a \in \mathrm{C}_u(\mathbb{G}) : (\pi \otimes \mathrm{id})(\Delta(a)) = 1 \otimes a\}. \tag{5.1}$$

Note that $\mathrm{C}_u(\mathbb{G}/\mathbb{H})$ is a unital C^*-algebra, which can be viewed as the fixed point space of the canonical *right* action $(\pi \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}}$ of $\mathbb{H}$ on $\mathrm{C}_u(\mathbb{G})$. Further the (universal) action of $\mathbb{G}$ on $\mathrm{C}_u(\mathbb{G})$ via the coproduct restricts to a (universal) action β of $\mathbb{G}$ on $\mathrm{C}_u(\mathbb{G}/\mathbb{H})$; the resulting action is ergodic. These observations go back to [Po$_2$].

Exercise 5.1. State purely algebraic versions of the above facts and prove both the algebraic and topological versions.

The above action is ergodic, and the unique invariant state (note that these notions, introduced in Section 1, have obvious versions for the right actions) is the Haar state on $\mathbb{G}$.

Classically ergodic actions of a compact group G are in one-to-one correspondence with homogeneous spaces for G (so also with compact subgroups of G). The quantum situation is far more complicated (again, for examples of this we refer to [Po$_2$], and for a recent analysis of the concept of quantum homogeneous spaces in the framework of locally compact quantum groups to the article [KaS]). A word of warning is in place – some authors use the terminology 'quantum homogeneous space of a quotient type' to describe the concept introduced in Definition 5.1.

Note the following, not too difficult result, shown in [BSS].

Proposition 5.2. *Let $\mathbb{G}$ be a compact quantum group with two quantum subgroups $\mathbb{H}_1$, $\mathbb{H}_2$ and the associated surjective unital *-homomorphisms $\pi_i : \mathrm{C}_u(\mathbb{G}) \to \mathrm{C}_u(\mathbb{H}_i)$ intertwining the respective coproducts ($i = 1, 2$). Then $\mathrm{C}_u(\mathbb{G}/\mathbb{H}_1) = \mathrm{C}_u(\mathbb{G}/\mathbb{H}_2)$ if and only if $\mathbb{H}_1$ is isomorphic to $\mathbb{H}_2$ as a quantum subgroup of $\mathbb{G}$, i.e. there exists an isomorphism $\gamma : \mathrm{C}_u(\mathbb{H}_1) \to \mathrm{C}_u(\mathbb{H}_2)$ such that $\gamma \circ \pi_1 = \pi_2$.*

In the rest of this section we will focus on quantum symmetry groups O_n^+ and S_n^+. As the treatment and results are identical in both cases (and in fact can be also extended to other situations, see [BSS]), we will write simply $\mathbb{G}_n$ to denote one of these quantum groups ($n \in \mathbb{N}$), and by G_n their classical versions; one only has to remember that for example when we write $\mathbb{G}_k \subset \mathbb{G}_n$ for $k < n$ we mean of course $S_k^+ \subset S_n^+$ and $O_k^+ \subset O_n^+$, not say $O_k^+ \subset S_n^+$.

Consider then $k, n \in \mathbb{N}$, $k < n$. We will always work with the *diagonal embeddings* of G_k into G_n and $\mathbb{G}_k$ into $\mathbb{G}_n$. This on the level of fundamental unitary representations means that we consider as the morphism identifying $\mathbb{G}_k$ as a quantum subgroup of $\mathbb{G}_n$ the map induced by the formula

$$u_{ij} \mapsto \begin{cases} v_{ij} & \text{if } i, j \leq k \\ \delta_{ij} 1_{\mathrm{C}_u(\mathbb{G}_k)} & \text{otherwise} \end{cases},$$

where $(u_{ij})_{i,j=1}^n$ is the canonical fundamental representation of $\mathbb{G}_n$ and $(v_{ij})_{i,j=1}^k$ the canonical fundamental representation of $\mathbb{G}_k$ (and do the same for the classical versions). This means that we can consider the quantum homogeneous spaces $\mathbb{G}_n/\mathbb{G}_k$, as well as the classical homogeneous spaces G_n/G_k. The following result offers alternative descriptions of the latter spaces.

Proposition 5.3. *Let $k, n \in \mathbb{N}$, $k < n$ and let $G_k \subset G_n$ denote the diagonal embedding of the symmetric or orthogonal groups. Denote the canonical fundamental unitary representation of G_n by $(u_{ij})_{i,j=1}^n$. Then the algebra $\mathrm{C}_u(G_n/G_k)$, defined as in* (5.1) *has the following alternative descriptions:*

- (a) *it coincides with the unital C^*-subalgebra $\mathrm{C}_\times(G_n/G_k)$ of $\mathrm{C}_u(G_n) = \mathrm{C}(G_n)$ generated by the elements $\{u_{ij} : i = k+1, \ldots n, j = 1, \ldots n\}$;*
- (b) *it is isomorphic to the universal unital commutative C^*-algebra $\mathrm{C}_+(G_n/G_k)$ generated by selfadjoint elements $(v_{ij}) : i = k+1, \ldots n, j = 1, \ldots n$ such that*
  - (a) *the rectangular matrix $V = (v_{ij})_{i=k+1,j=1}^n$ is a coisometry: $VV^* = I$;*
  - (b) *in case where $G_n = S_n$ each v_{ij} is a projection.*

Exercise 5.2. Prove the above statement.

Interestingly, both algebras $\mathrm{C}_\times(G_n/G_k)$ and $\mathrm{C}_+(G_n/G_k)$ have natural quantum versions, but the above proposition does not extend to the quantum setup, as we describe below.

Definition 5.4. Let $k, n \in \mathbb{N}$, $k < n$ and let $\mathbb{G}_k \subset \mathbb{G}_n$ denote the diagonal embedding of the symmetric or orthogonal quantum groups. Denote the canonical fundamental unitary representation of $\mathbb{G}_n$ by $(u_{ij})_{i,j=1}^n$. Let $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$ be the C^*-subalgebra of $\mathrm{C}_u(\mathbb{G}_n)$ generated by the elements $\{u_{ij} : i = k+1, \ldots n, j = 1, \ldots n\}$ and let $\mathrm{C}_+(\mathbb{G}_n/\mathbb{G}_k)$ be the universal unital C^*-algebra generated by selfadjoint elements $(v_{ij}) : i = k+1, \ldots n, j = 1, \ldots n$ such that

- (i) the rectangular matrix $V = (v_{ij})_{i=k+1,j=1}^n$ is a coisometry: $VV^* = I$;
- (ii) in case where $\mathbb{G}_n = S_n^+$ each v_{ij} is a projection.

It is easy to check that $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$ is a quotient of $\mathrm{C}_+(\mathbb{G}_n/\mathbb{G}_k)$ and that $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$ is a subalgebra of $\mathrm{C}(\mathbb{G}_n/\mathbb{G}_k)$. On the other hand we have the following theorem.

Theorem 5.5. *Let $k, n \in \mathbb{N}$, $n \geq 4$, $2 \leq k \leq n-1$, and let $\mathbb{G}_k \subset \mathbb{G}_n$ denote the diagonal embedding of the symmetric or orthogonal quantum groups. Then the inclusion $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k) \subset \mathrm{C}(\mathbb{G}_n/\mathbb{G}_k)$ is proper.*

Proof. Denote by $U = (u_{ij})_{i,j=1}^n$ the canonical fundamental representation of $\mathbb{G}_n$ and by $V = (v_{ij})_{i,j=1}^k$ the canonical fundamental representation of $\mathbb{G}_k$. Let $\alpha := (\pi \otimes \mathrm{id}) \circ \Delta_{\mathbb{G}_n}$ denote the canonical right action of $\mathbb{G}_k$ on $\mathrm{C}_u(\mathbb{G}_n)$, where $\pi : \mathrm{C}_u(\mathbb{G}_n) \to \mathrm{C}(\mathbb{G}_k)$ is the morphism inducing the inclusion $\mathbb{G}_k \subset \mathbb{G}_n$.

Consider first the case $2 \leq k \leq n-2$. Fix a nontrivial projection $p \in \mathrm{C}(\mathbb{Z}_2)$, and consider the following matrix:

$$\tilde{p} = \begin{pmatrix} p & p^\perp \\ p^\perp & p \end{pmatrix}.$$

Further define $\mathsf{B} = \mathrm{C}_u(\mathbb{G}_k) \star \mathrm{C}(\mathbb{Z}_2)$, let $\nu : \mathrm{C}_u(\mathbb{G}_n) \to \mathsf{B}$ be the surjection induced by the mapping $\mathrm{diag}(V, \tilde{p}, 1_{n-k-2}) \mapsto U$, and consider the right action $\beta : \mathsf{B} \to \mathrm{C}_u(\mathbb{G}_k) \otimes \mathsf{B}$ given by (the continuous linear extensions of) the formulas $\beta(p) = 1 \otimes p$ and

$$\beta(v_{ij}) = \sum_{s=1}^k v_{is} \otimes v_{sj}, \quad i,j = 1, \ldots, k.$$

We have then $\beta \circ \nu = (id \otimes \nu)\alpha$, and further

$$\mathrm{Fix}\, \beta = \nu(\mathrm{Fix}\, \alpha).$$

Indeed, the inclusion $\supset$ follows directly from the intertwining relation above and the opposite inclusion can be shown using conditional expectations onto the fixed point spaces, as

$$\mathrm{Fix}\, \beta = ((h \otimes \mathrm{id}) \circ \beta)(\mathsf{B}) = ((h \otimes \mathrm{id}) \circ \beta)(\nu(\mathrm{C}(\mathbb{G}_n))) = (h \otimes \nu)\alpha(\mathrm{C}(\mathbb{G}_n)) = \nu(\mathrm{Fix}\, \alpha),$$

where h denotes the Haar state on $\mathrm{C}_u(\mathbb{G}_k)$. Since $\nu(\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)) = \mathrm{C}(\mathbb{Z}_2) \subset \mathsf{B}$, as subalgebras of B, it suffices to find an element in $\mathrm{Fix}\, \beta$ which is not in $\mathrm{C}(\mathbb{Z}_2)$. Define then

$$x = (h \otimes id)\beta(v_{11} p v_{11}) = \frac{1}{k} \sum_{s=1}^k v_{s1} p v_{s1}$$

The last identity follows from an easily shown formula valid for all $i, j, s = 1, \ldots, k$:

$$h(v_{is} v_{js}) = \frac{1}{k} \delta_{ij}.$$

As $x \in \mathrm{Fix}\, \beta$, it remains to show that $x \notin \nu(\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k))$. Let q denote a non-trivial projection in $\mathrm{C}(\mathbb{Z}_2)$ and consider the unital *-homomorphism $\rho = \eta \star \mathrm{id} : \mathsf{B} \to \mathrm{C}(\mathbb{Z}_2) \star C(\mathbb{Z}_2)$, where $\eta : \mathrm{C}(\mathbb{G}_k) \to \mathrm{C}(\mathbb{Z}_2)$ is induced by the map $V \mapsto \mathrm{diag}(\tilde{q}, 1_{k-2})$, with $\tilde{q}$ defined analogously to $\tilde{p}$. If $x \in \nu(\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k))$, the element x would have to commute with p. Similarly $\rho(x)$ would have to commute with $p' = \rho(p)$. But $\rho(x) = qp'q + q^\perp p' q^\perp$, where q denotes the projection generating the first copy of $\mathrm{C}(\mathbb{Z}_2)$ in $\mathrm{C}(\mathbb{Z}_2) \star \mathrm{C}(\mathbb{Z}_2)$, and it is easy to see that $qp'q + q^\perp p' q^\perp$ does not commute with p', for instance by working with a concrete model of $\mathrm{C}(\mathbb{Z}_2) \star \mathrm{C}(\mathbb{Z}_2)$ given by $\mathrm{C}^*(\mathbb{Z}_2 \star \mathbb{Z}_2)$. Thus $x \notin \nu(\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k))$.

Let now $k = n-1$ and put:

$$y = (h \otimes id)\alpha(u_{kk} u_{nn} u_{kk}) = \frac{1}{k} \sum_{s=1}^k u_{sk} u_{nn} u_{sk}$$

Then $y \in \mathrm{C}(\mathbb{G}_n/\mathbb{G}_k)$, and we need to show that y is not in $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$. By 'passing to a quantum subgroup' argument we see it suffices to do it for the free permutation group. Assume then that we are in this case. As then one can show that $\mathrm{C}_\times(S_n^+/S_k^+)$ is commutative, it suffices to show that y does not commute with u_{nn}. So, consider the surjection $\rho' : \mathrm{C}(S_n^+) \to C(\mathbb{Z}_2) \star \mathrm{C}(\mathbb{Z}_2)$ induced by the following magic unitary matrix:

$$M = \begin{pmatrix} 1_{n-4} & 0 & 0 & 0 & 0 \\ 0 & p & 0 & p^\perp & \\ 0 & 0 & q & 0 & q^\perp \\ 0 & p^\perp & 0 & p & 0 \\ 0 & 0 & q^\perp & 0 & q \end{pmatrix}$$

Here p and q the free projections generating $\mathrm{C}(\mathbb{Z}_2) \star \mathrm{C}(\mathbb{Z}_2)$. Then $\rho'(u_{nn}) = q$, $\rho'(y) = p^\perp q p^\perp + pqp$, and we can finish as in the previous case.

□

The quantum group $\mathbb{G}_n$ acts in a natural way on each of the three C*-algebras $\mathrm{C}_u(\mathbb{G}_n/\mathbb{G}_k)$, $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$, $\mathrm{C}_+(\mathbb{G}_n/\mathbb{G}_k)$. Each of the respective actions is ergodic, and admits a unique invariant tracial state (the tracial property is related to the fact that $\mathbb{G}_n$ is of Kac type).

Exercise 5.3. Prove these statements.

The last facts allow us to consider the 'reduced' versions of the above C*-algebras, defined simply as the images of the algebras with respect to the GNS representations of the respective invariant states. The proof of the following result uses the connection between the quantum group representations and combinatorics of partitions mentioned in the last lecture.

Theorem 5.6 ([BSS])**.** *Let $k, n \in \mathbb{N}$, $k < n$ and let $\mathbb{G}_k \subset \mathbb{G}_n$ denote the diagonal embedding of the symmetric or orthogonal quantum groups. The reduced versions of the* C**-algebras* $\mathrm{C}_\times(\mathbb{G}_n/\mathbb{G}_k)$ *and* $\mathrm{C}_+(\mathbb{G}_n/\mathbb{G}_k)$ *are isomorphic.*

5.2. **Quantum partial permutations.** We finish this set of lectures by presenting another recently introduced in [BS$_4$] example of C*-algebras which are related to quantum symmetry groups – more specifically free permutation groups. We begin with some classical definitions.

Definition 5.7. Let $n \in \mathbb{N}$. A partial permutation of $\{1 \ldots, n\}$ is a bijection $\sigma : X \simeq Y$, with $X, Y \subset \{1, \ldots, n\}$. We denote by $\widetilde{S}_n$ the semigroup formed by all such partial permutations (with multiplication given by the 'partial' composition).

Note that the symmetric group S_n is a subgroup of $\widetilde{S}_n$. The embedding $S_n \subset M_n(0,1)$ given by permutation matrices can be extended to an embedding $\kappa : \widetilde{S}_n \subset M_n(0,1)$ defined as follows ($\sigma \in \widetilde{S}_n$, $i, j = 1, \ldots, n$):

$$\kappa(\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

This observation motivates the following definition.

Definition 5.8. Let $n \in \mathbb{N}$. Denote by $\widetilde{\mathsf{A}}_s(n)$ the universal unital C*-algebra generated by elements $(p_{ij} : i, j = 1, \ldots, n)$ such that

(i) each p_{ij} is an orthogonal projection;
(ii) for any $i, j, k = 1, \ldots, n$ such that $j \neq k$ we have $p_{ij}p_{ik} = p_{ji}p_{ki} = 0$.

We have then the following counterpart of Proposition 2.1.

Proposition 5.9. *Let $n \in \mathbb{N}$ and use the notation of the above definition. The prescription*

$$\Delta(p_{ij}) = \sum_{k=1}^{n} p_{ik} \otimes p_{kj}, \quad i, j = 1, \ldots, n$$

determines a unital *-homomorphism $\Delta : \widetilde{\mathsf{A}}_s(n) \to \widetilde{\mathsf{A}}_s(n) \otimes \widetilde{\mathsf{A}}_s(n)$ giving $\widetilde{\mathsf{A}}_s(n)$ the structure of an algebra of continuous functions on a compact quantum semigroup, to be denoted $\widetilde{S}_n^+$ and called the free semigroup of partial permutations on n-elements.

Exercise 5.4. Prove that the classical version of $\widetilde{S}_n^+$, defined by the analogy with Proposition 4.4, is the semigroup $\widetilde{S}_n$.

The difference between the 'generating matrix' of orthogonal projections in $\widetilde{\mathsf{A}}_s(n)$ and a magic unitary is that the sum of elements in each row and column need not be equal to 1. This implies that the corresponding algebra is highly non-commutative already at $n = 2$! In that case one can describe the C*-algebraic structure precisely.

Theorem 5.10 ([BS$_4$])**.** *The* C**-algebra $\tilde{A}_s(2)$ is isomorphic to the unitization of* $\mathsf{D} \oplus \mathsf{D}$*, where* D *is the universal* C**-algebra generated by two projections. It can be explicitly realised in the following isomorphic forms (below D_∞ denotes the infinite dihedral group, and $\epsilon : \mathrm{C}^*(D_\infty) \to \mathbb{C}$ is the counit character, given by the extension of the formula $\epsilon(\gamma) = 1$, $\gamma \in D_\infty$):*

(i) $\{(x, y) \in \mathrm{C}^*(D_\infty) \oplus \mathrm{C}^*(D_\infty) : \epsilon(x) = \epsilon(y)\}$;

(ii) $\{f \in \mathrm{C}([0, 1]; M_2 \oplus M_2) : f(0), f(1)$ diagonal and $f(1)_{2,2} = f(1)_{4,4}\}$.

It has been known for a long time that the free permutation groups are closely related to *Hadamard matrices*, i.e. matrices with entries in $\mathbb{T}$ having mutually orthogonal rows (and columns). In particular every Hadamard n by n matrix yields a representation of the algebra $\mathsf{A}_s(n)$, and via the *Hopf image construction* due to Banica and Bichon ([BB$_2$]) to a quantum subgroup of S_n^+. A rich source of the information on these facts and also on connections to the subfactor theory can be found in a recent survey [Ba$_3$]. It turns out that a very similar relationship connects *partial Hadamard matrices*, i.e. rectangular n by k matrices with entries in $\mathbb{T}$ and pairwise orthogonal rows, representations of $\widetilde{\mathsf{A}}_s(n)$ and quantum subsemigroups of $\widetilde{S}_n^+$ (see [BS$_4$]). Here we only formulate the corresponding definitions and results (with the obvious quantum semigroup analogues of the notions used earlier for quantum groups).

Definition 5.11. Let $m, n \in \mathbb{N}$, $m \leq n$. A matrix $H \in M_{M\times}(\mathbb{T})$ is said to be a partial Hadamard matrix, i.e. the matrix for which the vectors $\xi_1, \ldots, \xi_m \in \mathbb{C}^n$ defined as $(\xi_l)_i = H_{li}$, $i = 1, \ldots, n, k = 1, \ldots, m$, are mutually orthogonal. We define further vectors $\xi_l/\xi_k \in \mathbb{C}^n$ by coordinate-wise division and denote the rank-one projection in $B(\mathbb{C}^n)$ on ξ_l/ξ_k simply by P_{ξ_i/ξ_j}.

Theorem 5.12 ([BS$_4$])**.** *Let $m, n \in \mathbb{N}$, $m \leq n$. Every partial Hadamard matrix $H \in M_{M\times}(\mathbb{T})$ defines a representation $\pi_H : \widetilde{\mathsf{A}}_s(m) \to B(\mathbb{C}^n)$ defined by the formula*

$$\pi(p_{ij}) = P_{\xi_i/\xi_j}, i, j = 1, \cdots m,$$

where p_{ij} denote the standard generators of $\widetilde{\mathsf{A}}_s(m)$. Further there exists a largest quantum subsemigroup of $\widetilde{S}_m^+$, say $\mathbb{S}_H$, such that the algebra homomorphism π_H factorises via the quantum semigroup morphism $\gamma : \mathrm{C}(\widetilde{S}_m^+) \to \mathrm{C}(\mathbb{S}_H)$.

The word ‘largest’ above is understood via a certain universal property; for the details we refer to [BS_4].

5.3. **Outlook and a list of open problems.** In these lectures we only surveyed the landscape of quantum symmetry groups. In recent years they have become an arena and a source of examples for investigations related to areas such as classical and free (and more generally noncommutative) probability, theory of random matrices, noncommutative geometry and general theory of operator algebras. As mentioned before, the richness of related combinatorial structures is revealed in particular in the study of representation theory of quantum symmetry groups ([BSp]). These structures may in turn be used to analyse geometric properties of the dual discrete groups and to study $C(\mathbb{G})$ and $L^\infty(\mathbb{G})$ from the purely operator algebraic viewpoint – a perfect example of that is the article [Bra], which sparked the interest in the geometric theory of quantum groups (see [DFSW] and references therein). For more examples of the variety of contexts in which quantum symmetry groups play an important role we refer to the surveys [BBC_1] and [Ba_3]. Here we want to finish with the following list of open problems, related directly to the topics treated in these lectures.

- Which of the classical objects (such as finite graphs) admit quantum symmetries?
- More generally, when the structure of quantum symmetries is different from this of classical ones? Does there exists a finite graph $\mathcal{G}$ such that the action of the symmetry group of $\mathcal{G}$ on $\mathcal{G}$ is not ergodic, but the action of $\mathsf{QSYM}(\mathcal{G})$ on $\mathcal{G}$ is ergodic?
- Suppose that Γ is a finitely generated group, with two different generating sets S_1 and S_2. What properties are shared by the quantum isometry groups $\mathsf{QISO}(\widehat{\Gamma}, S_1)$ and $\mathsf{QISO}(\widehat{\Gamma}, S_2)$? Note that it is known that the latter quantum groups can be non-isomorphic, see [BhS] and [L-DS].
- Under what condition is a quantum symmetry group of a given finite structure (a graph, a filtration, etc.) finite?
- What is the ‘right’ notion of a quantum automorphism group of a given finite (quantum) group? One possible definition was proposed in [BhSS], but it was shown later in [KSW] that it always reduces to the classical symmetry group.
- How can one rigorously define *locally compact quantum groups* arising as quantum symmetry groups of some infinite objects?

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