77 ~~78~~

Zad 46 1) $f(x,y) = x^y \quad x>0, y\in\mathbb{R}$

$\frac{\partial f}{\partial x} = y x^{y-1} \qquad \frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2} \qquad \frac{\partial^2 f}{\partial y \partial x} = x^{y-1} + y \ln x \, x^{y-1}$

$\frac{\partial f}{\partial y} = \ln x \cdot x^y \qquad \frac{\partial^2 f}{\partial y^2} = (\ln x)^2 \cdot x^y \qquad \frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + \ln x \cdot y \cdot x^{y-1} = \frac{\partial^2 f}{\partial y \partial x}$

2) $u(x,y,z) = f(\overset{s}{\overset{\shortparallel}{x+y}}, \overset{t}{\overset{\shortparallel}{\sin(x+z)}})$

$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}(\cdot) + \frac{\partial f}{\partial t}(\cdot)\cos(x+z), \quad \frac{\partial u}{\partial y} = \frac{\partial f}{\partial s}, \quad \frac{\partial u}{\partial z} = \frac{\partial f}{\partial t}\cos(x+z)$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial s}(x+y, \sin(x+z)) + \frac{\partial f}{\partial t}(x+y, \sin(x+z))\cos(x+z)\right] = \frac{\partial^2 f}{\partial s^2}() + \frac{\partial^2 f}{\partial t \partial s}() \cdot \cos(x+z)$

$+ \frac{\partial^2 f}{\partial s \partial t}() \cdot \cos(x+z) + \frac{\partial^2 f}{\partial t^2}() \cos^2(x+z) + \frac{\partial f}{\partial t}()(-\sin(x+z))$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial s}(x+y, \sin(x+z)) = \frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial s \partial t} \cdot \cos(x+z)$

78

$$\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial t}(x+y, \sin(x+z))\cos(x+z)\right] = \frac{\partial^2 f}{\partial s \partial t}\cos(x+z) + \frac{\partial^2 f}{\partial t^2}\cos^2(x+z) - \frac{\partial f}{\partial t}\sin(x+z)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial s}(x+y, \sin(x+z))\right] = \frac{\partial^2 f}{\partial s^2}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \cancel{\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial t}\cos(x+z)\right]} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial t}(x+y, \sin(x+z))\cdot\cos(x+z)\right] = \frac{\partial^2 f}{\partial s \partial t}\cos(x+z)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial t}(x+y, \sin(x+z))\cdot\cos(x+z)\right] = \frac{\partial^2 f}{\partial t^2}\cos^2(x+z) + \frac{\partial f}{\partial t}\cdot(-\sin(x+z))$$

Zad 47 (79)

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ są ciągłe na $\mathbb{R}^2$: Poza $(0,0)$ ciągłe jako funkcje wymierne

$$\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0 = f(0,0), \text{ bo } \left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| \leq \left|\frac{1}{2}(x^2-y^2)\right| \underset{\substack{x\to 0\\ y\to 0}}{\to} 0$$

Zatem f ciągła w $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$$

dla $(x,y) \neq (0,0)$ $\qquad f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ $\qquad$ 80

$$\frac{\partial f}{\partial x} = \frac{(3x^2y - y^3)(x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{3x^4y + 3x^2y^3 + 2x^2y^3 - y^3x^2 - y^5 - 2x^4y}{(x^2+y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} = \lim_{\substack{x\to 0\\ y\to 0}} y\,\frac{(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} = \overset{\frac{\partial f}{\partial x}(0,0)}{\overset{\|}{0}}, \text{ bo } \left|\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}\right| < 2$$

Podobnie

$$\frac{\partial f}{\partial y} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial f}{\partial y}(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} x\,\frac{x^4 - 4x^2y^2 - y^4}{(x^2+y^2)^2} = 0 = \frac{\partial f}{\partial y}(0,0)$$

b) $\frac{\partial^2 f}{\partial x \partial y}$ i $\frac{\partial^2 f}{\partial y \partial x}$ istnieją na $\mathbb{R}^2$ i są ciągłe poza $(0,0)$

$(x,y) \neq (0,0)$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial x} \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} = \frac{(5x^4 - 12x^2y^2 - y^4)(x^2+y^2)^2 - (x^5 - 4x^3y^2 - xy^4)\cancel{f}\,2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

– ciągłe poza $(0,0)$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial y} \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} =$$

$$= \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2+y^2)^2 - (x^4y + 4x^2y^3 - y^5)\,2(x^2+y^2)\,2x}{(x^2+y^2)^2}$$

– ciągłe poza $(0,0)$

w $(0,0)$ $\qquad \frac{\partial f}{\partial x}(x,y) = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \qquad \frac{\partial f}{\partial y}(x,y) = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$ 82

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \frac{\partial f}{\partial y}(0,0)}{t} = \lim_{t\to 0} \frac{\frac{t^5}{t^4} - 0}{t} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t} = \lim_{t\to 0} \frac{\frac{-t^5}{t^4} - 0}{t} = -1$$

Zatem $\frac{\partial^2 f}{\partial x \partial y}$ i $\frac{\partial^2 f}{\partial y \partial x}$ istnieją też w zerze

c)

$$1 = \frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f(0,0)}{\partial y \partial x(0,0)} = -1$$

Zad 48 $\quad f(x,y) = x e^{xy^2}$ $\qquad$ Macierz Hessego $\qquad$ 83 $\quad$ 129

$D^2 f(1,1) h h' = h A h'$, gdzie $h, h' \in \mathbb{R}^2$

$$A = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(1,1) & \frac{\partial^2 f}{\partial x \partial y}(1,1) \\ \frac{\partial^2 f}{\partial y \partial x}(1,1) & \frac{\partial^2 f}{\partial y^2}(1,1) \end{pmatrix}$$

$\frac{\partial f}{\partial x} = e^{xy^2} + xy^2 e^{xy^2}$, $\frac{\partial f}{\partial y} = 2x^2 y e^{xy^2}$

$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy^2} + y^2 e^{xy^2} + xy^4 e^{xy^2} = 2y^2 e^{xy^2} + xy^4 e^{xy^2}$

$\frac{\partial^2 f}{\partial x \partial y} = 4xy e^{xy^2} + 2x^2 y^3 e^{xy^2}$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = 2x^2 e^{xy^2} + 4x^3 y^2 e^{xy^2}$

– pochodne cząstkowe drugiego rzędu są ciągłe, czyli f jest klasy C^2, a więc w szczególności dwukrotnie różniczkowalna

$$A = \begin{pmatrix} 3e & 6e \\ 6e & 6e \end{pmatrix}$$

$h = (h_1, h_2)$, $h' = (h_1', h_2')$ $\qquad$ $D^2 f(1,1) h h' = (h_1, h_2) \begin{pmatrix} 3e & 6e \\ 6e & 6e \end{pmatrix} \begin{pmatrix} h_1' \\ h_2' \end{pmatrix}$

$$D^2 f(1,1) h h' = h_1 h_1' \frac{\partial^2 f}{\partial x^2}(1,1) + h_1 h_2' \frac{\partial^2 f}{\partial x \partial y}(1,1) + h_2 h_1' \frac{\partial^2 f}{\partial y \partial x}(1,1) + h_2 h_2' \frac{\partial^2 f}{\partial y^2}(1,1)$$

$$D^2 f h h' = 3e h_1 h_1' + 6e h_1 h_2' + 6e h_2 h_1' + 6e h_2 h_2'$$

Zad 49. $g(t) = f(\overset{x}{\overset{\shortparallel}{\cos t}}, \overset{y}{\overset{\shortparallel}{\sin t}})$

$$g'(t) = \frac{\partial f}{\partial x}(\cos t, \sin t)(-\sin t) + \frac{\partial f}{\partial y}(\cos t, \sin t)\cos t$$

$$g''(t) = \frac{\partial^2 f}{\partial x^2}(\cos t, \sin t)\cdot \sin^2 t - \frac{\partial^2 f}{\partial x \partial y}(\cos t, \sin t)\cos t \sin t - \frac{\partial f}{\partial x}(\cos t, \sin t)\cdot \cos t$$

$$- \frac{\partial^2 f}{\partial x \partial y}\cos t \sin t + \frac{\partial^2 f}{\partial y^2}(\cos t, \sin t)\cos^2 t - \frac{\partial f}{\partial y}(\cos t, \sin t)\sin t =$$

$$= \sin^2 t \frac{\partial^2 f}{\partial x^2} - 2\sin t \cos t \frac{\partial^2 f}{\partial x \partial y} + \cos^2 t \frac{\partial^2 f}{\partial y^2} - \cos t \frac{\partial f}{\partial x} - \sin t \frac{\partial f}{\partial y}$$

Zad 50.

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ – operator Laplace'a $\quad u = \ln\sqrt{x^2+y^2} \quad \Delta u = ?$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\,\frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Podobnie $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$, $\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$. Zatem $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} = 0$

<u>Uwaga</u> Funkcje spełniające równanie $\Delta u = 0$ nazywamy <u>funkcjami harmonicznymi</u>